An overview of the algorithms for computing the Bernstein-Sato polynomial

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RIMS-Sing 2: Singularity theory and geometric topology
特異点論と幾何的トポロジー

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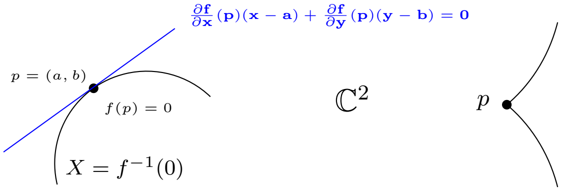
Motivation to Singularity Theory

Motivation

- Let $f \in \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial.
- $p \in \mathbb{C}^n$ is said to be singular if $p \in V(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$.



- To study singular points $\rightsquigarrow$ invariants. (Bernstein-Sato polynomial, monodromy zeta function, topological zeta function)
- Two hypersurfaces $X = V(f)$, $Y = V(g) \subseteq \mathbb{C}^n$ are called algebraically equivalent if there exists an algebraic isomorphism $\varphi : X \to Y$.

First part of the talk

- Basic notations and definitions
- History of the problem
- ...
- Well-known properties.
- Algorithms for computing $b_f(s)$

Basic notations

- $\mathbb{C}$ the field of the complex numbers.
- $\mathbb{C}[s]$ the ring of polynomials in one variable over $\mathbb{C}$.
- $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$ the ring of polynomials in n variables.
- $D_n = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$ the ring of $\mathbb{C}$-linear differential operators, i.e. the n-th Weyl algebra:

$$\partial_i x_i = x_i \partial_i + 1$$

- $D_n[s] = D_n \otimes_{\mathbb{C}} \mathbb{C}[s]$.

The Weyl algebra

- $W_n = \mathbb{C}\left\langle\left\{\phi_{x_1},\dots,\phi_{x_n},\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_1}\right\}\right\rangle \subset \mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathbf{x}])$

$$\begin{array}{cccc} & \mathbb{C}[x_1,\dots,x_n] & \longrightarrow & \mathbb{C}[x_1,\dots,x_n] \\ \phi_{x_i}\ : & f & \longmapsto & x_i f \\ \dfrac{\partial}{\partial x_i}\ : & f & \longmapsto & \dfrac{\partial f}{\partial x_i} \end{array}$$

- $D_n = \dfrac{\mathbb{C}\{x_1,\dots,x_n,\ \partial_1,\dots,\partial_n\}}{\langle\{x_i,x_j - x_j x_i,\quad \partial_i\partial_j - \partial_j\partial_i,\quad \partial_i x_j - x_j\partial_i - \delta_{ij}\}\rangle}$

The Weyl algebra

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$$\begin{array}{rcccc} & & \mathbb{C}[x_1,\ldots,x_n] & \longrightarrow & \mathbb{C}[x_1,\ldots,x_n] \\ \phi_{x_i} & : & f & \longmapsto & x_i f \\ \dfrac{\partial}{\partial x_i} & : & f & \longmapsto & \dfrac{\partial f}{\partial x_i} \end{array}$$

- $D_n = \dfrac{\mathbb{C}\{x_1,\ldots,x_n,\ \partial_1,\ldots,\partial_n\}}{\langle\{x_i x_j - x_j x_i,\quad \partial_i\partial_j - \partial_j\partial_i,\quad \partial_i x_j - x_j\partial_i - \delta_{ij}\}\rangle}$

Proposition

The natural map $x_i \mapsto \phi_{x_i},\ \partial_i \mapsto \frac{\partial}{\partial x_i}$ is a $\mathbb{C}$-algebra isomorphism between W_n and D_n.

- The non-commutative relations come from Leibniz rule.

$$\partial_i x_i = x_i \partial_i + 1$$

- The set of monomials $\{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\}$ forms a basis as $\mathbb{C}$-vector space.

$$P = \sum_{\alpha,\beta} a_{\alpha\beta} x^\alpha \partial^\beta \qquad (a_{\alpha\beta} \in \mathbb{C})$$

- To define a D_n-module, it is enough to give the action over the generators and then check that the relations are preserved.

- For any monomial order there exists a Gröbner basis.

- The Weyl algebra is simple, i.e. there are no two-sided ideals.

The $D_n[s]$-module $\mathbb{C}[\mathbf{x}, s, \frac{1}{f}] \cdot T$ (Bernstein, 1972)

- Let $f \in \mathbb{C}[\mathbf{x}]$ be a non-zero polynomial.
- By $\mathbb{C}[\mathbf{x}, s, \frac{1}{f}]$ we denote the ring of rational functions of the form
$$\frac{g(\mathbf{x}, s)}{f^k}$$
where $g(\mathbf{x}, s) \in \mathbb{C}[\mathbf{x}, s] = \mathbb{C}[x_1, \ldots, x_n, s]$.
- We denote by $\mathbb{C}[\mathbf{x}, s, \frac{1}{f}] \cdot T$ the free $\mathbb{C}[\mathbf{x}, s, \frac{1}{f}]$-module of rank one generated by the symbol T.
$$G(\mathbf{x}, s) \cdot T$$

The $D_n[s]$-module $\mathbb{C}[\mathbf{x}, s, \frac{1}{f}] \cdot T$ (Bernstein, 1972)

- $\mathbb{C}[\mathbf{x}, s, \frac{1}{f}] \cdot T$ has a natural structure of left $D_n[s]$-module.

$$
\begin{aligned}
x_i \bullet (G(\mathbf{x}, s) \cdot T) &= x_i G(\mathbf{x}, s) \cdot T \\
\partial_i \bullet (G(\mathbf{x}, s) \cdot T) &= \left(\frac{\partial G}{\partial x_i} + G(\mathbf{x}, s) s \frac{\partial f}{\partial x_i} \frac{1}{f} \right) \cdot T \\
s \bullet (G(\mathbf{x}, s) \cdot T) &= s G(\mathbf{x}, s) \cdot T
\end{aligned}
$$

The previous expression defines an action.

- $(x_i x_j - x_j x_i) \bullet (G(\mathbf{x}, s) \cdot T) = 0 \cdot T$
- $(\partial_i \partial_j - \partial_j \partial_i) \bullet (G(\mathbf{x}, s) \cdot T) = 0 \cdot T$
- $(\partial_i x_j - x_j \partial_i) \bullet (G(\mathbf{x}, s) \cdot T) = 0 \cdot T \qquad (i \neq j)$
- $(\partial_i x_i - x_i \partial_i - 1) \bullet (G(\mathbf{x}, s) \cdot T) = 0 \cdot T$

$\boxed{\text{Where does this action come from?}} \longrightarrow T = f^s$

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$$\partial_i \bullet (G(\mathbf{x}, s) \cdot T) = \left(\frac{\partial G}{\partial x_i} + G(\mathbf{x}, s) s \frac{\partial f}{\partial x_i} \frac{1}{f} \right) \cdot T$$

$$\Downarrow$$

$$\partial_i \bullet (G(\mathbf{x}, s) \cdot f^s) = \left(\frac{\partial G}{\partial x_i} + G(\mathbf{x}, s) s \frac{\partial f}{\partial x_i} \frac{1}{f} \right) \cdot f^s$$

Classical notation

- $\mathbb{C}[\mathbf{x}, s, \frac{1}{f}] \cdot f^s := \mathbb{C}[\mathbf{x}, s, \frac{1}{f}] \cdot T$
- $f^s := 1 \cdot T$
- $f^{s+k} := f^k \cdot T \qquad (k \in \mathbb{Z})$
- $0 := 0 \cdot T$

$$\partial_i \bullet f^s = s\frac{\partial f}{\partial x_i} f^{s-1} \quad \Longrightarrow \quad \partial_i \bullet T = s\frac{\partial f}{\partial x_i}\frac{1}{f} \cdot T$$

The global b-function

Theorem (Bernstein, 1972)

For every polynomial $f \in \mathbb{C}[\mathbf{x}]$ there exists a non-zero polynomial $b(s) \in \mathbb{C}[s]$ and a differential operator $P(s) \in D_n[s]$ such that

$$P(s) \bullet f^{s+1} = b(s) \cdot f^s \quad \in \quad \mathbb{C}\left[\mathbf{x}, s, \frac{1}{f}\right] \cdot f^s.$$

Definition (Bernstein & Sato, 1972)

The set of all possible polynomials $b(s)$ satisfying the previos equation is an ideal of $\mathbb{C}[s]$. The monic generator of this ideal is denoted by $b_f(s)$ and called the global Bernstein-Sato polynomial or global b-function.

Examples

1. Normal crossing divisor $f = x_1^{m_1} \cdots x_k^{m_k}$.

$$
\begin{aligned}
P(s) &= c \cdot \partial_1^{m_1} \cdots \partial_k^{m_k} \\
b_f(s) &= \prod_{i_1=1}^{m_1} \left(s + \frac{i_1}{m_1} \right) \cdots \prod_{i_k=1}^{m_k} \left(s + \frac{i_k}{m_k} \right)
\end{aligned}
$$

2. The classical cusp $f = x^2 + y^3$.

$$
\begin{aligned}
P(s) &= \frac{1}{12} y \partial_x^2 \partial_y + \frac{1}{27} \partial_y^3 + \frac{1}{4} \partial_x s + \frac{3}{8} \partial_x^2 \\
b_f(s) &= \left(s + 1 \right) \left(s + \frac{5}{6} \right) \left(s + \frac{7}{6} \right)
\end{aligned}
$$

The local b-function

Now assume that

- $f \in \mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$ is a convergent power series.
- $\mathcal{D}_n$ is the ring of differential operators with coefficients in $\mathcal{O}$.

Theorem (Kashiwara & Björk, 1976)

For every $f \in \mathcal{O}$ there exists a non-zero polynomial $b(s) \in \mathbb{C}[s]$ and a differential operator $P(s) \in \mathcal{D}_n[s]$ such that

$$P(s) \bullet f^{s+1} = b(s) \cdot f^s \quad \in \quad \mathcal{O}\left[s, \frac{1}{f}\right] \cdot f^s.$$

Definition

The monic polynomial in $\mathbb{C}[s]$ of lowest degree which satisfies the previous equation is denoted by $b_{f,0}(s)$ and called the local Bernstein-Sato polynomial of f at the origin or local b-function.

Some well-known properties of the b-function

1. The b-function is always a multiple of $(s+1)$. The equality holds if and only f is smooth.

2. The (resp. local) Bernstein-Sato polynomial is an (resp. analytic) algebraic invariant of the singularity $V = \{f = 0\}$.

3. The set $\{e^{2\pi i\alpha} \mid b_{f,0}(\alpha) = 0\}$ coincides with the eigenvalues of the monodromy of the Milnor fibration. (Malgrange, 1975 and 1983).

Some well-known properties of the b-function

1. Every root of $b_f(s)$ is negative and rational. (Kashiwara, 1976).

2. The roots of $b_f(s)$ belong to the real interval $(-n, 0)$. (Varchenko, 1980; Saito, 1994).

3. $b_f(s) = \mathrm{lcm}_{p \in \mathbb{C}^n}\{b_{f,p}(s)\} = \mathrm{lcm}_{p \in Sing(f)}\{b_{f,p}(s)\}$ (Briançon-Maisonobe and Mebkhout-Narváez, 1991).

Algorithms for computing the b-function

1. Functional equation, $P(s)f \bullet f^s = b_f(s) \cdot f^s$.
2. By definition, $(\mathrm{Ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{C}[s] = \langle b_f(s) \rangle$.
3. Now find a system of generator of the annihilator and proceed with the elimination.

Annihilator	Elimination
Oaku-Takayama (1997)	Noro (2002)
Briançon-Maisonobe (2002)	Andre-Levandovskyy-MM (2009)
Levandovskyy (2008)	

Second part of the talk

- Partial solution: the `checkRoot` algorithm
- Applications:
  1. Computation of b-functions via upper bounds.
  2. Integral roots of b-functions.
  3. Stratification associated with local b-functions without employing primary ideal decomposition.

Another idea for computing the b-function

1. Obtain an upper bound for $b_f(s)$: find $B(s) \in \mathbb{C}[s]$ such that $b_f(s)$ divides $B(s)$.

$$B(s) = \prod_{i=1}^{d} (s - \alpha_i)^{m_i}.$$

2. Check whether α_i is a root of the b-function.
3. Compute its multiplicity m_i.

Remark

There are some well-known methods to obtain such $B(s)$: Resolution of Singularities.

The main trick

- By definition, $(\mathrm{Ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{C}[s] = \langle b_f(s) \rangle$.
- $(\mathrm{Ann}_{D_n[s]}(f^s)+\langle f \rangle)\cap\mathbb{C}[s]+\langle q(s) \rangle = \langle b_f(s), q(s) \rangle, \; q(s) \in \mathbb{C}[s]$

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Proposition

$$(\mathrm{Ann}_{D_n[s]}(f^s) + \langle f, q(s) \rangle) \cap \mathbb{C}[s] = \langle b_f(s), q(s) \rangle$$
$$= \langle gcd(b_f(s), q(s)) \rangle$$

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Corollary

- m_α the multiplicity of α as a root of $b_f(-s)$.
- $J_i = \mathrm{Ann}_{D_n[s]}(f^s) + \langle f, (s+\alpha)^{i+1} \rangle \subseteq D_n[s]$.

The following conditions are equivalent:

1. $m_\alpha > i$.
2. $(s+\alpha)^i \notin J_i$.

Algorithm (compute the multiplicity of α as a root of $b_f(-s)$)

Input: $I = \mathrm{Ann}_{D_n[s]}(f^s)$, f a polynomial in R_n, α in $\mathbb{Q}$;
Output: m_α, the multiplicity of α as a root of $b_f(-s)$;

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for $i = 0$ to n do
1. $J := I + \langle f, (s+\alpha)^{i+1} \rangle$; $\qquad \triangleright\ J_i \subseteq D_n[s]$
2. G a reduced Gröbner basis of J w.r.t. any term ordering;
3. r normal form of $(s+\alpha)^i$ with respect to G;
4. if $r = 0$ then $\qquad \triangleright\ r = 0 \Longleftrightarrow (s+\alpha)^i \in J_i$
    $m_\alpha = i$; break $\qquad \triangleright$ leave the for block
  end if

end for

return m_α

Comments

The `checkRoot` algorithm

This algorithm is much faster, than the computation of the whole Bernstein polynomial via Gröbner bases because:

- No elimination ordering is needed. (cf. [Noro])
- The element $(s+\alpha)^{i+1}$ seems to simplify tremendously the computation.

Applications

1. Computation of the *b*-functions via upper bounds.
    - Embedded resolutions.
    - Topologically equivalent singularities.
    - A'Campo's formula.
    - Spectral numbers.
2. Intergral roots of *b*-functions.
    - Logarithmic comparison problem.
    - Intersection homology D-module.
3. Stratification associated with local *b*-functions.
4. Bernstein-Sato polynomials for Varieties.
5. Narvaez's paper.

Computation of the b-functions via embedded resolutions

Resolution of Singularities

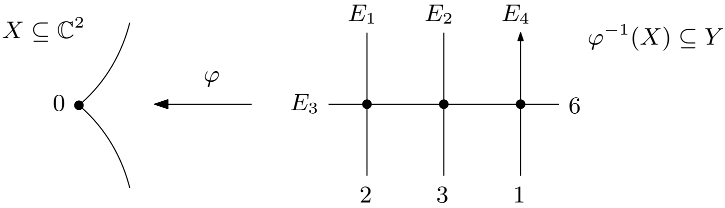
- Let $f \in \mathcal{O}$ be a convergent power series, $f : \Delta \subseteq \mathbb{C}^n \to \mathbb{C}$.
- Assume that $f(0) = 0$, otherwise $b_{f,0}(s) = 1$.
- Let $\varphi : Y \to \Delta$ be an embedded resolution of $\{f = 0\}$.
- If $F = f \circ \varphi$, then $F^{-1}(0)$ is a normal crossing divisor.

Theorem (Kashiwara).

There exists an integer $k \geq 0$ such that $b_f(s)$ is a divisor of the product $b_F(s) b_F(s+1) \cdots b_F(s+k)$.

Example

- Let us consider $f = y^2 - x^3 \in \mathbb{C}\{x, y\}$.



- From Kashiwara, the possible roots of $b_f(-s)$ are:

$$\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1, \frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{11}{6}.$$

- Using our algorithm, we have proven that the numbers in red are the roots of $b_f(s)$, all of them with multiplicity one.

Hard example

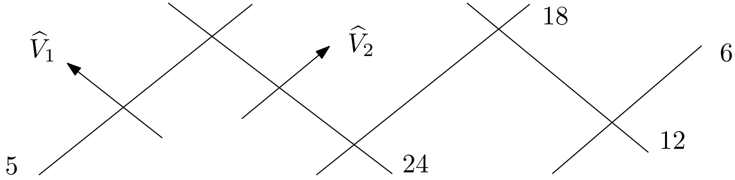


Figure: Embedded resolution of $V((xz+y)(x^4+y^5+xy^4))$

$$\begin{aligned} b_f(s) = {} & (s+1)^2(s+17/24)(s+5/4)(s+11/24)(s+5/8) \\ & (s+31/24)(s+13/24)(s+13/12)(s+7/12)(s+23/24) \\ & (s+5/12)(s+3/8)(s+11/12)(s+9/8)(s+7/8) \\ & (s+19/24)(s+3/4)(s+29/24)(s+25/24) \end{aligned}$$

Integral roots of b-functions

The minimal integral root of $b_f(s)$

Let us consider the following example:

- $A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \end{pmatrix}$
- Δ_i determinant of the minor resulting from deleting the i-th column of A, $i = 1, 2, 3, 4$.
- $f = \Delta_1 \Delta_2 \Delta_3 \Delta_4 \in \mathbb{C}[x_1, \ldots, x_{12}]$.

From Kashiwara, the possible integral roots of $b_f(-s)$ are

$$11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1.$$

Using our algorithm, we have proven that the minimal integral root of $b_f(s)$ is -1.

Stratification associated with local b-functions

Stratratification associated with local b-functions

Theorem

- $\mathrm{Ann}_{D[s]}(f^s) + D[s]\langle f \rangle = D[s]\langle \{P_1(s), \ldots, P_k(s), f\} \rangle$
- $I_{\alpha,i} = \left(I : (s+\alpha)^i\right) + D[s]\langle s+\alpha \rangle$, $(i = 0, \ldots, m_\alpha - 1)$

$$m_\alpha(p) > i \iff p \in V(I_{\alpha,i} \cap \mathbb{C}[\mathbf{x}])$$

1. $V_{\alpha,i} = V(I_{\alpha,i} \cap \mathbb{C}[\mathbf{x}])$
2. $\emptyset =: V_{\alpha,m_\alpha} \subset V_{\alpha,m_\alpha - 1} \subset \cdots \subset V_{\alpha,0} \subset V_{\alpha,-1} := \mathbb{C}^n$
3. $m_\alpha(p) = i \iff p \in V_{\alpha,i-1} \setminus V_{\alpha,i}$

(cf. [Nishiyama-Noro])

Experiments

- $f = (x^2 + 9/4y^2 + z^2 - 1)^3 - x^2z^3 - 9/80y^2z^3 \in \mathbb{C}[x,y,z]$
- $b_f(s) = (s+1)^2(s+4/3)(s+5/3)(s+2/3)$
- $V_1 = V(x^2 + 9/4y^2 - 1, z)$
- $V_2 = V(x, y, z^2 - 1) \longrightarrow$ two points
- $V_3 = V(19x^2 + 1, 171y^2 - 80, z) \longrightarrow$ four points
- $V_3 \subset V_1, \quad V_1 \cap V_3 = \emptyset$
- $Sing(f) = V_1 \cup V_2$

$$\begin{array}{rclcccccl} \alpha & = & -1, & \emptyset & \subset & V_1 \subset V(f) & \subset & \mathbb{C}^3; \\ \alpha & = & -4/3, & \emptyset & \subset & V_1 \cup V_2 & \subset & \mathbb{C}^3; \\ \alpha & = & -5/3, & \emptyset & \subset & V_2 \cup V_3 & \subset & \mathbb{C}^3; \\ \alpha & = & -2/3, & \emptyset & \subset & V_1 & \subset & \mathbb{C}^3. \end{array}$$

From this, one can easily find a stratification of $\mathbb{C}^3$ into constructible sets such that $b_{f,p}(s)$ is constant on each stratum.

$$\begin{cases} 1 & p \in \mathbb{C}^3 \setminus V(f), \\ s+1 & p \in V(f) \setminus (V_1 \cup V_2), \\ (s+1)^2(s+4/3)(s+2/3) & p \in V_1 \setminus V_3, \\ (s+1)^2(s+4/3)(s+5/3)(s+2/3) & p \in V_3, \\ (s+1)(s+4/3)(s+5/3) & p \in V_2. \end{cases}$$

ありがとうございました

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特異点論と幾何的トポロジー

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