Monodromy conjecture and zeta functions via resolution of singularities

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Joint work with

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RIMS-Sing 2: Singularity theory and geometric topology

特異点論と幾何的トポロジー

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Introduction

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ.

Consider an embedded resolution of singularities $h : Y \to \mathbb{C}^n$ of $D = \{f = 0\}$, with $h^{-1}(D) = \bigcup_{i \in S} E_i$ normal crossings, E_i smooth.

- $\rightsquigarrow$ Numerical data: $\{(N_i, \nu_i)\}_{i \in S} \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ given by the multiplicities

$$\mathrm{div}(h^* f) = \sum_{i \in S} N_i E_i \quad \text{and} \quad \mathrm{div}(h^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)) = \sum_{i \in S} (\nu_i - 1) E_i.$$

- $\rightsquigarrow$ Stratification of Y: For any $I = \{i_1, \ldots, i_m\} \subset S$:

$$Y_I = \bigcap_{i \in I} E_i \setminus \bigcup_{j \notin I} E_j$$
$$= \left\{ q \in Y \;\middle|\; \begin{array}{l} h^* f = u(y)\, y_1^{N_{i_1}} \cdots y_m^{N_{i_m}} \\ h^*(\wedge_k \mathrm{d}x_k) = v(y)\, y_1^{\nu_{i_1} - 1} \cdots y_m^{\nu_{i_m} - 1} (\wedge_k \mathrm{d}y_k) \end{array} \text{ around } q \right\}$$

Definition

The (local) topological zeta function of f at $0 \in \mathbb{C}^n$:

$$Z_{\mathrm{top},0}(f;s) = \sum_{I \subset S} \chi(Y_I \cap h^{-1}(0)) \cdot \prod_{i \in I} \frac{1}{N_i s + \nu_i} \in \mathbb{Q}(s).$$

- (Denef-Loeser) it does not depend on the chosen resolution.
- (Artal et al.) it is an analytic invariant, but not topological.
- $\left\{ \text{Poles } s_0 \text{ of } Z_{\mathrm{top},0}(f;s) \right\} \subset \{-\nu_i/N_i\}_{i=1}^{r} \subset \mathbb{Q}_{\leq 0}$.

The Monodromy conjecture

Fix $x_0 \in f^{-1}(0)$,

- Consider the locally trivial fibration

$$f_| : \mathbb{B}_\varepsilon(x_0) \cap f^{-1}(\mathbb{D}_\eta^*) \longrightarrow \mathbb{D}_\eta^*,$$

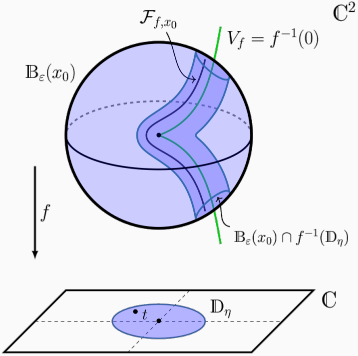
  for some $0 < \eta \ll \varepsilon \ll 1$.

- Milnor fiber at x_0:

$$\mathcal{F}_{t,x_0} = f_|^{-1}(t), \quad t \in \mathbb{D}_\eta^*.$$

- The monodromy action

$$H^\bullet(\mathcal{F}_{t,x_0};\mathbb{C}) \overset{\sigma^*}{\to} H^\bullet(\mathcal{F}_{t,x_0};\mathbb{C})$$



Conjecture (Igusa, Denef-Loeser)

If $s_0 \in \mathbb{C}$ is a pole of $Z_{\text{top},0}(f;s)$, then $e^{2\pi s_0}$ is an eigenvalue of some $H^j(\mathcal{F}_t;\mathbb{C}) \overset{\sigma^*}{\to} H^j(\mathcal{F}_t;\mathbb{C})$, at some point $x_0 \in f^{-1}(0)$ closed to the origin.

Theorem (A'Campo'75)

Let $h : Y \to \mathbb{C}^n$ be an embedded $\mathbb{Q}$-resolution of $(D, 0)$. Consider the subset of strata $\{S_m\}_{m\geq 0}$ of $\{Y_k\}_{k\geq 0}$ verifying that for any $q \in S_m$, h^*f is locally a germ

$$x^{N_m} : \mathbb{C}^n \longrightarrow \mathbb{C},$$

being N_m constant along the stratum. Then:

$$\zeta_{f,0}(t) = \prod_{m\geq 0} \left(1 - t^{N_m}\right)^{\chi\left(S_m \cap h^{-1}(0)\right)}.$$

The Monodromy conjecture: state of the art

Proven for:

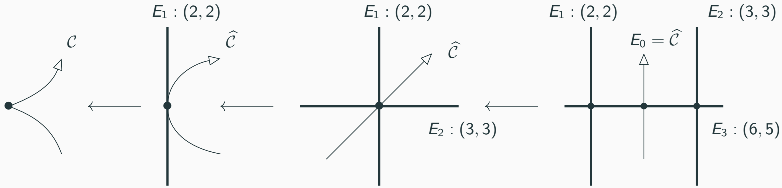
- (Loeser'88): curves, $n = 2$.
- (Loeser'90): non-degenerate surface singularities with respect to its Newton polyhedron $\Gamma(f)$.
- (Rodrigues-Veys'01): $n = 3$ and f homogeneous + some condition.
- (Artal et al.'02-'05): SIS, quasi-ordinary singularities....
- ...
- No counterexample is known so far!

Usual strategy of proof

- Study the combinatorics of the resolutions,
- Determine E_i providing true poles,
- Compare the poles to the eigenvalues. (A'Campo's formula)

Example

The cusp $\mathcal{C} : f(x, y) = y^2 - x^3$.



$$\operatorname{div}(h^* f) = \widehat{\mathcal{C}} + 2E_1 + 3E_2 + 6E_3 \quad \text{and} \quad \operatorname{div}(h^*(\mathrm{d}x \wedge \mathrm{d}y)) = E_1 + 2E_2 + 4E_3,$$

$$Z_{\mathrm{top},0}(f; s) = \frac{4s+5}{(s+1)(6s+5)}, \qquad \zeta_{f,0}(t) = \frac{1-t}{t^2 - t + 1}.$$

Embedded resolutions are hard and costly to compute in general!

Question

Do simpler models exists for $f^{-1}(0)$ to determine $Z_{\text{top},0}(f;s)$?

$\rightsquigarrow$ $Z_{\text{top},0}(f;s)$ from embedded $\mathbb{Q}$-resolutions.

Embedded $\mathbb{Q}$-resolutions of singularities

Definition

A complex analytic space Y is called a V-manifold if $Y = \bigcup_k U_k$ such that each open $U_k \simeq \mathbb{C}^n/G_k$, for some finite group $G_k \subset \mathrm{GL}_n(\mathbb{C})$.

Notation: for the dth-cyclic group $G = \{z^d = 1\} = \langle \xi_d \rangle$:

$$\frac{1}{d}(q_1, \ldots, q_n) = \mathbb{C}^n/G,$$

given by the action $\xi_d \cdot (x_1, \ldots, x_n) = (\xi_d^{q_1} x_1, \ldots, \xi_d^{q_n} x_n)$.

Definition (Steenbrink'76)

A hypersurface $D \subset Y$ has $\mathbb{Q}$-normal crossings if it is locally given by

$$x_1^{a_1} \cdots x_k^{a_k} : \mathbb{C}^n / A \longrightarrow \mathbb{C},$$

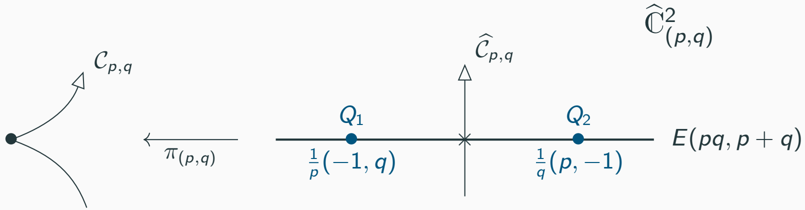
for some finite abelian group $A \subset \mathrm{GL}_n(\mathbb{C})$ acting diagonally.

Definition

An embedded $\mathbb{Q}$-resolution of $(D, 0) \subset (\mathbb{C}^n, 0)$ is a proper analytic map $h : Y \to (\mathbb{C}^n, 0)$:

1. Y is a V-manifold with only abelian quotient singularities.
2. h restricted to $Y \setminus h^{-1}(D_{\mathrm{sing}})$ an isomorphism.
3. $h^{-1}(D) = \bigcup_{i \in S} E_i$ is $\mathbb{Q}$-normal crossings on Y.

Cusp: $f(x,y) = x^q - y^p$



$$\mathrm{Sing}\big(\widehat{\mathbb{C}}^2_{(p,q)}\big) = \{Q_1, Q_2\}$$

Remark: Considering the different isotropy abelian groups acting in Y, we can define a finite refined stratification

$$Y = \bigsqcup_{k \geq 0} Y_k,$$

with

$$Y_k = \left\{ q \in Y \;\middle|\; \begin{array}{l} h^* f = y_1^{N_{1,k}} \cdots y_n^{N_{n,k}} \\ h^*(\wedge_i \mathrm{d}x_i) = y_1^{\nu_{1,k}-1} \cdots y_n^{\nu_{n,k}-1} (\wedge_i \mathrm{d}y_i) \end{array} \; G_k\text{-invariants, arround } q \right\}$$

such that $N_{i,k}, \nu_{j,k}$ and the diagonal action by G_k are constant along Y_k.

Warning! We can have "fake actions"

We can assume that the groups are small, i.e. they do not contain pseudo-reflexions around hyperplanes.

McQuillan'19 – Abramovich, Tëmkin, W łodarczyk'20

"Very fast, very functorial, and very easy resolution of singularities".

Theorem (———'11)

Let $h : Y \to \mathbb{C}^d$ be an embedded $\mathbb{Q}$-resolution of $(D, 0)$. Consider the subset of strata $\{S_m\}_{m\geq 0}$ of $\{Y_k\}_{k\geq 0}$ verifying that for any $q \in S_m$, h^*f is locally a germ

$$z^{N_m} : \mathbb{C}^n/G_m \longrightarrow \mathbb{C},$$

being constant both N_m and a (small) G_m along the stratum. Then:

$$\zeta_{f,0}(t) = \prod_{m\geq 0} \left(1 - t^{N_m/|G_m|}\right)^{\chi\left(S_m \cap h^{-1}(0)\right)}.$$

A new formula for $Z_{\text{top},0}(f;s)$ from $\mathbb{Q}$-resolutions of singularities

Let $h : Y \to (\mathbb{C}^n, 0)$ be an embedded $\mathbb{Q}$-resolution of $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$.

- Consider a stratification $Y = \bigsqcup_{k \geq 0} Y_k$ with

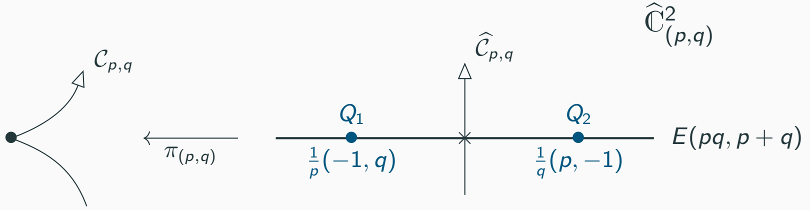
$$(N_{1,k}, \cdots, N_{n,k}), \quad (\nu_{1,k}, \cdots, \nu_{n,k}), \quad G_k \text{ acting diag.},$$

  constant along Y_k as before.

Theorem (León-Cardenal, ———, Veys, Viu-Sos'20)

$$Z_{\text{top},0}(f;s) = \sum_{k \geq 0} \chi\left(Y_k \cap h^{-1}(0)\right) \cdot |G_k| \cdot \prod_{i=1}^{n} \frac{1}{N_{i,k}s + \nu_{i,k}}$$

Example: $f(x,y) = y^p - x^q$ with $\gcd(p,q) = 1$,



$$\operatorname{div}(h^* f) = \widehat{\mathcal{C}}_{p,q} + (pq)E \quad \text{and} \quad \operatorname{div}(h^*(\mathrm{d}x \wedge \mathrm{d}y)) = (p+q)E,$$

$$\text{Strata:} \quad E \setminus \{Q_1, Q_2, \widehat{\mathcal{C}}_{p,q}\},\ E \cap \widehat{\mathcal{C}}_{p,q},\ Q_1,\ Q_2.$$

$$Z_{\text{top},0}(f;s) = \frac{1}{(pq)s + (p+q)} \left(-1 + \frac{1}{s+1} + p + q \right) = \frac{(p+q-1)s + (p+q)}{(s+1)\big((pq)s + (p+q)\big)},$$

$$\zeta_{f,0}(t) = (1 - t^{pq})^{-1} \cdot \left(1 - t^{pq/p}\right) \cdot \left(1 - t^{pq/q}\right) = \frac{(1-t^p)(1-t^q)}{1 - t^{pq}}.$$

Motivic ideas: motivic zeta function and change of variables

Motivic ideas: intrinsic definition as motivic integral

Kontsevich's motivic integral :

contact order of f

$$Z_{\mathrm{mot},0}(f;s) = \int_{\left\{ \substack{\text{arcs on } \mathbb{C}^n \\ \text{with origin in } 0} \right\}} \mathbb{L}^{-\operatorname{ord}_t f \cdot s} \, \mathrm{d}\mu_{\mathrm{mot}}$$

geometric infinitesimal structure around 0

"universal" additive invariant of varieties

specialization by $\chi(\cdot)$

$$Z_{\mathrm{top},0}(f;s)$$

Definition

$(K_0(\mathrm{Var}_{\mathbb{C}}), +, \cdot)$ is the ring:

- generated by classes $[X]$ of isomorphism of complex varieties.
- relations:
  - for any (Zariski) closed subset $F \subset X$: $[X] = [X \setminus F] + [F]$,
  - $[X \times Y] = [X] \cdot [Y]$.

The unit elements: $0 = [\emptyset]$ and $1 = [\mathrm{pt}]$, respectively. Denote $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}]$.

Example : $[\mathbb{P}^1] = [\mathbb{C} \sqcup \{\infty\}] = \mathbb{L} + 1$. In fact, as $\mathbb{P}^n = \mathbb{C}^n \sqcup \mathbb{P}^{n-1}$, for $n \geq 1$,

$$[\mathbb{P}^n] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + \mathbb{L} + 1.$$

Definition

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$\chi : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}$ $\qquad$ $\chi(X) =$ Euler characteristic

$P : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[t]$ $\qquad$ $P_X(t) =$ Poincaré polynomial

$H : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[u, v]$ $\qquad$ $H_X(u, v) =$ Hodge-Deligne polynomial

$N_p : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}$ $\qquad$ $N_p(X) =$ Number of $\mathbb{F}_p$-points

Let X be an algebraic variety.

Consider the localization $\mathcal{M}_{\mathbb{C}} = K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$, there exists a normalized measure based in X:

$$\exists \mu_{\mathrm{mot},\mathcal{L}(X)} : \quad \{\text{“cylinders” on } \mathcal{L}(X)\} \longrightarrow \widehat{\mathcal{M}}_{\mathbb{C}}$$
$$A \longmapsto \lim_{m} \frac{[\pi_m(A)]}{\mathbb{L}^{n(m+1)}}$$

in a completion $\mathcal{M}_{\mathbb{C}} \to \widehat{\mathcal{M}}_{\mathbb{C}}$.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic germ. Consider $s \in \mathbb{C}$ and define:

$$\mathcal{L}(\mathbb{C}^n)_0 = \{\gamma \in \mathcal{L}(\mathbb{C}^n) \mid \gamma(0) = 0\}$$

Definition

The (local) motivic zeta function of f is given by

$$\begin{aligned} Z_{\mathrm{mot},0}(f;s) &= \int_{\mathcal{L}(\mathbb{C}^n)_0} \mathbb{L}^{-\operatorname{ord}_t f \cdot s} \mathrm{d}\mu_{\mathrm{mot}} \\ &= \sum_{m \geq 0} \mu_{\mathrm{mot}} \Big(\{\gamma \in \mathcal{L}(\mathbb{C}^n)_0 \mid \operatorname{ord}_t(f \circ \gamma) = m\} \Big) \cdot \mathbb{L}^{-m \cdot s} \end{aligned}$$

Main tool in Motivic integration: Consider

- $h : Y \to X$ proper map between general varieties X and Y.
- $A \subset \mathcal{L}(X)$ and $B \subset \mathcal{L}(Y)$ measurables such that h induces a bijection between them.
- $\alpha : A \to \mathbb{Z} \cup \{\infty\}$ integrable in this context.
- $\mathrm{Jac}(h) =$ Jacobian ideal sheaf of h.

Theorem (Kontsevich, Denef-Loeser)

$$\int_A \mathbb{L}^{-\alpha} \mathrm{d}\mu_{\mathrm{mot},\mathcal{L}(X)} = \int_B \mathbb{L}^{-\alpha\circ h - \mathrm{ord}_t \mathrm{Jac}(h)} \mathrm{d}\mu_{\mathrm{mot},\mathcal{L}(Y)}$$

If both X, Y are smooth, $\mathrm{Jac}(h)$ is K_h the relative canonical divisor.

If $h : Y \to \mathbb{C}^n$ is an embedded resolution of D defined by f,

$$h^{-1}(D) = \bigcup_{i \in S} E_i \quad \rightsquigarrow \quad \{(N_i, \nu_i)\}_{i \in S} \quad \text{and} \quad \{Y_I\}_{I \subset S}$$

change of vars

$$Z_{\mathrm{mot},0}(f;s) \overset{\downarrow}{=} \mathbb{L}^{-n} \sum_{I \subset S} \left[Y_I \cap h^{-1}(0) \right] \cdot \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{N_i s + \nu_i} - 1}$$

specialization by $\chi(\cdot)$

$(\mathbb{L} \to 1)$

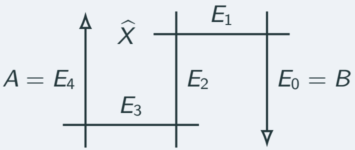
$$Z_{\mathrm{top},0}(f;s) = \sum_{I \subset S} \chi(Y_I \cap h^{-1}(0)) \cdot \prod_{i \in I} \frac{1}{N_i s + \nu_i}$$

Motivic zeta functions of $\mathbb{Q}$-Gorenstein varieties and $\mathbb{Q}$-resolutions

- (Veys'91): $Z_{\text{top}}(f;s)$ for log-canonical models in $n=2$ & using determinants of p-deformations in Igusa's p-adic zeta functions.

Example

Let $X = \frac{1}{7}(1,3)$ and consider $f = x^{N_1}y^{N_2} \in \mathcal{O}_X$, $\omega = x^{\nu_1 - 1}y^{\nu_2 - 1}\mathrm{d}x\mathrm{d}y \in \Omega_X^2$. Let D_1 and D_2 be the divisors associated with f and g.



E_1 $\left(\frac{N_1+3N_2}{7}, \frac{\nu_1+3\nu_2}{7}, 3\right)$

E_2 $\left(\frac{3N_1+2N_2}{7}, \frac{3\nu_1+2\nu_2}{7}, 2\right)$

E_3 $\left(\frac{5N_1+N_2}{7}, \frac{5\nu_1+\nu_2}{7}, 2\right)$

$$Z_{\text{mot},0}(f,\omega;s) \overset{[Veys]}{=} \frac{\mathbb{L}^{-2}(\mathbb{L}-1)^2\mathcal{D}_r}{(\mathbb{L}^{N_1 s+v_1}-1)(\mathbb{L}^{N_1 s+v_1}-1)}$$

Example (continued)

$$\mathcal{D}_r = \begin{vmatrix} K_1 & -\mathbb{L}^{<3>} & \mathbb{L}^{<2>} - 1 \\ -\mathbb{L}^{<0>} & K_2 & -\mathbb{L}^{<1>} \\ 0 & -\mathbb{L}^{<4>} & K_3 \end{vmatrix}$$

where $K_1 = 1 + \mathbb{L}^{<1>} + \mathbb{L}^{2<1>}$, $K_2 = 1 + \mathbb{L}^{<2>}$, $K_3 = 1 + \mathbb{L}^{<3>}$.

$$\begin{aligned} <1> &= \frac{N_1 + 3N_2}{7} s + \frac{\nu_1 + 3\nu_2}{7}, \\ <2> &= \frac{3N_1 + 2N_2}{7} s + \frac{3\nu_1 + 2\nu_2}{7}, \\ <3> &= \frac{5N_1 + N_2}{7} s + \frac{5\nu_1 + \nu_2}{7}, \end{aligned} \qquad \begin{aligned} <0> &= N_2 s + \nu_2, \\ <4> &= N_1 s + \nu_1. \end{aligned}$$

$$\begin{aligned} \mathcal{D}_r = 1 &+ \mathbb{L}^{\frac{N_1+3N_2}{7}s + \frac{\nu_1+3\nu_2}{7}} + \mathbb{L}^{\frac{2N_1+6N_2}{7}s + \frac{2\nu_1+6\nu_2}{7}} + \mathbb{L}^{\frac{3N_1+2N_2}{7}s + \frac{3\nu_1+2\nu_2}{7}} \\ &+ \mathbb{L}^{\frac{4N_1+5N_2}{7}s + \frac{4\nu_1+5\nu_2}{7}} + \mathbb{L}^{\frac{5N_1+N_2}{7}s + \frac{5\nu_1+\nu_2}{7}} + \mathbb{L}^{\frac{6N_1+4N_2}{7}s + \frac{6\nu_1+4\nu_2}{7}}. \end{aligned}$$

- (Denef-Loeser'01): Study of motivic measures for quotient singularities $\mathbb{C}^n/G$, for finite $G \subset \mathrm{GL}_n(\mathbb{C})$, in terms of "fractional" arcs in $\mathbb{C}^n$.

  $\rightsquigarrow$ motivic orbifold measures in terms of $\pi : \mathbb{C}^n \to \mathbb{C}^n/G$.

  $\rightsquigarrow$ $\mathrm{ord}_t \, \mathrm{Jac}(h)$ very complicated to compute.

Example

- $G_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$
- $G_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & -1 \end{pmatrix} \right\}$

$$\mu^{\mathrm{orb}}(\mathcal{L}(\mathbb{C}^2/G_1)_0) = \mathbb{L}^{-2} + \mathbb{L}^{-1} = \mathbb{L}^{-2}(1 + \mathbb{L})$$

$$\mu^{\mathrm{orb}}(\mathcal{L}(\mathbb{C}^2/G_2)_0) = \mathbb{L}^{-2}(1 + \mathbb{L}^{1/2})(1 + \mathbb{L}^{3/4})$$

Then $\mu^{\mathrm{orb}}(\mathcal{L}(\mathbb{C}^2/G_1)_0) \neq \mu^{\mathrm{orb}}(\mathcal{L}(\mathbb{C}^2/G_2)_0)$ but $\mathbb{C}^2/G_1 \cong \mathbb{C}^2/G_2$.

- <u>New ideas</u>:
  - $\rightsquigarrow$ $\mathbb{Q}$-Gorenstein measure $\mu^{\mathbb{Q}\,\mathrm{Gor}}(A) = \int_A \mathbb{L}^{-\frac{1}{r}\operatorname{ord}_t(rK_X)} \mathrm{d}\mu_{\mathrm{mot}}$.
  - $\rightsquigarrow$ $Z_{\mathrm{mot},0}$ for $\mathbb{Q}$-divisors in $\mathbb{Q}$-Gorenstein vareties.
  - $\rightsquigarrow$ Change of variables in terms of the relative divisor $\operatorname{ord}_t K_h$ and $\mu^{\mathbb{Q}\,\mathrm{Gor}}$.
  - $\rightsquigarrow$ **New formulas from $\mathbb{Q}$-resolution of singularities**.

Let X be a $\mathbb{Q}$-Gorenstein variety having at worst log-terminal singularities and (D_1, D_2) two $\mathbb{Q}$-Cartier divisors.

Definition

The ($\mathbb{Q}$-Gorenstein) motivic zeta function of the pair (D_1, D_2) with respect to a subvariety W is

$$Z_{\mathrm{mot},W}(D_1, D_2; s) = \int_{\mathcal{L}(X)_W^{\mathrm{reg}}} \mathbb{L}^{-(\mathrm{ord}_t D_1 \cdot s + \mathrm{ord}_t D_2)} \mathrm{d}\mu_{\mathcal{L}(X)}^{\mathbb{Q}\,\mathrm{Gor}},$$

whenever the right-hand side converges in $\widehat{\mathcal{M}}_{\mathbb{C}}[\mathbb{L}^{1/r}][\![\mathbb{L}^{-s/r}]\!]$.

Theorem (León-Cardenal, ———, Veys, Viu-Sos'20)

Let $h : Y \to X$ be a proper birational map between $\mathbb{Q}$-Gorenstein varsieties. Then

$$Z_{\mathrm{mot},W}(D_1, D_2; s) = Z_{\mathrm{mot},h^{-1}W}(h^* D_1, h^* D_2 + K_h; s).$$

Let $h : Y \to X$ be an embedded $\mathbb{Q}$-resolution of $(X, D_1 + D_2)$.

- Consider a stratification $Y = \bigsqcup_{k\geq 0} Y_k$ with

$$(N_{1,k}, \cdots, N_{n,k}), \quad (\nu_{1,k}, \cdots, \nu_{n,k}), \quad G_k \text{ acting diag.},$$

  constant along Y_k as before. Also:
  - For every $q \in Y_k$, there is local coordinates s.t. $\pi^* D_1$ and $\pi^* D_2 + K_h$ have local ecuations given by monomials and any $g \in G_k$ acts as

$$\operatorname{diag}(\xi^{\varepsilon_{1,g}}, \ldots, \xi^{\varepsilon_{n,g}}) \quad \text{with} \quad 0 \leq \varepsilon_{i,g} < |G_k|.$$

  - Define the expression:

$$S_k(\mathbb{L}) = \sum_{g\in G_k} \mathbb{L}^{\frac{1}{|G_k|}\left(\varepsilon_{1,g}\cdot(N_{1,k}\cdot s+\nu_{1,k})+\cdots+\varepsilon_{n,g}\cdot(N_{n,k}\cdot s+\nu_{n,k})\right)}.$$

Theorem (León-Cardenal, ———, Veys, Viu-Sos'20)

$$Z_{\mathrm{mot},W}(D_1, D_2; s) = \mathbb{L}^{-n} \sum_{k\geq 0} \left[Y_k \cap h^{-1}(W)\right] \cdot S_k(\mathbb{L}) \cdot \prod_{i=1}^{n} \frac{\mathbb{L} - 1}{\mathbb{L}^{N_{i,k}s+\nu_{i,k}} - 1}.$$

Let $h : Y \to X$ be an embedded $\mathbb{Q}$-resolution of $(X, D_1 + D_2)$.

- Consider a stratification $Y = \bigsqcup_{k \geq 0} Y_k$ with

$$(N_{1,k}, \cdots, N_{n,k}), \quad (\nu_{1,k}, \cdots, \nu_{n,k}), \quad G_k \text{ acting diag.},$$

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  - Define the expression:

$$S_k(\mathbb{L}) = \sum_{g \in G_k} \mathbb{L}^{\frac{1}{|G_k|}\left(\varepsilon_{1,g}\cdot(N_{1,k}\cdot s+\nu_{1,k})+\cdots+\varepsilon_{n,g}\cdot(N_{n,k}\cdot s+\nu_{n,k})\right)}.$$

Corollary: specializing by the Euler Characteristic

$$Z_{\text{top},W}(D_1, D_2; s) = \sum_{k \geq 0} \chi\left(Y_k \cap h^{-1}(W)\right) \cdot |G_k| \cdot \prod_{i=1}^{n} \frac{1}{N_{i,k} s + \nu_{i,k}}$$

Examples and applications

Example: Brieskorn-Pham surface singularity

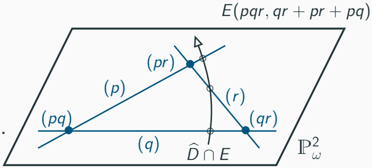
Let $g(x,y,z) = x^p + y^q + z^r$, $p,q,r \in \mathbb{N}$ pairwise coprimes, $\omega = (qr, pr, pq)$

$D = V(g) \subset \mathbb{C}^3$, has an isolated singularity at the origin.

$$E \subset \widehat{\mathbb{C}}^3_\omega \simeq U_1 \cup U_2 \cup U_3, \quad U_i = \mathbb{C}^3/G_i$$

$$\pi_\omega \downarrow \quad \{0\} \subset \mathbb{C}^3$$

- $E \simeq \mathbb{P}^2_\omega$.
- Sing$(\widehat{\mathbb{C}}^3_\omega) \simeq L_x \cup L_y \cup L_z \subset \mathbb{P}^2_\omega$.



- $\widehat{D} \cap E \simeq \mathcal{C}$ where $\mathcal{C} : g(x,y,z) = 0$ in $\mathbb{P}^2_\omega$.
- $E \setminus \widehat{D} \simeq \mathbb{P}^2_\omega \setminus \mathcal{C}$ and $(N_E, \nu_E) = (pqr, qr+pr+pq)$.
- Strat. by isotropy of $E \simeq \left(\mathbb{P}^2_\omega \setminus \{\text{axis}\}\right) \cup \underbrace{L_x^* \cup L_y^* \cup L_z^*}_{\text{punctured axis}} \cup \underbrace{O_1 \cup O_2 \cup O_3}_{\text{origins}}$.

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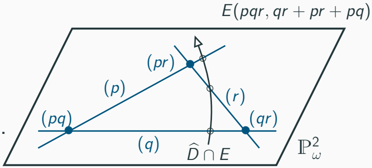

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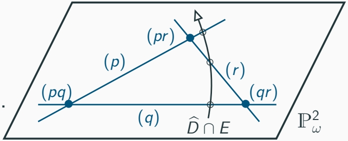

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Example: What about nonabelian quotient singularities?

- Batyrev'00: Canonical abelianization method using blowing-ups.
  and ξ primitive $\xi^d = 1$.
  Consider $X = \mathbb{C}^3 / G_{d,q}$, where

$$G_{d,q} = \left\langle A = \begin{pmatrix} 1 & & \\ & \xi & \\ & & \xi^q \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle \subset \mathrm{GL}_3(\mathbb{C})$$

  - X is Gorensteis canonical sing. $\iff q = d - 1$.
  - Ito: For $d = q - 1$, construction of crepant resolution + McKay correspondence.
  - Assume $d \mid q^3 + 1 \implies G_{d,q}$ small.
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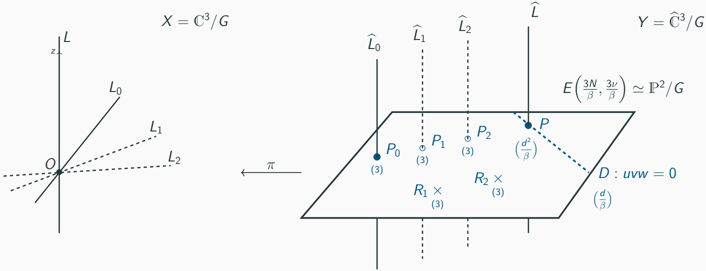
Define $\beta = \gcd(d, q^2 - q + 1)$. Resolution induced by an ordinary blow-up:



The singular locus depends on whether $3 \mid d$.

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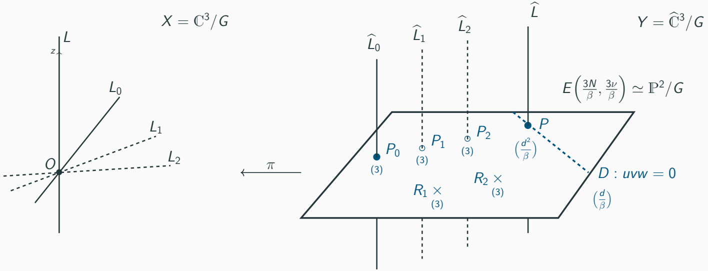

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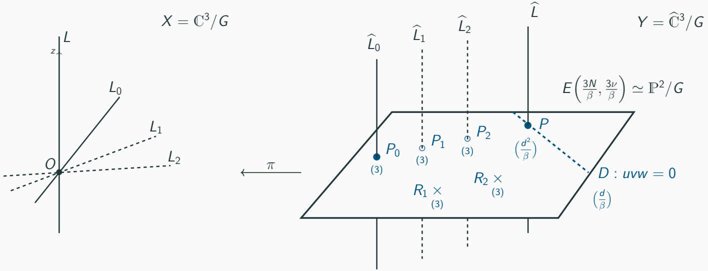

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ありがとうございました