Singularities of flat fronts and their caustics, and an example arising from the hyperbolic Schwarz map of a hypergeometric equation

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Dedicated to the late Professor Katsumi Nomizu

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Abstract. For parallel families of flat fronts and their caustic surfaces in the hyperbolic 3-space, singularities such as cuspidal edges and swallowtails are studied. An example arising from a hypergeometric differential equation is closely studied.

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Contents

Introduction

The hyperbolic Schwarz map of an equation $u'' - q(x)u = 0$ in complex variable x defines a surface F in the hyperbolic 3-space $\boldsymbol{H}^3$. The singularities of the parallel family of F gives the caustic surface S in $\boldsymbol{H}^3$. Studying the singularities of the parallel surfaces of F and those of S, we present a criterion for cuspidal edges and swallowtails on S in terms of the coefficient q. When the equation is the hypergeometric equation with a finite monodromy group, we study in detail the singularities of the members of the parallel family and of the caustic S. We actually treat a typical example when the monodromy group is the dihedral group of order 12, because other cases are quite similar. The paper [SYY2] studies this example, and states that the hyperbolic Schwarz map is a generic member of the parallel family by showing pictures as in Figure 2, and presents a picture of the caustic surface what appears to be a cuspidal line that just ends at some point, as in Figure 6, without including a discussion of the actual behavior of the singular set. This paper clarifies what is really happening at the apparent end of the cuspidal line of the caustic surface.

Part 1. Local theory of hyperbolic Schwarz maps, parallel surfaces and caustics

1. The three kinds of Schwarz maps

We start from a differential equation

$$u'' - q(x)u = 0, \quad x \in X$$

where $q(x)$ is holomorphic in a domain X. For two linearly independent solutions u_0 and u_1 to this equation, we define the *Schwarz map*

$$S : X \ni x \longmapsto u_0(x) : u_1(x) \in \boldsymbol{P}^1, \tag{S}$$

the *derived Schwarz map*

$$DS : X \ni x \longmapsto u_0'(x) : u_1'(x) \in \boldsymbol{P}^1, \tag{DS}$$

and the *hyperbolic Schwarz map*

$$HS : X \ni x \longmapsto H(x) = U(x)\ {}^t\bar{U}(x) \in \boldsymbol{H}^3, \tag{HS}$$

where

$$U = \begin{pmatrix} u_0 & u_0' \\ u_1 & u_1' \end{pmatrix}.$$

The image F (of X) under HS lies in the three-dimensional hyperbolic space $\boldsymbol{H}^3$ identified with the space of positive 2×2-hermitian matrices modulo diagonal ones, while the images under S and DS lie in the complex projective line $\boldsymbol{P}^1$, which is regarded as the ideal boundary of $\boldsymbol{H}^3$; refer to [SYY1] and [SYY2]. It is known that along the curve defined by

$$|q(x)| = 1,$$

the map *HS* degenerates; since generically the singularities are cuspidal edges, this curve (and its image also) is called the *cusp line*. It is important to note that even at a singular point of F the unit normal vector is well-defined, which we denote by ν. Then the pair of maps *HS* and ν is an immersion into the unit tangent bundle of $\boldsymbol{H}^3$. Such a map is called a *flat front* [KUY], [KRSUY].

2. Parallel surfaces

The collection of points on oriented normal geodesics of F with a constant distance from F is called a parallel surface of F. Any parallel surface is given by

$$HS_k : X \ni x \longmapsto V_k(x)\ {}^t\bar{V}_k(x) \in \boldsymbol{H}^3,$$

where

$$V_k = U \begin{pmatrix} \sqrt{k} & 0 \\ 0 & 1/\sqrt{k} \end{pmatrix},$$

for some $k \in (0, \infty)$; its image will be denoted by F_k. We refer to [KUY, SYY2] for these maps. Note that when k tends to ∞ then the map HS_k tends to the Schwarz map S, when k tends to zero then the map tends to the derived Schwarz map *DS*, and when $k = 1$ the map is the hyperbolic Schwarz map *HS* of course. Since we have

$$dV_k = V_k \begin{pmatrix} 0 & (q/k)dx \\ kdx & 0 \end{pmatrix},$$

by introducing the new coordinate y by

$$y = kx,$$

we get

$$dV_k = V_k \begin{pmatrix} 0 & q/k^2 \\ 1 & 0 \end{pmatrix} dy;$$

that is, the map HS_k giving a parallel surface is, relative to the coordinate y, the hyperbolic Schwarz map of the differential equation

$$\ddot{v} - Q(y)v = 0, \quad \text{where} \quad \cdot = d/dy \quad \text{and} \quad Q(y) = \frac{1}{k^2} q\left(\frac{y}{k}\right).$$

Note that for any solution $u(x)$ of the original equation, the function $v(y) = u(y/k)$ is a solution of the new equation.

3. Singularities on parallel surfaces

Let the notation be as above. We quote from [KRSUY] and [SSY] a characterization of the singularities of the Schwarz map of an equation $\ddot{v} - Q(y)v = 0$:
(1) The map is singular along the curve

$$\{y \mid Q\overline{Q} = 1\},$$

(2) the map has cuspidal edge singularity at a point y satisfying

$$Q\overline{Q} = 1, \quad \dot{Q} \neq 0, \quad Q^3\overline{\dot{Q}} - \dot{Q} \neq 0,$$

(3) the map has swallowtail singularity at a point y satisfying

$$Q\overline{Q} = 1, \quad \dot{Q} \neq 0, \quad Q^3\overline{\dot{Q}} - \dot{Q} = 0, \quad \mathrm{Re}\left(\frac{\ddot{Q}}{Q^2} - \frac{3}{2}\frac{\dot{Q}^2}{Q^3}\right) \neq 0,$$

(4) and the map has a singularity of type A_4 at a point y satisfying

$$Q\overline{Q} = 1, \quad \dot{Q} \neq 0, \quad Q^3\overline{\dot{Q}} - \dot{Q} = 0, \quad \mathrm{Re}\left(\frac{\ddot{Q}}{Q^2} - \frac{3}{2}\frac{\dot{Q}^2}{Q^3}\right) = 0, \quad \mathrm{Im}\left(\frac{\dddot{Q}}{Q^{5/2}} - 5\frac{\dot{Q}\ddot{Q}}{Q^{7/2}}\right) \neq 0.$$

Since, by differentiation, we have

$$\dot{Q} = q'/k^3, \quad \ddot{Q} = q''/k^4, \quad \dddot{Q} = q'''/k^5,$$

we rewrite these conditions in terms of $q = q(x)$ as in the next lemma.

Lemma 3.1. *For singularities of the parallel surface HS_k the following holds:* (1) *The singular locus is given by the curve*

$$\{x \in X \mid q\overline{q} = k^4\}.$$

(2) *Cuspidal edge singularity is characterized by*

$$q' \neq 0 \qquad \text{and} \qquad q^3\overline{q'} - (q\overline{q})^{3/2}q' \neq 0.$$

(3) *Swallowtail is characterized by*

$$q' \neq 0, \qquad q^3\overline{q'} - (q\overline{q})^{3/2}q' = 0, \qquad \text{and} \qquad \mathrm{Re}\left(\frac{q''}{q^2} - \frac{3}{2}\frac{(q')^2}{q^3}\right) \neq 0.$$

(4) *Singularity of type A_4 is characterized by*

$$q' \neq 0, \quad q^3\overline{q'} - (q\overline{q})^{3/2}q' = 0, \quad \mathrm{Re}\left(\frac{q''}{q2} - \frac{3}{2}\frac{(q')^2}{q^3}\right) = 0, \quad \mathrm{Im}\left(\frac{q'''}{q^{5/2}} - 5\frac{q'q''}{q^{7/2}}\right) \neq 0.$$

Remark 3.2. The condition $q^3\overline{q'} - (q\overline{q})^{3/2}q' \neq 0$ can be written as

$$\mathrm{Im}\frac{q'}{q^{3/2}} \neq 0,$$

which is an invariant form relative to the coordinate change $y = kx$.

Correction. In one of previous papers [SSY], we misstated a condition on the singularity called a pair of cuspidal beaks. Let us here correct it as follows: The condition in (6) in Lemma 2.3 is stated as $q' = 0$, $q'' \neq 0$, and $\overline{q'}(x) \neq (q'/q^3)(x)$, but it should be as $q' = 0$, $q'' \neq 0$, and $\overline{q''}(x) \neq (q''/q^4)(x)$.

4. Caustic surface and its singularities

The caustic surface C of the family of parallel surfaces $F_k = HS_k(X)$ is the union of the cusp lines of F_k. On the other hand, [KRSUY] tells that the caustic surface C is given as the image of the map $X \ni x \mapsto C(x) \in \boldsymbol{H}^3$, where

$$C = U \begin{pmatrix} |q|^{1/2} & 0 \\ 0 & |q|^{-1/2} \end{pmatrix} {}^t\overline{U}.$$

Note that if we set

$$U_c = U \begin{pmatrix} q^{1/4} & 0 \\ 0 & q^{-1/4} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

then U_c satisfies

$$C = U_c\, {}^t\overline{U_c} \quad \text{and} \quad dU_c = U_c \begin{pmatrix} 0 & \theta_c \\ \omega_c & 0 \end{pmatrix},$$

where

$$\omega_c = \sqrt{q}dx - \frac{i}{4} d\log q, \quad \theta_c = \sqrt{q}dx + \frac{i}{4} d\log q.$$

By introducing the coordinate z as

$$dz = \sqrt{q}dx - (i/4)d\log q,$$

we find that the caustic surface $x \mapsto C(x)$ is the hyperbolic Schwarz map of the differential equation

$$\ddot{u} - R(z)u = 0, \quad \text{where} \quad R(z) = \frac{4 + iq^{-3/2}q'}{4 - iq^{-3/2}q'},$$

where we denote by $''\cdot''$ the derivation with respect to z, while $'$ denotes the derivation with respect to x. This implies in particular that the caustic surface is also a flat front, locally; the coefficient R is not single-valued, and the jacobians $x \leftrightarrow z$ vanish where $16 + q^{-3}(q')^2 = 0$.

A characterization of singular points of the hyperbolic Schwarz map in terms of the coefficient is stated in Section 3. In the following we apply this to the equation $\ddot{u} - R(z)u = 0$ and express the conditions in terms of the original coefficient $q(x)$.

Proposition 4.1. *The caustic surface C is singular at $C(x)$ if and only if*

$$\mathrm{Im}\frac{q'}{q^{3/2}} = 0.$$

The singular point lies on the parallel surface F_k with parameter $k = (q(x)\overline{q(x)})^{1/4}$.

Proof. Rewrite the condition $R\overline{R} = 1$, which is equivalent to

$$(4 + iq^{-3/2}q')(4 - i\overline{q}^{-3/2}\overline{q'}) = (4 - iq^{-3/2}q')(4 + i\overline{q}^{-3/2}\overline{q'}),$$

and then to

$$q^{-3/2}q' - \overline{q}^{-3/2}\overline{q'} = 0.$$

□

Proposition 4.2. *The caustic surface C has cuspidal edge singularity at $C(x)$ if and only if*

$$\mathrm{Im}\frac{q'}{q^{3/2}} = 0, \quad \text{and} \quad \mathrm{Re}\left(\frac{q''}{q^2} - \frac{3}{2}\frac{(q')^2}{q^3}\right) \neq 0.$$

The point lies on the parallel surface F_k with parameter $k = (q(x)\overline{q(x)})^{1/4}$ and the point is a swallowtail of the parallel surface if $q(x) \neq 0$ and $q'(x) \neq 0$.

Proof. We first check the condition $\dot{R} \neq 0$. Since

$$\begin{aligned}
\dot{R} &= \frac{dx}{dz}\frac{dR}{dx} \\
&= \frac{1}{\sqrt{q} - i\frac{q'}{4q}} \frac{(6q^{1/2}q' + iq'')(4q^{3/2} - iq') - (4q^{3/2} + iq')(6q^{1/2}q' - iq'')}{(4q^{3/2} - iq')^2} \\
&= \frac{32iq^{9/2}}{(4q^{3/2} - iq')^3}\left(\frac{q''}{q^2} - \frac{3}{2}\frac{(q')^2}{q^3}\right),
\end{aligned}$$

the condition $\dot{R} \neq 0$ is equivalent to $\frac{q''}{q^2} - \frac{3}{2}\frac{(q')^2}{q^3} \neq 0$. We next compute $R^3\dot{\overline{R}} - \dot{R}$:

$$R^3\dot{\overline{R}} - \dot{R} = \left(\frac{4q^{3/2} + iq'}{4q^{3/2} - iq'}\right)^3 \frac{-32i\overline{q}^{9/2}}{(4\overline{q}^{3/2} + i\overline{q'})^3}\left(\frac{\overline{q''}}{\overline{q^2}} - \frac{3}{2}\frac{(\overline{q'})^2}{\overline{q}^3}\right) - \frac{32iq^{9/2}}{(4q^{3/2} - iq')^3}\left(\frac{q''}{q^2} - \frac{3}{2}\frac{(q')^2}{q^3}\right).$$

Hence, the condition $R^3\dot{\overline{R}} - \dot{R} = 0$ is equivalent to

$$\overline{q}^{9/2}(4q^{3/2} + iq')^3\left(\frac{\overline{q''}}{\overline{q^2}} - \frac{3}{2}\frac{(\overline{q'})^2}{\overline{q}^3}\right) + q^{9/2}(4\overline{q}^{3/2} + i\overline{q'})^3\left(\frac{q''}{q^2} - \frac{3}{2}\frac{(q')^2}{q^3}\right) = 0.$$

On the other hand, the condition $R\dot{R} = 1$ can be written as $q^{3/2}(4\overline{q}^{3/2} + i\overline{q'}) = \overline{q}^{3/2}(4q^{3/2} + iq')$. By the help of this identity, a calculation shows that $R^3\dot{\overline{R}} - \dot{R} \neq 0$ is equivalent to the condition

$$\mathrm{Re}\left(\frac{q''}{q^2} - \frac{3}{2}\frac{(q')^2}{q^3}\right) \neq 0.$$

For the last statement, refer to Lemma 3.1. □

Proposition 4.3. *The caustic surface C has swallowtail singularity at $C(x)$ if and only if*

$$\mathrm{Im}\frac{q'}{q^{3/2}} = 0, \quad \frac{q''}{q^2} - \frac{3}{2}\frac{q'^2}{q^3} \neq 0, \quad \mathrm{Re}\left(\frac{q''}{q^2} - \frac{3}{2}\frac{q'^2}{q^3}\right) = 0, \quad \text{and} \quad \mathrm{Im}\left(\frac{q'''}{q^{5/2}} - 5\frac{q'q''}{q^{7/2}}\right) \neq 0,$$

provided that $16 + q(x)^{-3}q'(x)^2 \neq 0$. *The point lies on the parallel surface* F_k *with parameter* $k = (q(x)\overline{q(x)})^{1/4}$ *and it is a singular point of type* A_4 *of the parallel surface if* $q(x) \neq 0$ *and* $q'(x) \neq 0$.

Proof. The first three conditions follow from the proof of Proposition 4.2. We need to check furthermore the condition

$$\mathrm{Re}\left(\frac{\ddot{R}}{R^2} - \frac{3}{2}\frac{\dot{R}^2}{R^3}\right) \neq 0.$$

An actual computation shows that this is equivalent, under the assumption that the first three conditions are satisfied, to the last identity. □

Remark 4.4. The propositions above imply that cuspidal edges, swallowtails, and A_4-singularities of a family of parallel surface correspond to ordinary points, cuspidal edges, and swallowtails of the caustic surface, respectively.

Note that $16 + q^{-3}(q')^2$ is positive if $q'/q^{3/2}$ is real and that the condition $16 + q^{-3}(q')^2 \neq 0$ is that the jacobian of $z \to x$ is not degenerate, because $16 + q^{-3}(q')^2 = (4 + iq^{-3/2}q')(4 - iq^{-3/2}q')$.

Part 2. Example: A hypergeometric differential equation

The hypergeometric differential equation

$$x(1-x)u'' + \{c - (a+b+1)x\}u' - abu = 0$$

can be tranformed by multiplying $\sqrt{x^c(1-x)^{a+b+1-c}}$ to the unknown u into the so-called SL-type $u'' - qu = 0$, where

$$q(x) = -\frac{1}{4}\left(\frac{1-\mu_0^2}{x^2} + \frac{1-\mu_1^2}{(1-x)^2} + \frac{1+\mu_\infty^2 - \mu_0^2 - \mu_1^2}{x(1-x)}\right),$$

and

$$\mu_0 = 1 - c, \quad \mu_1 = c - a - b, \quad \mu_\infty = b - a.$$

In this paper we treat this SL-type equation in the case where the monodromy group is the dihedral group of order 12, which is realized for the parameters

$$\mu_1 = 1/2, \quad \mu_2 = 1/2, \quad \mu_3 = 1/3; \quad \text{namely,} \quad a = 1/6, \quad b = -1/6, \quad c = 1/2.$$

We refer to [SYY1] for these expressions and related matters. We study the cusp lines and the swallowtails on the parallel surfaces, and the singularities of the caustic surface. When the monodromy group is finite or Fuchsian, we can get parallel results.

5. Cusp lines of the parallel surfaces

The equation of the parallel surface F_k is given by $\ddot{v} - Q(y)v = 0$, where

$$Q(y) = -\frac{1}{4}\left(\frac{3}{4y^2} + \frac{3}{4(k-y)^2} + \frac{11}{18y(k-y)}\right).$$

We draw the figures of the cusp lines for several values of k as in Figure 1, where we use the coordinate $y = u + iv$. When k is large, the cusp line has two components, after

$$k = \sqrt{19}/3 = 1.4529663145\cdots,$$

the number of connected components becomes one, and after

$$k = \frac{4\sqrt{3}}{27} = 0.2566001196\cdots,$$

it becomes three.

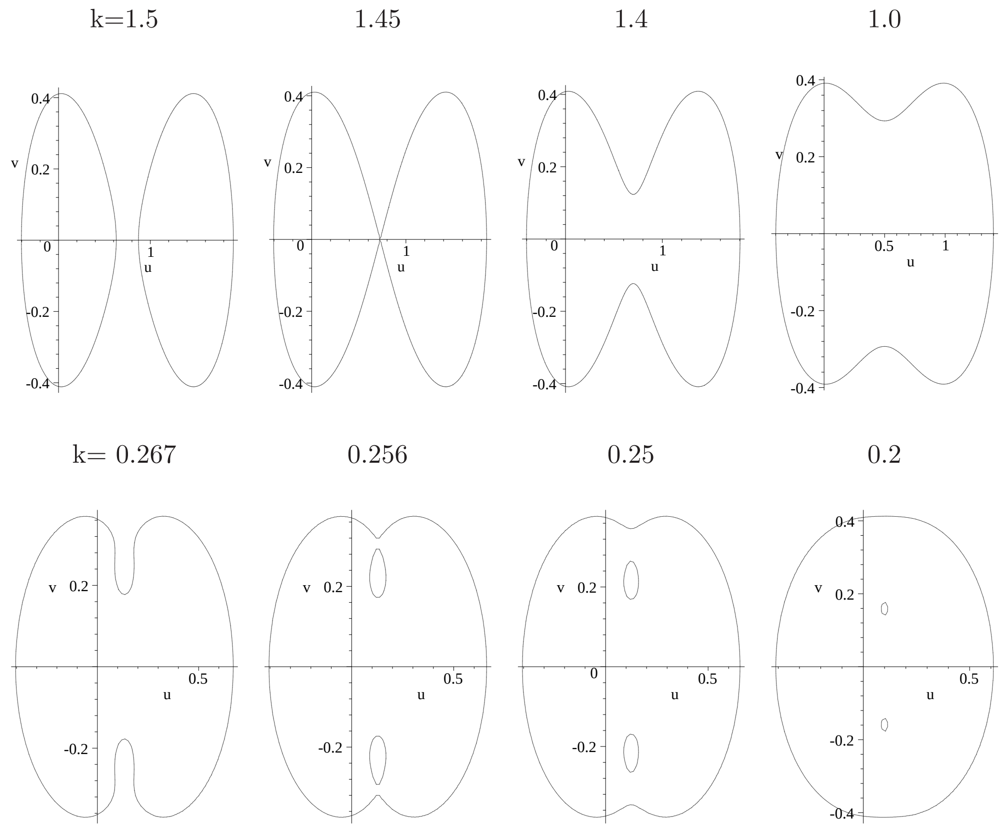

FIGURE 1. Cusp lines in the y-plane

The images of the upper half part are drawn for several values of k in Figure 2. For the last one, when $k = 0.25$, it would be difficult from this picture to see what is happening.

k=1.5 1.45 1 0.25

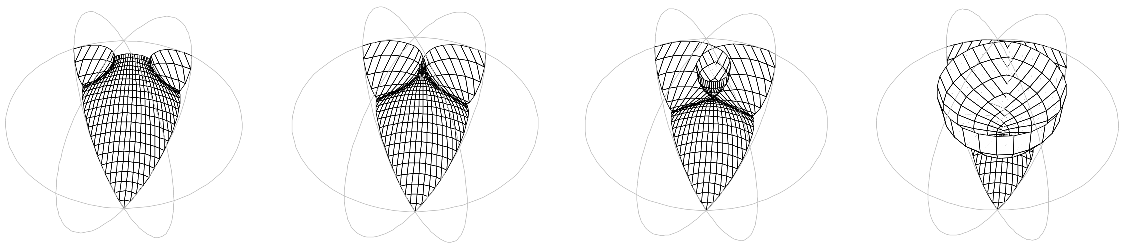

FIGURE 2. Parallel family: images of the upper half palne

Remark 5.1. The jacobian of $z \to x$ degenrates, i.e. $16 + q'^2/q^3 = 0$, if and only if $x = 0,\ 1,\ 0.1233799246\cdots,\ 0.8766200753\cdots$.

Remark 5.2. An exact computation of the critical values of k above is done as follows. First, we note that the swallowtails are defined by three equations $cusp = 0$, $swr = 0$, and $swi = 0$ that are defined in the next section. Second, we eliminate the variable v of these equations to get relations between k and u: the result is $A = 0$, $B = 0$, or $C = 0$, where

$$
\begin{aligned}
A &= 2006122600857600k^4u^8 - 8024490403430400k^5u^7 + (9654465016627200k^6 - 1203673560514560k^4 \\
&\quad -3329851637366784k^2)u^6 + (-877678637875200k^7 + 3611020681543680k^5 + 9989554912100352k^3)u^5 \\
&\quad +(-6850791135682560k^8 - 6676626780979200k^6 - 1446156635524300 8k^4 + 200777373057024k^2 \\
&\quad -11424613007360)u^4 + (5802474530488320k^9 + 7334885759385600k^7 + 12273874523652096k^5 \\
&\quad -4015547461140 48k^3 + 22849226014720k)u^3 + (-1417356718283520k^{10} - 5509803968630784k^8 \\
&\quad -6493402335707136k^6 + 438677645819904k^4 - 23920283484160k^2)u^2 + (-292745252701440k^{11} \\
&\quad +2444197869195264k^9 + 2021390892564480k^7 - 237900272762880k^5 + 12495670476800k^3)u \\
&\quad +442574057234529k^{12} - 792402203507904k^{10} - 255891118429440k^8 + 46926482374656k^6 \\
&\quad -2409879306240k^4, \\
B &= 2u - k, \\
C &= 144u^4 - 288ku^3 + (144k^2 + 32)u^2 - 32ku + 27k^2.
\end{aligned}
$$

We then compute resultants relative to u of these equations separately, to know where the multiple roots will appear. Here, we only need to be concerned about real solutions. In this way, we get the values $k = \sqrt{19}/3$ and $k = 4\sqrt{3}/27$.

6. Swallowtail points of parallel surfaces

To find the coordinates of swallowtails, we use a mathematical software called `risa/asir`; referred to in *Acknowledgments*. Define polynomials of u and v, where

$y = u + iv$, as follows. The dot $''\cdot''$ denotes the derivation relative to y.

$$\begin{aligned} cusp &= \text{the numerator of } Q\overline{Q} - 1, \\ swr &= \text{the real part of the numerator of } Q^3\overline{\dot{Q}} - \dot{Q}, \\ swi &= \text{the imaginary part of the numerator of } Q^3\overline{\dot{Q}} - \dot{Q}, \\ rrr &= \text{the real part of the numerator of } \frac{\ddot{Q}}{Q^2} - \frac{3}{2}\frac{(\dot{Q})^2}{Q^3}. \end{aligned}$$

The coordinates of the swallowtails can be computed as follows: for each given k

- first, compute a primary ideal decomposition of the ideal $\langle cusp, swr, swi\rangle$ and check that the ideal $\langle cusp, swr, swi, rrr\rangle$ coincides with the full polynomial ring.
- then, compute the set of common zeros of each component of the primary decomposition.

We refer to [NSYY] for a detailed treatment of such procedures. Note that, because of the symmetry of $Q(y)$, swallowtails are located symmetrically relative to the lines $\{u = k/2\}$ and $\{v = 0\}$. Figure 3 shows the swallowtails on the cusp lines lying in the part $v > 0$; the value of k is indicated in the figure.
At about $k = 0.256$, the cusp line bears another component – let us call this the *raindrop*, and below $k = 0.20$ only the raindrop is drawn.

On the small neighborhood of the raindrop the parallel surface behaves just like the hyperbolic Schwarz map of the Airy differential equation around the unit disc. Compare figures in [SYairy] with the top left one of the following pictures in Figure 4. The images of a few concentric rings around the raindrop are also shown; as the ring goes farther from the raindrop, the image behaves not like that of Airy. Refer also to [HRSY] for a discrete treatment of flat fronts.

These pictures suggest that the self-intersection curve around the raindrop looks like the picture in Figure 5, where the bold curves denote the cusp lines.

7. Caustic surface as the union of the cusp lines of the parallel surfaces

A global and rough view of the caustic surface is shown in Figure 6 (left). It looks as if a cusp line of the caustic surface just ends at a point; this cannot occur, since caustics are flat fronts. A bit finer picture of the caustic surface is given in Figure 6 (right) as the union of the images of the cusp lines on the parallel surfaces F_k for $0.18 < k < 0.28$. At the end of the cusp line, something seems to be happening.

Figure 7 shows some other views from different angles. These show that the cusp line does not end at a point, but rather a triangular horn is coming out.

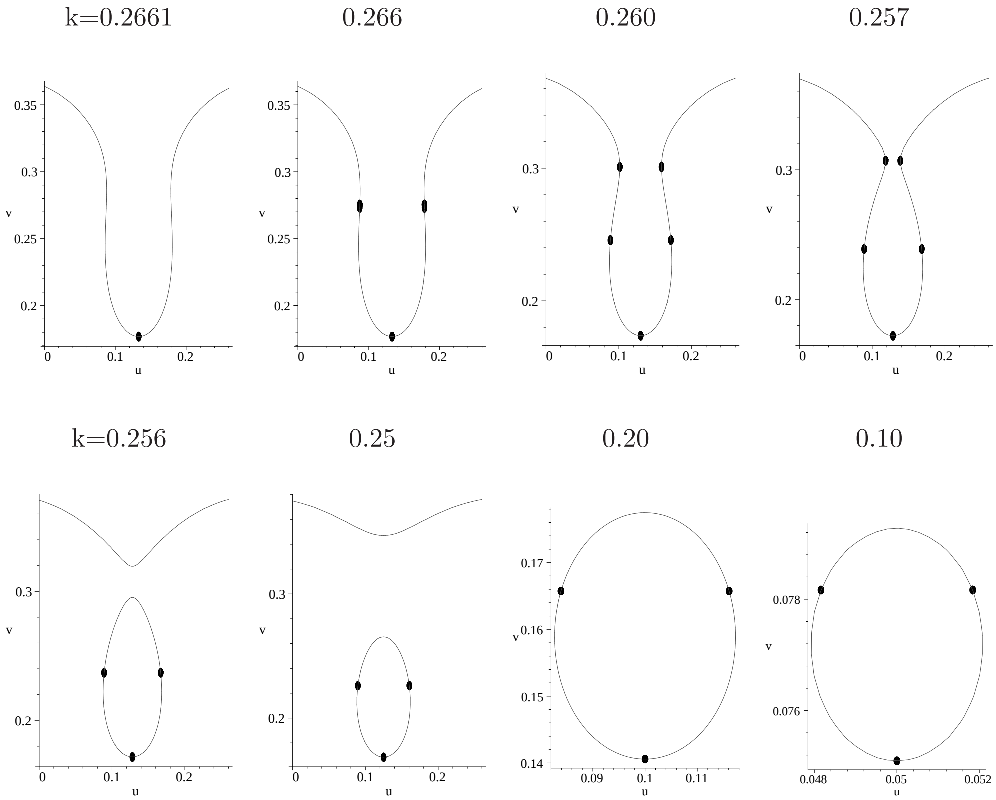

FIGURE 3. Cusp lines and swallowtails in part of the y-plane for $v > 0$

8. Singularities of the caustic surface

The condition $|R| = 1$, which defines the cusp line of the caustic surface C, is equivalent to

$$q^{3/2}\overline{q'} - \overline{q}^{3/2}q' = 0,$$

unless $q = 0$. We use the coordinate $x = s + it$ instead of the coordinate z. This cusp line is drawn in Figure 8(left), and the upper half is re-drawn on the right showing that the cusp line is the totality of the swallowtails of the parallel surfaces.

Some critical points and values are listed:

(A) The cusp line has a triple point at

$$A := (1/2, \sqrt{38}/8) = (0.5, 0.7705517504)$$

where $q = 0$.

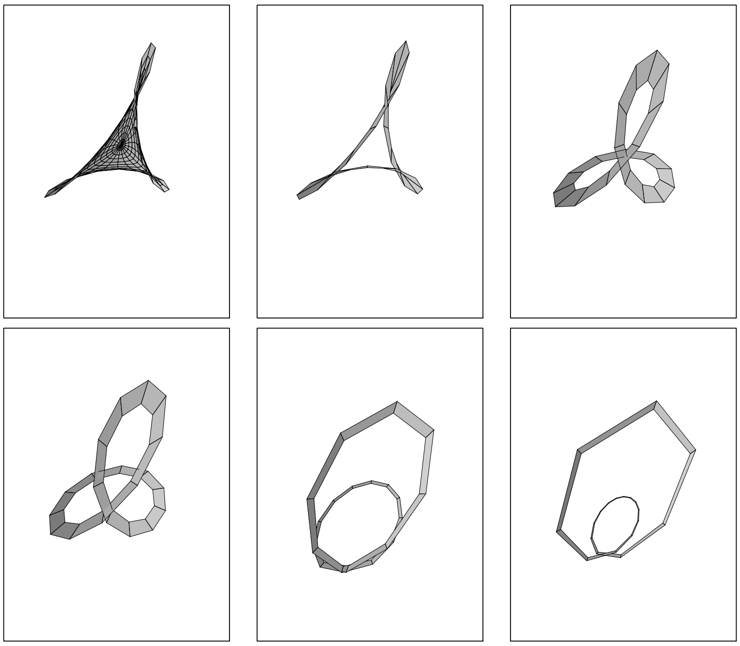

FIGURE 4. Images of the raindrop and a few concentric rings under HS_k $(k = 0.1)$

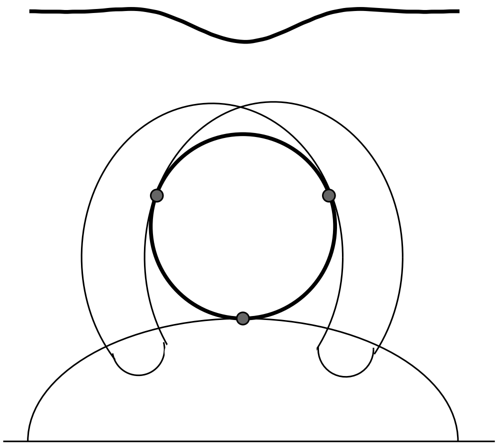

FIGURE 5. Self-intersection curve around the raindrop

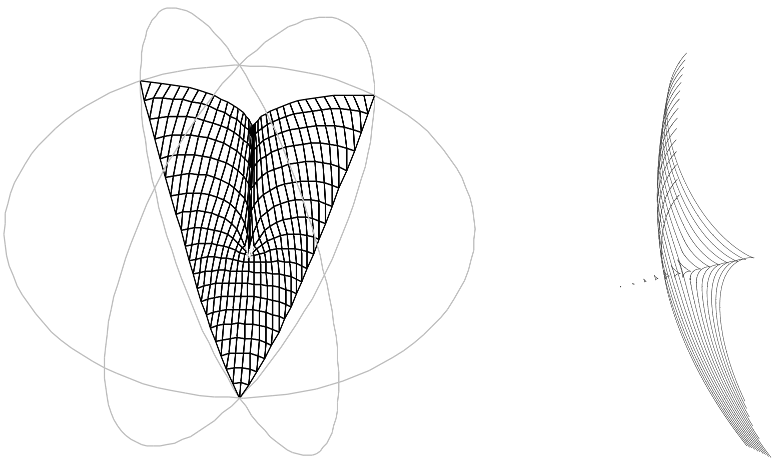

FIGURE 6. Images of the caustic surface

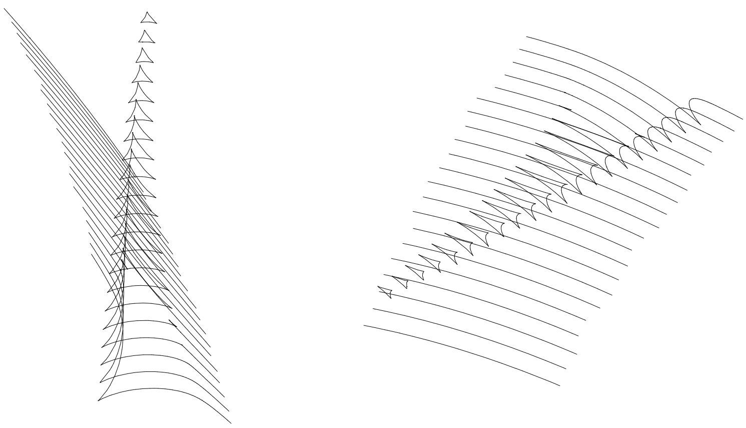

FIGURE 7. Union of the images of the cusp lines with $0.18 \leq k \leq 0.28$ from various angles

(S) The cusp line of a parallel surface F_k tangents the cusp line of the caustic surface C at the value

$$k = 0.2660117903 \cdots ,$$

which is a positive real solution of the polynomial equation

$$435848050125k^8 + 119481222825k^4 - 609206272 = 0.$$

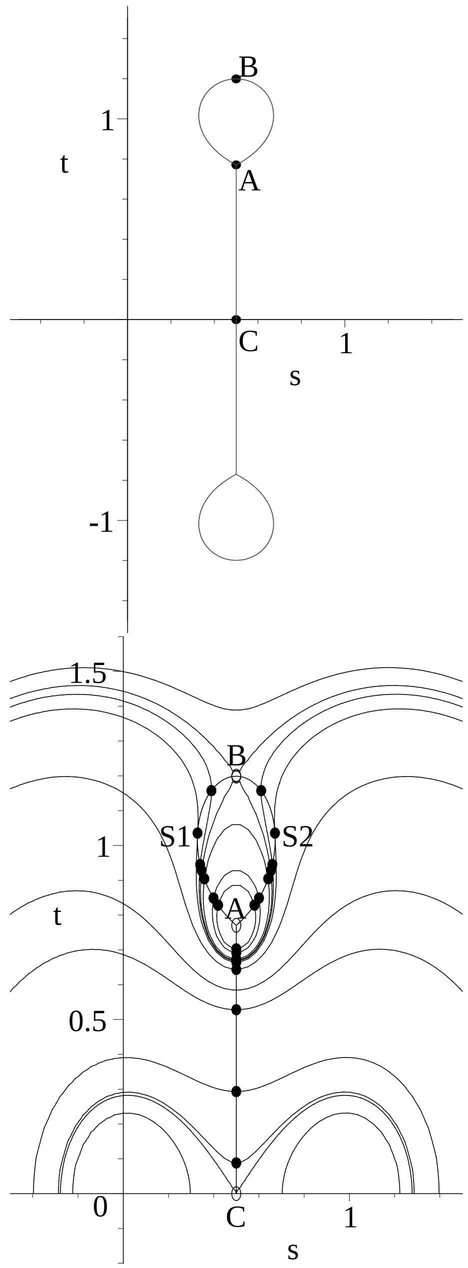

FIGURE 8. Cusp line of the caustic surface

It is tangent at the two points:

$$S_1 := (0.3283904\cdots, 1.0310487\cdots), \quad S_2 := (0.6716095\cdots, 1.0310487\cdots).$$

(B) The cusp line of F_k self-touches at

$$B := (1/2, \sqrt{23}/4) = (0.5, 1.198957881\cdots)$$

for the value

$$k = \frac{4\sqrt{3}}{27} = 0.2566001196\cdots,$$

where $q' = 0$. This cusp line of F_k meets the cusp line of the caustic C at the two points: the coordinates are

$$\begin{aligned} B_1 &:= (0.3469275914\cdots, 0.9279648433\cdots), \\ B_2 &:= (0.6530724085\cdots, 0.9279648433\cdots), \end{aligned}$$

other than B. Their values of s are two among three real solutions of the polynomial equation

$$\begin{aligned} -3276800s^6 + 9830400s^5 + 1279295488s^4 - 2574974976s^3 \\ +2574545344s^2 - 1285419456s + 225022617 = 0, \end{aligned}$$

and the value of t is the real solution greater than $\sqrt{38}/8$, the second coordinate of the point A, of the polynomial equation

$$3276800t^6 + 1236574208t^4 - 1053072576t^2 - 12223143 = 0.$$

Now we study singularities of the caustic surface. Points on the cusp line can have worse singularities at most at

$$S_1, \quad S_2, \quad B, \quad B_1, \quad B_2;$$

Note that the point A is mapped to a boundary point of $\boldsymbol{H}^3$.

Proposition 8.1. *The caustic has swallowtails at S_1 and S_2.*

Proof. We solve the first two conditions of Proposition 4.3. The solutions when $s > 0$ and $t > 0$ turns out to be $\{S_1, S_2, A\}$. The last condition of Proposition 4.3 excludes the point A. □

Proposition 8.2. *The caustic has cuspidal edges at B_1, B_2 and B, that is, no special behavior occurs.*

Proof. We only need to check the conditions of Proposition 4.2 are satisfied for $k = 4\sqrt{3}/27$ at these points. At B, we have $q' = 0$ and $q'' \neq 0$. □

Figure 9 draws a part of the caustic surface, which is the image of the region $\{(s,t) \in 0 \leq s \leq 1, \quad 0.5 \leq t \leq 2.5\}$ consisting of the cusp lines with $0.205 \leq k \leq 0.285$.

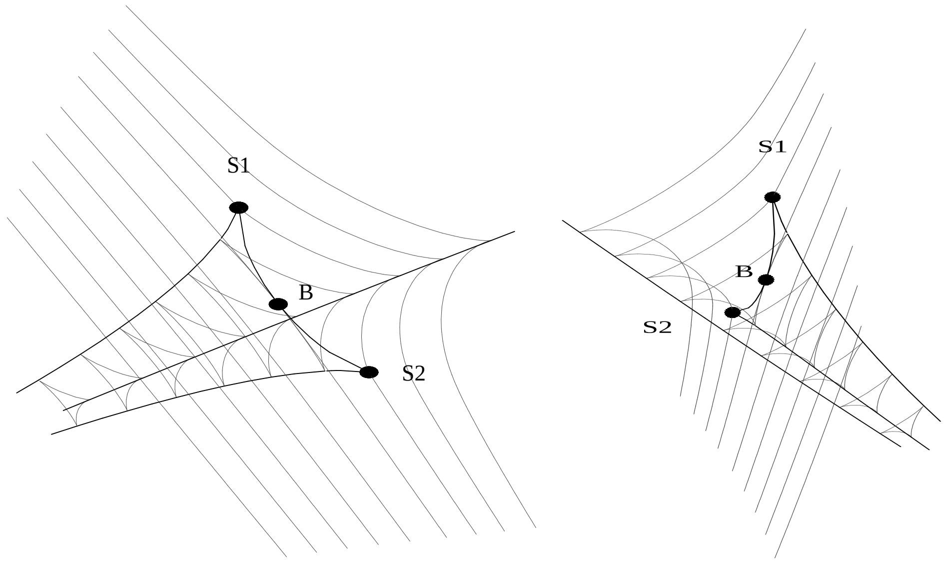

FIGURE 9. Caustic surface and its cuspidal edge curve

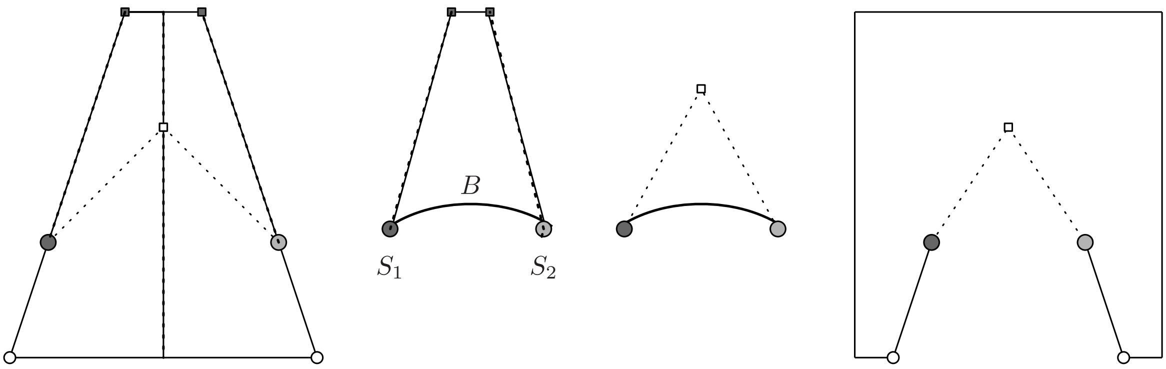

FIGURE 10. A paper model

A paper model can be made by gluing the four pieces shown in Figure 10 after folding the left one along the center line. The thick curves are cusp lines, dotted lines are self-intersection curves.

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