HYPERBOLIC SCHWARZ MAP OF THE CONFLUENT HYPERGEOMETRIC DIFFERENTIAL EQUATION

KENTARO SAJI, TAKESHI SASAKI, AND MASAAKI YOSHIDA

ABSTRACT. The hyperbolic Schwarz map is defined in [SYY1] as a map from the complex projective line to the three-dimensional real hyperbolic space by use of solutions of the hypergeometric differential equation. Its image is a flat front ([GMM, KUY, KRSUY]), and generic singularities are cuspidal edges and swallowtail singularities. In this paper, we study creations/eliminations of the swallowtails on the image surfaces of the two-parameter family of the confluent hypergeometric differential equations, and give a stratification of the parameter space according to types of singularities. Such a study was made for a 1-parameter family of hypergeometric differential equation in [NSYY].

CONTENTS

1. INTRODUCTION

The *confluent hypergeometric differential equation* is defined as

$$xu'' + (\gamma - x)u' - \alpha u = 0. \tag{1.1}$$

It is regular singular at $x = 0$ and irregular singular at $x = \infty$. By a change of the unknown u by multiplying a non-zero function $\rho = \exp(-x/2)x^{\gamma/2}$, and a change of parameters

$$a = 2\alpha - \gamma, \quad b = \gamma^2 - 2\gamma, \tag{1.2}$$

this equation transforms to the SL-form:

$$u'' - q(x)u = 0, \quad \text{where} \quad q = \frac{x^2 + 2ax + b}{4x^2}. \tag{E}$$

(Note that (E) is the SL-form of the Bessel equation if and only if $a = 0$.) Let us recall the definition of the hyperbolic Schwarz map associated with (E). For two linearly independent solutions u_0 and u_1 to this equation, we define the (multi-valued) map

$$\mathscr{S} : X = \boldsymbol{C} - \{0\} \ni x \longmapsto H(x) = U(x)\ {}^t\overline{U}(x), \tag{HS}$$

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where

$$U = \begin{pmatrix} u_0 & u_0' \\ u_1 & u_1' \end{pmatrix}.$$

Its target can be regarded as the three-dimensional hyperbolic space $\boldsymbol{H}^3$ identified with the space of positive 2×2-hermitian matrices modulo diagonal ones. The map $\mathscr{S}$ is called the *hyperbolic Schwarz map*. Its image is a surface with singularity, which is known to be a flat front in $\boldsymbol{H}^3$. We remark that the ordinary (multi-valued) *Schwarz map* is defined as

$$S : X \ni x \longmapsto u_0(x) : u_1(x) \in \boldsymbol{P}^1,$$

and the (multi-valued) *derived Schwarz map* as

$$DS : X \ni x \longmapsto u_0'(x) : u_1'(x) \in \boldsymbol{P}^1,$$

where $\mathbf{P}^1$ denotes the complex projective line, the ideal boundary of $\boldsymbol{H}^3$. The maps S and DS are connected by a one-parameter family of flat fronts in $\boldsymbol{H}^3$ and the map $\mathscr{S}$ is a member of this family. We refer to [GMM, KUY, SYY2] for these maps.

In this paper, we study the singularities on the image surface of the hyperbolic Schwarz map of the differential equation (E) with the coefficient q with real parameters a and b and show how the singularities depend on the parameters. It is well known that generic singularities of the image surfaces are cuspidal edges and swallowtails. Owing to the criteria of certain singularities in terms of the coefficient q, confluences, namely creations/eliminations, of swallowtail singularities are classified; we thus have a stratification of the parameter space according to types of singularities. Refer to Figures 6 and 7. Typical variations of cuspidal edges are picturized. The point $x = 0$ is a regular singular point of the differential equation and it is mapped into the boundary of $\boldsymbol{H}^3$. The behavior of the map around $x = \infty$ may be complicated because the differential equation is irregular singular at that point; the asymptotic behavior of the surface will be studied in the forthcoming paper [SY2].

We refer to [E, Y] for the hypergeometric differential equation and its solutions.

2. Criteria on singularities of flat fronts

2.1. **Singularities of flat fronts.** A smooth map f from a domain $U \subset \boldsymbol{R}^2$ to a Riemannian 3-manifold N^3 is called a *front* if there exists a unit vector field $\nu\colon U \to T_1N$ along the map f such that df and ν are perpendicular and the map $\nu\colon U \to T_1N$ along f is an *immersion*, where T_1N is the unit tangent bundle of N. We call ν the *unit normal vector field* of f. Note that, if we identify T_1N with the unit cotangent bundle T_1N^*, the condition $df \perp \nu$ is equivalent to the corresponding map $L\colon U \to T_1^*N$ to be Legendrian with respect to the canonical contact structure T_1^*N. A point $p \in U$ is called a *singular point* of f if $\mathrm{rank}(df)$ is less than 2 at p; it is called a singular point of rank one if $\mathrm{rank}(df)$ is equal to one. Relative to the coordinates (u, v) on U, define a function λ by

$$\lambda(u, v) = \Omega(f_u, f_v, \nu),$$

where Ω is the volume form. A singular point in U is said to be nondegenerate if $d\lambda \neq 0$. Here, let us recall some terminologies in singularity theory: Let f_i be a map germ at p_i defined on the source space S_i into the target space T_i, for $i = 1, 2$. Then, they are said to be equivalent if there exist a local diffeomorphism ϕ from S_1 to S_2 with $\phi(p_1) = p_2$ and a local diffeomorphism ψ from T_1 to T_2, such that $\psi \circ f_1 = f_2 \circ \phi$. It is well-known that generic singularities of fronts are *cuspidal* *edges* and *swallowtails* [A]; refer also to [SY1] for an elementary description. Recall that the cuspidal edge is (the equivalence class of) the map germ

$$(u,v)\mapsto(u^2,u^3,v)$$

at the origin, and the swallowtail singularity is the map germ

$$(u,v)\mapsto(3u^4+u^2v,4u^3+2uv,v)$$

at the origin. The generic (in Arnold's sense) confluence of swallowtail singularities are classified into five types called A_4, a pair of cuspidal lips, a pair of cuspidal beaks, and two types where $\mathrm{rank}(df)=0$; Refer to [LLR](p. 547) and also to [S, IST]. The first three are defined as the map germs at the origin as follows:

$$\begin{aligned} A_4:&\qquad (u,v)\mapsto(5u^5+2uv,4u^5+u^2v-v^2,v),\\ \text{Cuspidal lips:}&\qquad (u,v)\mapsto(u^3+uv^2,3u^4+2u^2v^2,v),\\ \text{Cuspidal beaks:}&\qquad (u,v)\mapsto(u^3-uv^2,3u^4-2u^2v^2,v). \end{aligned}$$

Each belongs to a family of the map germs

$$\begin{aligned} (u,v)&\mapsto(5u^5+2uv+3cu^2,4u^5+u^2v+2cu^3-v^2,v),\\ (u,v)&\mapsto(u^3+uv^2+cu,3u^4+2u^2v^2+2cu^2,v),\\ (u,v)&\mapsto(u^3-uv^2+cu,3u^4-2u^2v^2+2cu^2,v), \end{aligned}$$

respectively, where c is the parameter of the family. We furthermore introduce a family of the map germs of a higher-order singularity as

$$(u,v)\quad\mapsto\quad(6u^5+4cu^3+2uv,5u^6+3cu^4+u^2v,v),$$

which defines the singularity of type A_5 at the origin when $c=0$:

$$A_5:\qquad (u,v)\mapsto(6u^5+2uv,5u^6+u^2v,v).$$

Is A_5 the precise naming or not

In this subsection, we review the criteria of these singularities in terms of λ, and paraphrase them in terms of the coefficient q.

Assume that the map f is of rank one at a point p. Then there exits a nonvanishing vector field η around p so that $(\eta f)(q)=0$ for any singular point q around p.

Lemma 2.1. ([KRSUY, S]*; see also* [SUY]). *Let p be a nondegenerate singular point of the front f. Then, the map f at p is equivalent to*

1. *a cuspidal edge if and only if $\eta(\lambda)(p)\neq 0$,*
2. *a swallowtail singularity if and only if $\eta(\lambda)(p)=0$ and $\eta\eta(\lambda)(p)\neq 0$, and*
3. *a singularity of type A_4 if and only if $\eta(\lambda)(p)=0$, $\eta\eta(\lambda)(p)=0$, and $\eta\eta\eta(\lambda)(p)\neq 0$.*
4. *a singularity of type A_5 if and only if $\eta(\lambda)(p)=0$, $\eta\eta(\lambda)(p)=0$, $\eta\eta\eta(\lambda)(p)=0$, and $\eta\eta\eta\eta(\lambda)(p)\neq 0$.*

When the map f is of rank one and degenerate, we have the following characterization of a pair of cuspidal lips or a pair of cuspidal beaks:

Lemma 2.2. ([IST]). *Let p be a degenerate singular point of rank one of the front f. Then, it is equivalent to*

1. *a pair of cuspidal lips if and only if $d\lambda(p)=0$ and $\det(\mathrm{Hess}(\lambda))>0$, and*
2. *a pair of cuspidal beaks if and only if $d\lambda(p)=0$, $\det(\mathrm{Hess}(\lambda))<0$, and $\eta(\eta\lambda(p)\neq 0$.*

We apply the criteria above to the hyperbolic Schwarz map $f = \mathscr{S}$ associated to the equation (E). The inner product $\langle\ ,\ \rangle$ on the tangent bundle $T\boldsymbol{H}^3$ is given as $\langle X, Y\rangle = \mathrm{tr}(X\tilde{Y})/2$, where $\tilde{Y}$ is the cofactor matrix of Y, and the cross-product at $p \in \boldsymbol{H}^3$ is given as $X \times Y = i(Xp^{-1}Y - Yp^{-1}X)/2$. Refer, *e.g.*, to [KRSUY], p. 319. The normal vector field ν is given by the equation

$$\nu = U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} {}^t\overline{U},$$

which we regard as a map to $T\boldsymbol{H}^3$. Then the function λ is equal to $2i\langle \nu, f' \times \overline{f'}\rangle$ up to a constant multiple. Here and in the following, f' denotes the derivative $\partial f/\partial x$ and $f'' = \partial^2 f/\partial x^2$, and so on. Since $f' = U'{}^t\overline{U}$ and $U' = U\begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}$, we see that

$$\lambda = q\overline{q} - 1. \tag{2.1}$$

By definition, the set of singular points is

$$C\!E := \{x \in X; q(x)\overline{q}(x) - 1 = 0\}.$$

A simple computation shows that the vector field η can be chosen as

$$\eta = i\left((1 + \overline{q})\partial_x - (1 + q)\partial_{\overline{x}}\right)$$

around the point where $q \neq -1$ and

$$\eta = (1 - \overline{q})\partial_x + (1 - q)\partial_{\overline{x}}$$

around the point where $q \neq 1$. Then a computation using the first expression yields

$$\begin{aligned}
\eta(\lambda) &= 2\mathrm{Re}\{i\overline{q}(1 + \overline{q})q'\}, \\
\eta\eta(\lambda) &= 2\mathrm{Re}\{-(1 + \overline{q})^2\overline{q}q'' + (1 + \overline{q})(1 + 2q)q'\overline{q'}\}, \\
\eta\eta\eta(\lambda) &= 2\mathrm{Re}\left\{i(1 + \overline{q})(-(1 + \overline{q})^2\overline{q}q''' + (3 + 4\overline{q})\overline{q'}q'^2\right. \\
&\qquad \left.+(2 + 3q + 3\overline{q} + 4q\overline{q})\overline{q'}q'' - (1 + q)(1 + 3q)\overline{q''}q')\right\} \\
\eta\eta\eta\eta(\lambda) &= 2\mathrm{Re}\left\{(1 + \overline{q})^4\overline{q}q'''' - 5(1 + \overline{q})^2(2 + 3\overline{q})\overline{q'}q'q''\right. \\
&\qquad -(1 + \overline{q})^2(4 + 5q + 10\overline{q} + 11q\overline{q})\overline{q'}q''' \\
&\qquad +3(4 + 9q + 5\overline{q} + 11q\overline{q} + 5q^2 + 6q^2\overline{q})\overline{q'}^2q'' \\
&\qquad +(1 + \overline{q})(7 + 8q)\overline{q'}^2q'^2 \\
&\qquad \left.+(1 + q)(1 + \overline{q})(3 + 5q + 5\overline{q} + 7q\overline{q})\overline{q''}q''\right\}.
\end{aligned} \tag{2.2}$$

Now, we are prepared to paraphrase Lemmas 2.1 and 2.2 as follows:

Lemma 2.3. (1) *A point $x \in X$ is a singular point of the hyperbolic Schwarz map $\mathscr{S}$ if and only if $|q(x)| = 1$,*

(2) *a singular point $x \in X$ of $\mathscr{S}$ is equivalent to the cuspidal edge if and only if $q'(x) \neq 0$ and $\overline{q'}(x) \neq (q'/q^3)(x)$,*

(3) *a singular point $x \in X$ of $\mathscr{S}$ is equivalent to the swallowtail if and only if $q'(x) \neq 0$, $\overline{q'}(x) = (q'/q^3)(x)$, and $\overline{q''}(x) \neq -(q'/q^3)'(x)/q(x)$,*

(4) *a singular point $x \in X$ of $\mathscr{S}$ is equivalent to A_4 if and only if $q'(x) \neq 0$, $\overline{q'}(x) = (q'/q^3)(x)$, and $\overline{q''}(x) = -(q'/q^3)'(x)/q(x)$, $\overline{q'''}(x) \neq ((q'/q^3)'/q)'(x)/q(x)$,*

(5) *a singular point $x \in X$ of $\mathscr{S}$ is equivalent to A_5 if and only if $q'(x) \neq 0$, $\overline{q'}(x) = (q'/q^3)(x)$, and $\overline{q''}(x) = -(q'/q^3)'(x)/q(x)$, $\overline{q'''}(x) = ((q'/q^3)'/q)'(x)/q(x)$, $\overline{q''''}(x) \neq -(((q'/q^3)'/q)'/q)'(x)/q(x)$,*

(6) *any degenerate singular point of rank one cannot be cuspidal lips,*

(7) *and a degenerate singular point of rank one is equivalent to a pair of cuspidal beaks if and only if $q' = 0$, $q'' \neq 0$, and $\overline{q'}(x) \neq (q'/q^3)(x)$.*

Proof. The claim (1) is what we observed above. To see the claim (2), we rewrite the first identity of (2.2) by use of $\overline{q} = 1/q$:

$$\eta(\lambda) = iq(1+q)\left(\frac{q'}{q^3} - \overline{q'}\right),$$

from which we can see that $\eta(\lambda) \neq 0$ if and only if $\overline{q'} \neq q'/q^3$ when $1+q \neq 0$. We obtain the same condition also when $1-q \neq 0$; hence, we have (2). For (3), rewrite the second identity of (2.2) by use of $\overline{q} = 1/q$ and $\overline{q'} = q'/q^3$. Then we see

$$\eta\eta\lambda = -q(1+q)^2\left(\overline{q''} + \frac{q''}{q^4} - \frac{3q'^2}{q^5}\right),$$

which implies the third condition of (3) in case $1+q \neq 0$. In case $1-q \neq 0$, we have the same expression. The claims (4) and (5) are similarly shown.

Since q is a rational function of x, it is easy to see that

$$\det \mathrm{Hess}\lambda = -|q''|^2$$

on the singular set *CE*. This implies that $\det \mathrm{Hess}\lambda \leq 0$. Hence we have (6) and (7). (Note that $q'' = 0$ if and only if $2ax + 3b = 0$.) □

We remark that the condition $\overline{q''}(x) \neq -(q'/q^3)'(x)/q(x)$ in (3) of the lemma above is rewritten as $\mathrm{Re}(2q''/q^2 - 3(q')^2/q^3)(x) \neq 0$, which is the expression given in [KRSUY].

2.2. **Cuspidal edge.**

In the following, we use the real coordinates (s,t): $x = s+it$ for the sake of simplicity. In these coordinates,

$$\eta = \mathrm{Im}(q)\partial_s + (1+\mathrm{Re}(q))\partial_t \quad \text{or} \quad (1-\mathrm{Re}(q))\partial_s + \mathrm{Im}(q)\partial_t.$$

The set *CE* is defined explicitly as

$$(2.3)\quad ce := 15t^4 - (4as - 30s^2 + 4a^2 - 2b)t^2 + (b + 2as + 5s^2)(-b - 2as + 3s^2) = 0.$$

It is a plane quartic curve symmetric relative to the change $t \to -t$ for each fixed (a,b). The point $x = 0$ is a singularity of the differential equation; the point $(s,t) = (0,0)$ is on *CE* only if $b = 0$.

Lemma 2.4. *Every connected component of CE intersects the s-axis.*

Proof. Suppose that some component does not intersect the s-axis; namely, there exists a floating island component for some (a,b). Then, we can change the parameters continuously in such a way that the island should shrink to a point, since such a component does not exists when $(a,b) = (0,0)$. This implies that $ce(t,s)$ would attain a local maxima/minima at a certain point. We can show that it does not occur away from the s-axis, by the following calculus: Set $y = t^2$. The system $\partial ce/\partial y = \partial ce/\partial s = 0$ has a unique solution

$$y = -\frac{(3b+a^2)(a^4+10a^2b+5b^2)}{(a^2+15b)^2}, \quad s = -\frac{a(a^2+7b)}{a^2+15b}.$$

At this point, we have

$$ce = 16\frac{b^2(a^2-b)}{a^2+15b},$$

and the Hessian of ce equals $-16(a^2+15b)$. Since y can not be negative, the completion of the proof is now immediate. □

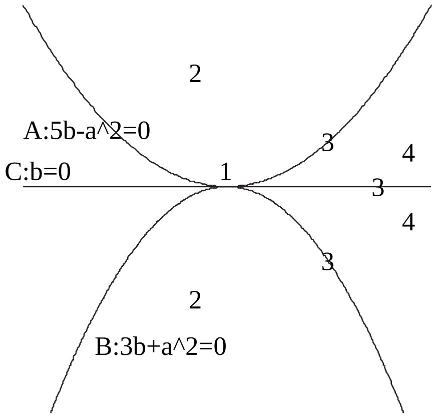


Figure 1. Stratification of ab-plane according to cardinalities of $CE \cap \{t = 0\}$

In view of this lemma, we consider the intersection $CE \cap \{t = 0\}$ to get a global view of the set CE. The intersection is defined by the equation

$$(b + 2as + 5s^2)(-b - 2as + 3s^2) = 0,$$

which is solved as

$$s = -\frac{a}{5} \pm \frac{\sqrt{a^2 - 5b}}{5}, \quad \frac{a}{3} \pm \frac{\sqrt{a^2 + 3b}}{3}.$$

When $(a, b) = (0, 0)$, four roots coincide; in this case, the coefficient reduces to $q = 1/4$, which we exclude from our consideration since $\mathscr{S}$ does not define a surface. On the line $b = 0$, the roots are $2a/3$, 0(double), $-2a/5$. On the curve $a^2 = 5b$, the first equation has double roots and, on the curve $a^2 = -3b$, the second equation has double roots. Thus, we have the stratification of the parameter plane as in Figure 1. Each numeral in the figure shows the cardinality of the intersection $CE \cap \{t = 0\}$.

The shape of the set CE depends on the parameter (a, b). Here, we notice that the polynomial ce has the homogeneity

$$ce(ks, kt, ka, k^2 b) = k^4 ce(s, t, a, b),$$

that comes from $q(kx; ka, k^2 b) = q(x; a, b)$. Hence, it is enough to consider the cases $a = 1$ and $a = 0$. When $a = 1$, the degenerate singular points, where $q' = 0$, are

$$(b; s, t) = (-1/3; 1/3, 0), (1/5; -1/5, 0), (0; 0, 0).$$

When $a = 0$, the equation still has the symmetry, it is enough to consider the case $b = \pm 1$. Remark that the case $b = 0$ was already excluded.

In Figure 2, we exhibit the set CE for the case $(a, b) = (1, -0.31)$ and $(a, b) = (0, -1)$. Refer to Figures 3 and 4 for other cases.

2.3. **Swallowtail singularities.**

To find swallowtail singularities, we need to solve the equation $q^3\overline{q'} - q' = 0$. This equation turns out to be

$$swr = 0 \quad \text{and} \quad swi = 0,$$

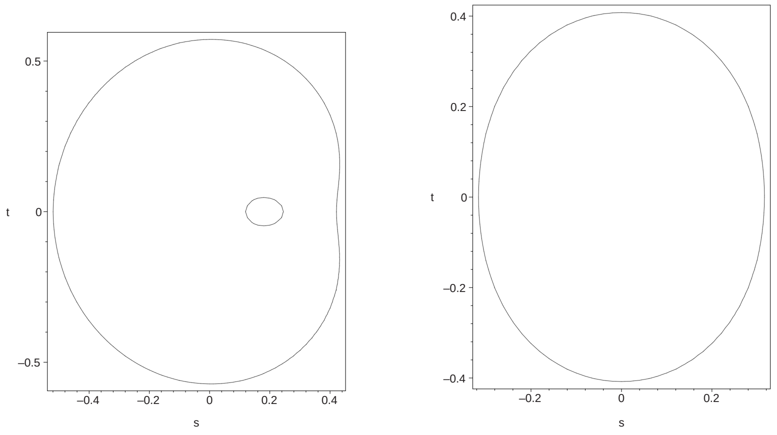


FIGURE 2. The curve *CE* when $(a, b) = (1, -0.31)$ (Left) and $(a, b) = (0, -1)$ (Right)

where

$$\begin{aligned}
swr &= 118at^6s - 12a^2 + 130bt^6 + 394at^4s^3 + (60a^2 + 354b)t^4s^2 \\
&\quad +6a(-7b + 12a^2)t^4s + (16a^4 - 6b^2)t^4 + 402at^2s^5 + (60a^2 + 414b)t^2s^4 \\
&\quad +12a(4a^2 + 11b)t^2s^3 + 36b(4a^2 + b)t^2s^2 + 6ab(4a^2 + 11b)t^2s \\
&\quad +6b^2(2a^2 + b)t^2 + 126as^7 + (-12a^2 + 126b)s^6 - 6a(4a^2 + 3b)s^5 \\
&\quad -(16a^4 + 48a^2b + 6b^2)s^4 - 10ab(4a^2 + 3b)s^3 - 6b^2(6a^2 + b)s^2 - 14b^3as - 2b^4 \\
swi &= t\,\big[126at^6 + 402at^4s^2 + (48a^2 - 12b)t^4s + 6a(4a^2 - b)t^4 \\
&\quad +394at^2s^4 + 40bt^2s^3 - 12a(4a^2 - 9b)t^2s^2 - 32a^4 + 24b^2 + 48a^2bt^2s \\
&\quad -2ab(4a^2 - 9b)t^2 + 118as^6 - (48a^2 + 12b)s^5 - 6a(12a^2 + 13b)s^4 \\
&\quad -(32a^4 + 24b^2 + 144a^2b)s^3 - 6ab(12a^2 + 13b)s^2 - 12b^2(4a^2 + b)s - 10b^3a\big]
\end{aligned}$$

For a point in the set $SW = \{(s, t); ce = swr = swi = 0\}$ to be a swallowtail singularity, it is necessary to check the third condition in (3) of Lemma 2.3. We denote by *swexc* the numerator of the real part of $2q''/q^2 - 3(q')^2/q^3$, which is a polynomial of (s, t) of total degree nine; its explicit expression is given in the appendix.

We first compute the exceptional set $SWE := \{ce = swr = swi = swexc = 0\}$. By relying on the primary decomposition of the corresponding ideal $\langle ce, swr, swi, swexc\rangle$ in the polynomial ring $\mathbf{R}[s, t, b]$ ($a = 1$ is assumed), we have the following. (Refer to, e.g., [Jac] for primary decomposition.)

Lemma 2.5. *Assume* $a = 1$. *Then the set defined by the ideal* $\langle ce, swr, swi, swexc\rangle$ *is the union of the sets defined by the ideals*

(1) $[b, s^2 + t^2]$,
(2) $[b - 1, 4t^2 + 1, 2s + 1]$,
(3) $[25b^2 - 10b - 27, 25t^2 + 10b + 7, 10s + 5b + 1]$,
(4) $[27b^2 - 70b - 21, t, 8s + 3b - 3]$,
(5) $[P_1, P_2, P_3]$,

where

$$\begin{aligned}
P_1 &= 49005b^5 - 91665b^4 + 51270b^3 - 6414b^2 - 147b - 1, \\
P_2 &= 3011952t^2 + 288933480b^4 - 567654615b^3 + 356174169b^2 - 68910105b + 3483983, \\
P_3 &= 16063744s + 396597465b^4 - 830462220b^3 + 605893890b^2 - 162782468b + 17102389.
\end{aligned}$$

By this lemma, the exceptional real points are

$$\begin{aligned}
p_1 :&\quad b = b_1; \quad (s,t) \sim (0.3291502622, 0.2516350726), \\
p_2 :&\quad b = b_2; \quad (s,t) \sim (0.4768336246, 0), \\
p_3 :&\quad b = b_3; \quad (s,t) \sim (-0.1696551154, 0.03711109674), \\
p_4 :&\quad b = b_4; \quad (s,t) \sim (-0.6990558469, 0),
\end{aligned}$$

where b_1 is one of solutions of the equation $25b^2 - 10b - 27 = 0$ of (3):

$$b_1 = \frac{1 - \sqrt{28}}{5} \sim -0.8583005244;$$

Note that the second solution $(1 + \sqrt{28})/5$ is excluded because the value t cannot be real. The values b_2 and b_4 are solutions of the equation $27b^2 - 70b - 21 = 0$ of (4):

$$b_2 = \frac{35 - \sqrt{1792}}{27} \sim -0.2715563324, \quad b_4 = \frac{35 + \sqrt{1792}}{27} \sim 2.8641489250,$$

and b_3 is the unique real solution of the equation $P_1 = 0$:

$$b_3 \sim 0.2081942455.$$

In the case $a = 0$, we see that when $b = 1$ the swallowtail points are $(s,t) = (0, \pm 1/\sqrt{5}), (\pm 1/\sqrt{3}, 0)$ and that when $b = -1$ there exist no swallowtail points.

2.4. **Types of confluence of swallowtail singularities.** The types of the above exceptional points can be identified by using the criteria in Lemma 2.3. Let C be one of the ideals in the previous lemma for the cases (3)–(5) when $a = 1$. By computing the primary decomposition of the ideal generated by the polynomials in C and the numerator *num* of the expression of $\eta\eta\eta(\lambda)$ in (2.2), we can see that p_1 and p_3 are of type A_4 and that $\eta\eta\eta(\lambda) = 0$ for p_2 and p_4. The polynomial *num* is given in the appendix.

When b passes through b_2 (resp. b_4), three swallowtail points get together to the point p_2 (resp. p_4) and then reappears a single swallowtail point. At these extreme values of b, the derivatives $\eta^k(\lambda)$ vanishes for $0 \le k \le 3$ as we have seen. However, we can examine $\eta^4(\lambda) \neq 0$. Hence, the type of singularity is A_5. We remark that this type of confluence is not generic in Arnold's sense.

When $a = 0$, no confluence occurs.

We next treat degenerate points of rank one, which can be a pair of cuspidal beaks in view of Lemma 2.3. In fact, by solving the equation $q' = 0$ and checking $q'' \neq 0$ and $q^3(x)\overline{q'}(x) - q'(x) \neq 0$, we can show that $(s,t) = (1/3, 0)$ when $b = -1/3$ and $(s,t) = (-1/5, 0)$ when $b = 1/5$ are actually cuspidal beaks.

Remark 2.6. When b passes through the value 0, three swallowtail points get together to the origin and then reappear three swallowtail points again. Such a phenomenon was observed also in the study of Gauss hypergeometric equation ([NSYY]). A similar phenomenon is known in the study of confluence of swallowtail points by Arnold; refer to the third case in the list of classification given in [LLR], p. 547. However, the present type of confluence seems to be different from Arnold's since the point $x = 0$ is a singularity of the equation and the map $\mathscr{S}$ itself is multi-valued at this point.

2.5. **Figures of the cuspidal edge.** Summarizing the above, in the case $a = 1$, we have critical values of b where the shape of the cuspidal edge and the location of the swallowtail points on it make changes: $b = b_1, -1/3, b_2, 1/5, b_3, b_4$.

In Figure 3-4, we exhibit the shape of the cuspidal edge for several values of b including these. The balck ball indicates a swallowtail point, the white quadrangle a pair of cuspidal beaks, the white ball a singularity of type A_4, and the crossed ball a singularity of type A_5.

When $a = 0$, we show two figures in Figure 5.

In Figure 6, we give a finer stratification of the ab-plane according to the cardinality of swallowtail poins, which are also described in the figure. The curves are named as

$$E1: \ b = b_1 a^2, \quad E2: \ b = b_2 a^2, \quad E3: \ b = 0.4a^2, \quad E4: \ b = b_4 a^2;$$

Here we remark that the two curves $b = b_3 a^2$ and $b = (1/5)a^2$ are very nearly situated; so, we draw $b = 0.4a^2$ instead of $b = b_3 a^2$ so that the stratification is better observed. In Figure 7, the stratification is transported to the $\alpha\gamma$-plane. The lines C_1 and C_2 are the pullback of the line C.

3. Surfaces for particular values of the parameter b

We show the image surface of $\mathscr{S}$ for some parameters $(a, b) = (1, b)$. Since $\mathscr{S}$ is multi-valued in general, we cut the (s, t)-plane along the negative s-axis. The hyperbolic Schwarz map is defined by use of solutions

$$u_0 = \rho(x)\ {}_1F_1(\alpha, \gamma; x) \quad \text{and} \quad u_1 = \rho(x) x^{1-\gamma}\ {}_1F_1(\alpha - \gamma + 1, 2 - \gamma; x),$$

where

$${}_1F_1(\alpha, \gamma; x) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)n!} x^n$$

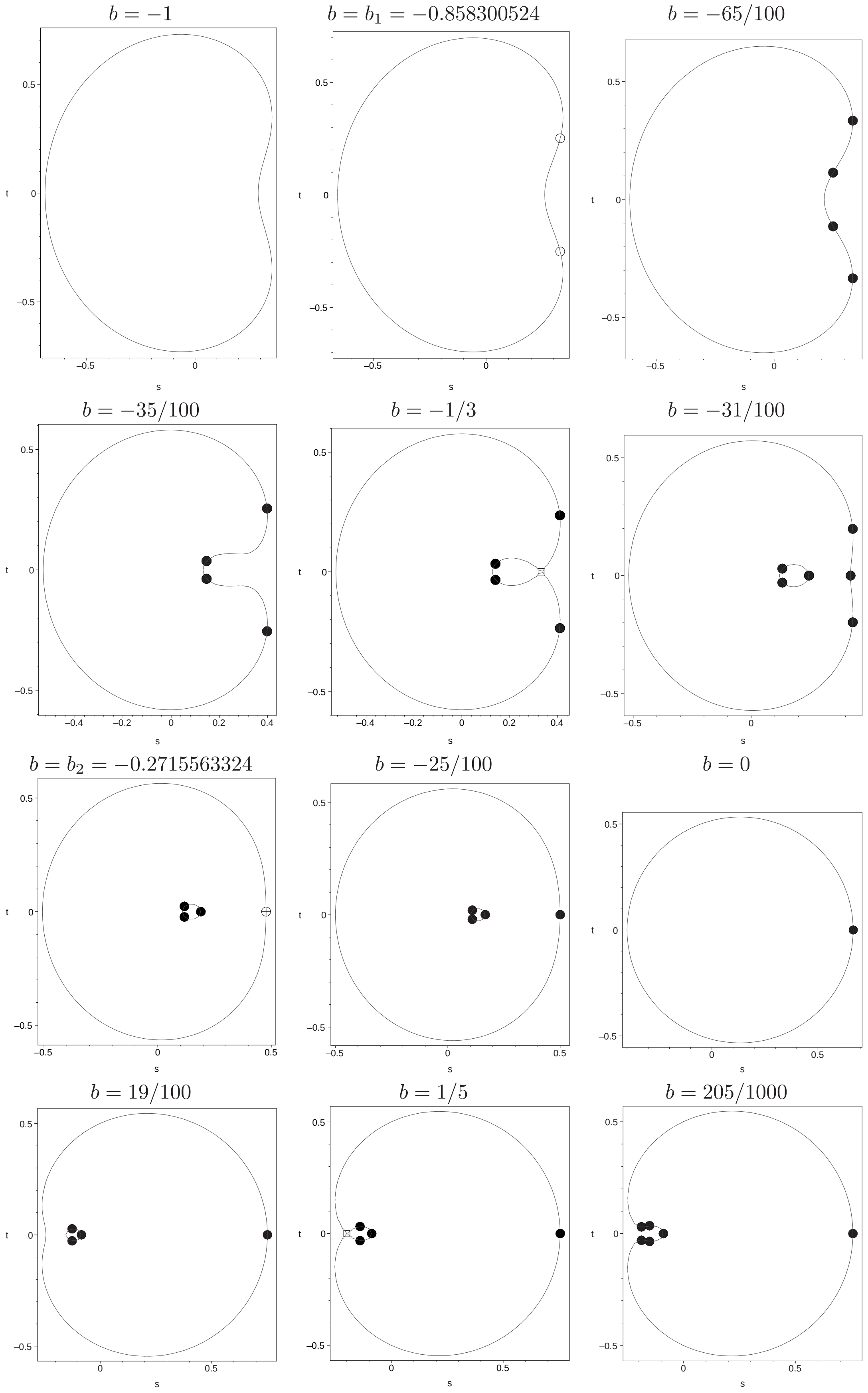
is the confluent hypergeomtric function and $\rho(x) = \exp(-x/2)x^{\gamma/2}$ is the multiplier that changes the equation (1.1) into the SL-form (E). For a given b, we have generally two sets of (α, γ) determined by (1.2); however, the both define the same map up to interchange of u_0 and u_1.

In drawing the surfaces in Figure 8, we choose five values of b so that they are lying in intervals separated by the exceptional values b_i $(1 \le i \le 4)$. Area of drawing is chosen to be a part of annuli around the origin or a quadrangle by referring the shape of the cuspidal edge in Figures 3-4. The right picture in each row is an enlargement of a part of the left picture.

The following is the list of data realtive to the coordinate $x = s + it = r\exp(i\theta)$, where $-19/20\pi \le \theta \le 19/20\pi$ is assumed.

- $b = -0.9$: $0.2 \le r \le 1.2$
- $b = -0.65$: $0.2 \le r \le 0.7$, and $0.2 \le s \le 0.4, -0.7 \le t \le 0.7$
- $b = -0.31$: $0.05 \le r \le 0.6$, and $0.05 \le s \le 0.4, -0.4 \le t \le 0.4$
- $b = -0.25$: $0.09 \le s \le 0.55$, $-0.1 \le t \le 0.1$, and $0.09 \le s \le 0.22$, $-0.1 \le t \le 0.1$
- $b = 1$: $0.2 \le r \le 1.1$, and $0.6 \le s \le 1.4, -0.5 \le t \le 0.5$

Drawing is done by Maple 9.5.



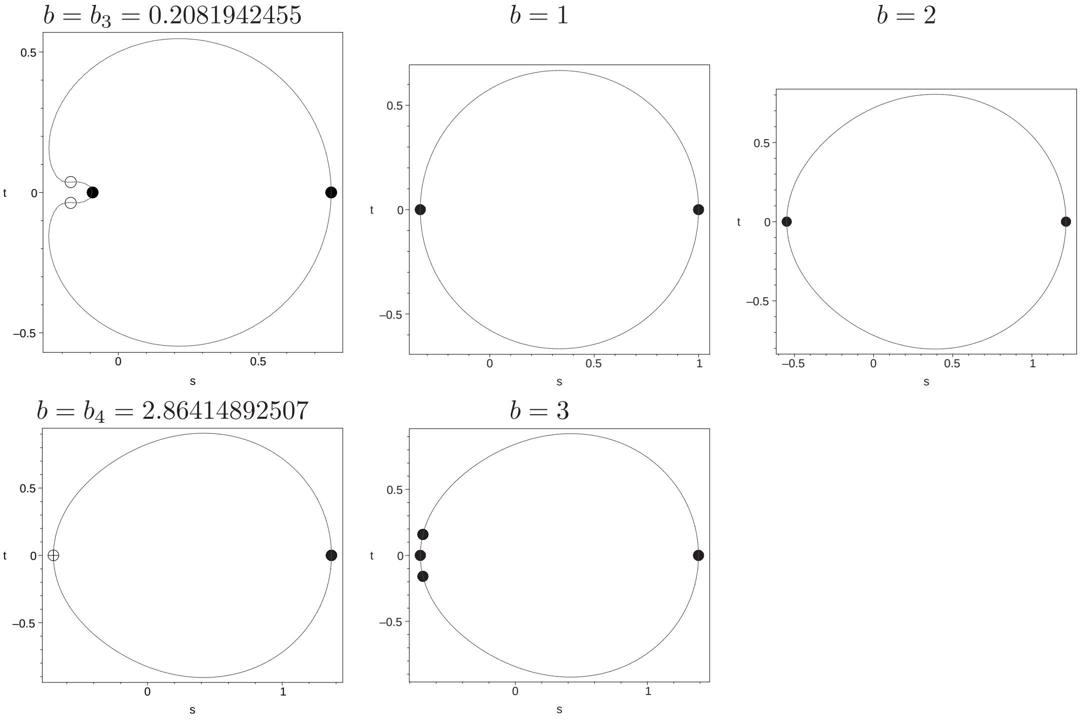
$b = -1$ $\quad$ $b = b_1 = -0.858300524$ $\quad$ $b = -65/100$

$b = -35/100$ $\quad$ $b = -1/3$ $\quad$ $b = -31/100$

$b = b_2 = -0.2715563324$ $\quad$ $b = -25/100$ $\quad$ $b = 0$

$b = 19/100$ $\quad$ $b = 1/5$ $\quad$ $b = 205/1000$

Figure 3. Shapes of CE when $a = 1$


$b = b_3 = 0.2081942455$ $b = 1$ $b = 2$

$b = b_4 = 2.86414892507$ $b = 3$

FIGURE 4. Shapes of *CE* when $a = 1$ continued

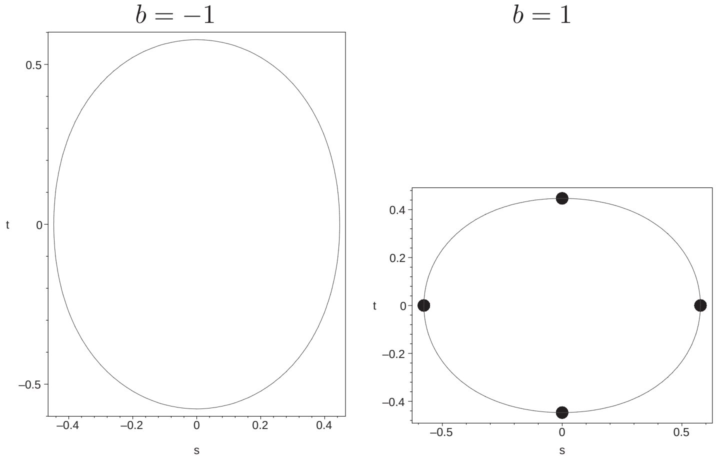

$b = -1$ $b = 1$

FIGURE 5. Shapes of *CE* when $a = 0$

APPENDIX

Two polynomials *swexc* and *num* are computed as follows.

$$\begin{aligned} swexc \;=\; & 32t^6a^4 - 88t^8a^2 + 64a^5t^4s - 144s^5abt^2 - 656s^3at^4b + 576s^3at^2b^2 + 16s^9a \\ & +104s^8a^2 + 240s^7a^3 + 224s^6a^4 + 64a^5s^5 + 24s^8b + 24s^2b^4 + 72s^4b^3 + 72s^6b^2 \\ & -24b^4t^2 + 24bt^8 - 72b^2t^6 + 72b^3t^4 + 208s^3ab^3 + 384s^5ab^2 + 208s^7ab + 672s^5a^3b \\ & +600s^4a^2b^2 + 600s^6a^2b - 96s^5at^4 - 128s^3at^6 + 160s^6a^2t^2 + 528s^5a^3t^2 \\ & +336s^3a^3t^4 + 480s^4a^4t^2 - 80s^4a^2t^4 - 48ast^8 + 48a^3st^6 + 288a^4s^2t^4 \\ & +408s^4a^2t^2b + 576s^3a^3t^2b - 224a^2s^2t^6 + 104a^2s^2b^3 + 224a^4s^4b + 240a^3s^3b^2 \\ & +128a^5s^3t^2 + 16asb^4 - 304at^6bs - 32a^4t^4b - 24a^2t^6b + 88a^2t^2b^3 + 24a^2t^4b^2 \\ & -96s^2bt^6 - 96s^6bt^2 - 240s^4bt^4 + 72s^4b^2t^2 - 72s^2b^2t^4 + 144s^2b^3t^2 \\ & -96a^3t^4bs + 192a^4t^2bs^2 - 216a^2t^4bs^2 + 624a^2t^2b^2s^2 + 240a^3t^2b^2s \\ & +192at^4b^2s + 144at^2b^3s \end{aligned}$$

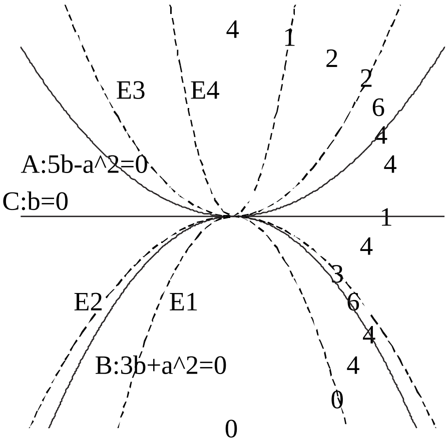


FIGURE 6. Stratification of ab-plane and cardinalities of swallowtail points

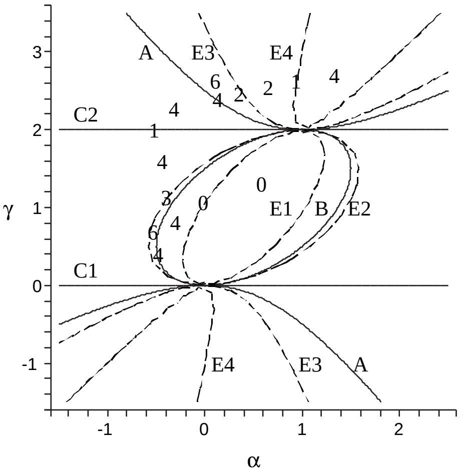
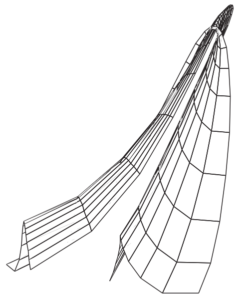


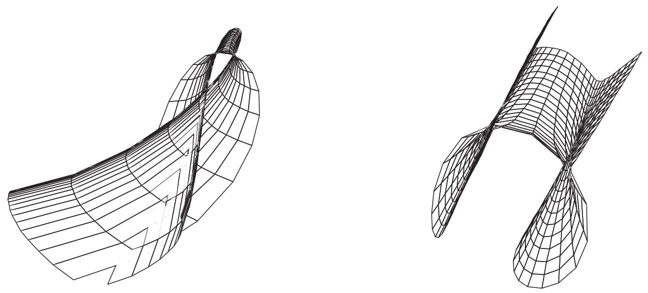
FIGURE 7. Stratification of $\alpha\gamma$-plane and cardinalities of swallowtail points

$$
\begin{aligned}
num \quad = \quad & -81t^{12} + 162s^2t^{10} - (810b+90)st^{10} + (45b-108)t^{10} + 1377s^4t^8 \\
& -(810b+450)s^3t^8 + (-45b+204)s^2t^8 - (135b^2+1068b+648)st^8 \\
& +(-99b^2+324b+192)t^8 + 2268s^6t^6 + (2268b-900)s^5t^6 + (-630b+1896)s^4t^6 \\
& -(540b^2+2592+1296b)s^3t^6 + (324b^2-1296b+768)s^2t^6 \\
& -b(81b^2+958b-768)st^6 + b^2(27b+400)t^6 + 1377s^8t^4 + (3564b-900)s^7t^4 \\
& +(-1170b+3384)s^6t^4 + (-810b^2+2520b-3888)s^5t^4 + (2598b^2-5832b+1152)s^4t^4 \\
& +b(273b^2-4594b+2304)s^3t^4 - b^2(985b-1936)s^2t^4 - b^3(25b-556)st^4 \\
& +28b^4t^4 + 162s^{10}t^2 + (1134b-450)s^9t^2 + (-855b+2436)s^8t^2 \\
& +(-540b^2+4656b-2592)s^7t^2 + (3828b^2-6480b+768)s^6t^2 \\
& +b(789b^2-6314b+2304)s^5t^2 - b^2(2323b-2672)s^4t^2 - 2b^3(127b-676)s^3t^2 \\
& +272b^4s^2t^2 + 11b^5st^2 - 81s^{12} + (-162b-90)s^{11} + (636-225b)s^{10} \\
& +(-135b^2+1908b-648)s^9 + (192-2268b+1653b^2)s^8 + b(435b^2-2678b+768)s^7 \\
& -b^2(-1136+1311b)s^6 - b^3(229b-796)s^5 + 268b^4s^4 + 35b^5s^3
\end{aligned}
$$

$b = -0.9$

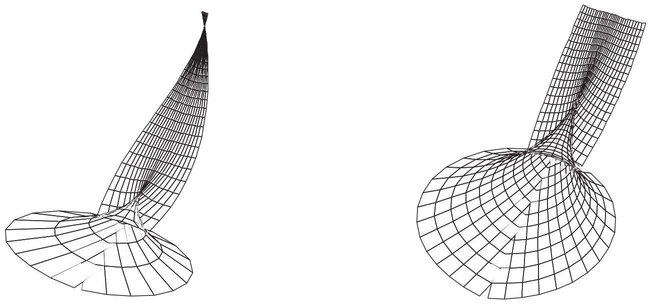


$b = -0.65$



$b = -0.31$

$b = -0.25$



$b = 1$

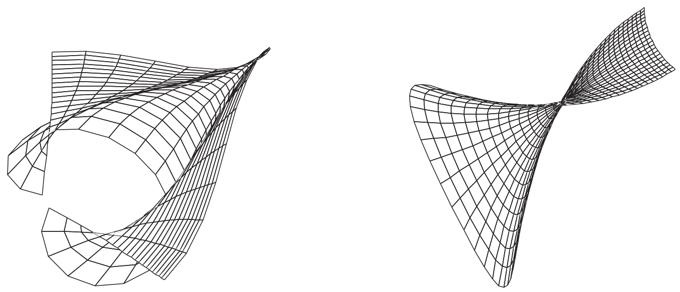


FIGURE 8. Pictures of image surfaces when $a = 1$

References

[A] V. I. Arnol'd, *Wave front evolution and equivariant Morse lemma*, Comm. pure appl. Math., **29**(1976), 557–582.

[E] A. Erdélyi, Higher Transcendental Functions, vol. 1, McGraw-Hill, New York, 1953.

[GMM] J. A. Gálvez, A. Martínez and F. Milán, *Flat surfaces in hyperbolic 3-space*, Math. Annalen, **316** (2000), 419–435.

[IST] S. Izumiya, K. Saji, and M. Takahashi, *Horospherical flat surfaces in hyperbolic 3-space*, Hokkaido University Preprint Series in Mathematics #838, 2007.

[Jac] N. Jacobson, *Lectures in abstract algebra 3 : Theory of fields and Galois theory*, Graduate Texts in Math. 32, 1975, Springer.

[KUY] M. Kokubu, M. Umehara and K. Yamada, *An elementary proof of Small's formula for null curves in* $PSL(2,\boldsymbol{C})$ *and an analogue for Legendreian curves in* $PSL(2,\boldsymbol{C})$, Osaka J. Math. **40**(2003), 697–715.

[KRSUY] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic space*, Pacific J. Math. **221**(2005), 303–351.

[LLR] R. Langevin, G. Levitt and H. Rosenberg, *Classes d'homotopie de surfaces avec rebroussements et queues d'aronde dans* $\mathbf{R}^3$, Can. J. Math. **47**(1995), 544–572.

[NSYY] M. Noro, T. Sasaki K. Yamada and M. Yoshida, *Confluence of swallowtail singularities of the hyperbolic Schwarz map defined by the hypergeometric differential equation*, preprint 2007.

[S] K. Saji, in preparation.

[SUY] K. Saji, M. Umehara and K. Yamada, in preparation.

[SY1] T. Sasaki and M. Yoshida, *Surface singularities appeared in the hyperbolic Schwarz map for the hypergeometric equation*, preprint, 2007.

[SY2] T. Sasaki and M. Yoshida, *Hyperbolic Schwarz map of the Airy differential equation*, preprint, 2007.

[SYY1] T. Sasaki K. Yamada and M. Yoshida, *Hyperbolic Schwarz map for the hypergeometric equation*, preprint, math.CA/0609196; revised 2007.

[SYY2] T. Sasaki K. Yamada and M. Yoshida, *Derived Schwarz map of the hypergeometric differential equation and a parallel family of flat fronts*, to appear in International J. Math.

[Y] M. Yoshida, *Hypergeometric Functions, My Love*, Vieweg Verlag, Wiesbaden, 1997.

(Saji) Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: `saji@math.sci.hokudai.ac.jp`

(Sasaki) Department of Mathematics, Kobe University, Kobe 657-8501, Japan
E-mail address: `sasaki@math.kobe-u.ac.jp`

(Yoshida) Faculty of Mathematics, Kyushu University, Fukuoka 810-8560, Japan
E-mail address: `myoshida@math.kyushu-u.ac.jp`