

We can easily check

Lemma A.1. *Matrices v^σ and V_σ have the property*

$$(1) \quad \sum_{\sigma} v_{ij}^{\sigma} V_{\sigma}^{k\ell} = \frac{1}{2}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}); \quad \text{in particular} \quad \sum_{\sigma,j} v_{ij}^{\sigma} V_{\sigma}^{jk} = \frac{n+1}{2}\delta_{ik}.$$

$$(2) \quad \sum_{i,j} v_{ij}^{\sigma} V_{\tau}^{ij} = \delta_{\tau}^{\sigma}.$$

After Weise [7], the sets $\{v^\sigma\}$ and $\{V_\sigma\}$ are called the *inverse* of each other provided the above two properties (1) and (2) hold between them. If a set of matrices $\{h^\sigma\}$ is related to $\{v^\sigma\}$ by

$$h_{ij}^{\sigma} = \sum a_i^p v_{pq}^{\rho} a_j^q B_{\rho}^{\sigma}$$

for nonsingular matrices $a = (a_i^j)$ and $b = (b_{\rho}^{\sigma})$, where $B = b^{-1}$, then define $\{H_{\sigma}\}$ by

$$(3) \quad H_{\sigma}^{ij} = \sum A_p^i V_{\rho}^{pq} A_q^j b_{\sigma}^{\rho},$$

where $A = a^{-1}$. With this definition, the pair $\{h^\sigma\}$ and $\{H_{\sigma}\}$ also satisfies (1) and (2), that is, $\{h^\sigma\}$ and $\{H_{\sigma}\}$ are the inverse of each other.

References

1. Decruyenaere, F., Dillen, F., Verstraelen, L., and Vrancken, L., Affine maximal surfaces in \mathbf{R}^5 , preprint, 1994.
2. Decruyenaere, F., Dillen, F., Verstraelen, L., and Vrancken, L., Affine surfaces in \mathbf{R}^5 , *International J. Math.* **5**(1994), 657–679.
3. Dillen, F., Mys, G., Verstraelen, L., and Vrancken, L., The affine mean curvature vector for surfaces in \mathbf{R}^4 , *Math. Nachr.* **166**(1994), 155–165.
4. Nomizu, K. and Sasaki, T., *Affine Differential Geometry*, Cambridge Univ. Press, Cambridge, 1994.
5. Nomizu, K. and Vrancken, V., A new equiaffine theory for surfaces in \mathbf{R}^4 , *International J. Math.* **4**(1993), 127–165.
6. Sasaki, T., On the Veronese embedding and related system of differential equations, *Proceedings of Global Differential Geometry and Global Analysis, Berlin 1990*; *Lecture Notes in Math.* **1481** ed. by D. Ferus et al., Springer, 1991, 210–247.
7. Weise, K.H., Der Berührungstensor zweier Flächen und die Affingeometrie der F_p im A_n , *Math. Z.* **43**(1938), 469–480; *ibid.* **44**(1939), 161–184.

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because ∇ is torsion-free. Then, by definition,

$$\begin{aligned}
& nd\beta(X_1, \dots, X_n) \\
&= \sum_j (-1)^{j-1} X_j(\beta(X_1, \dots, \widehat{X}_j, \dots, X_n)) \\
&\quad + \sum_{j < k} (-1)^{j+k} \beta([X_j, X_k], X_1, \dots, \widehat{X}_j, \dots, \widehat{X}_k, \dots, X_n) \\
&= \sum_j (-1)^{j-1} X_j(\theta(Z, X_1, \dots, \widehat{X}_j, \dots, X_n)) \\
&\quad + \sum_{j < k; \ell} (-1)^{j+k} \theta(Z, (\Gamma_{jk}^\ell - \Gamma_{kj}^\ell) X_\ell, X_1, \dots, \widehat{X}_j, \dots, \widehat{X}_k, \dots, X_n) \\
&= \sum_j (-1)^{j-1} X_j(\theta(Z, X_1, \dots, \widehat{X}_j, \dots, X_n)) \\
&\quad + \sum_{j < k; \ell} (-1)^{j+k} \theta(Z, (\Gamma_{jk}^\ell - \Gamma_{kj}^\ell) X_\ell, X_1, \dots, \widehat{X}_j, \dots, \widehat{X}_k, \dots, X_n) \\
&= \sum_j (-1)^{j-1} X_j(w^j \theta(X_j, X_1, \dots, \widehat{X}_j, \dots, X_n)) \\
&\quad + \sum_{j < k} (-1)^{j+k} \{w^j (\Gamma_{jk}^k - \Gamma_{kj}^k) \theta(X_j, X_k, X_1, \dots, \widehat{X}_j, \dots, \widehat{X}_k, \dots, X_n) \\
&\quad\quad + w^k (\Gamma_{jk}^j - \Gamma_{kj}^j) \theta(X_k, X_j, X_1, \dots, \widehat{X}_j, \dots, \widehat{X}_k, \dots, X_n)\} \\
&= \left(\sum X_j(w^j) + \sum_{j < k} A_{jk} \right) \theta(X_1, \dots, X_n),
\end{aligned}$$

where

$$A_{jk} = -w^j (\Gamma_{jk}^k - \Gamma_{kj}^k) + w^k (\Gamma_{jk}^j - \Gamma_{kj}^j).$$

Here we see $A_{jj} = 0$ and $A_{jk} = A_{kj}$ and $\sum_{j < k} A_{jk} = \frac{1}{2} \sum_{j, k} A_{jk}$. Then, since $\sum_i \Gamma_{ji}^i = 0$, we see

$$\sum_{j < k} A_{jk} = \sum_{j, k} w^j \Gamma_{kj}^k$$

and get the identity (3.11).

Appendix

Let E_{ij} be the $n \times n$ -matrix with 1 in (ij) -th component and 0's in the others. The letters σ, τ, \dots are used to denote pairs (ij) for $1 \leq i, j \leq n$. Define symmetric matrices v^σ and V_σ as follows.

$$\begin{aligned}
& v^{(ij)} = E_{ij} + E_{ji}, \\
\text{(A.1)} \quad & V_{(ij)} = \begin{cases} \frac{1}{2} E_{ii} & j = i \\ \frac{1}{2} (E_{ij} + E_{ji}) & j \neq i. \end{cases}
\end{aligned}$$

Now, from (2.11), we have

$$T_{ij}^\sigma = X_i(u_j^\sigma) + u_j^\rho(H_\rho^{pq}C_{ipq}^\sigma + 2H_\rho^{pq}h_{rq}^\sigma\Gamma_{ip}^r) - \Gamma_{ij}^k u_k^\sigma.$$

Hence

$$\begin{aligned} T &= X_i(H_\sigma^{ij}u_j^\sigma) + u_j^\rho H_\rho^{pq}C_{ipq}^\sigma H_\sigma^{ij} + 2u_j^\rho H_\rho^{pq}H_\sigma^{ij}h_{rq}^\sigma\Gamma_{ip}^r - H_\sigma^{ij}\Gamma_{ij}^k u_k^\sigma \\ &= X_i(H_\sigma^{ij}u_j^\sigma) + 2u_j^\rho H_\rho^{pq}H_\sigma^{ij}h_{rq}^\sigma\Gamma_{ip}^r - H_\sigma^{ij}\Gamma_{ij}^k u_k^\sigma \quad (\text{apolarity}) \\ &= X_i(H_\sigma^{ij}u_j^\sigma) + u_j^\rho H_\rho^{pq}(\delta_{ir}\delta_{jq} + \delta_{iq}\delta_{jr})\Gamma_{ip}^r - H_\sigma^{ij}\Gamma_{ij}^k u_k^\sigma \quad (\text{see Appendix}) \\ &= X_i(H_\sigma^{ij}u_j^\sigma) + u_j^\rho H_\rho^{kj}\Gamma_{ik}^i. \end{aligned}$$

Namely,

$$(3.10). \quad T = X_i(w^i) + \Gamma_{ik}^i w^k \quad \text{where} \quad w^i = H_\sigma^{ij}u_j^\sigma.$$

We now prove the identity

$$\int_D T\theta = 0$$

when w is compactly supported so that $\text{supp } w \subset D$. In fact, it is sufficient to see that, if we define an $(n-1)$ -form β by

$$\beta(V_1, \dots, V_{n-1}) = \theta(Z, V_1, \dots, V_{n-1}) \quad \text{where} \quad Z = \sum w^j X_j,$$

then it holds

$$(3.11) \quad d\beta = \frac{1}{n}T\theta.$$

Summing up the computation above, we have proved

$$(3.12) \quad \frac{d}{dt} \left(\int_D \theta_t \right) \Big|_{t=0} = -\frac{n+1}{2(n+2)} \int_D p^\sigma \text{tr } S_\sigma \theta$$

and conclude the proof of Theorem 3.1.

It remains to show (3.11). For this, remark that

$$[X_j, X_k] = (\Gamma_{jk}^\ell - \Gamma_{kj}^\ell)X_\ell$$

Since the matrix b is the identity when $t = 0$, we have

$$\delta(\det b) = \text{tr } \delta b,$$

which implies

$$(3.5) \quad \delta\theta_t = (\text{tr } \delta b)\theta.$$

We also see

$$(3.6) \quad \text{tr } \delta a + \text{tr } \delta b = 0.$$

On the other hand, the identity (1.15) is now

$$v_{ij}^\sigma = a_i^k \bar{h}_{k\ell}^\rho a_j^\ell B_\rho^\sigma$$

and we have

$$0 = \delta a_i^k h_{kj}^\sigma + \delta \bar{h}_{ij}^\sigma + h_{i\ell}^\sigma \delta a_j^\ell + h_{ij}^\rho \delta B_\rho^\sigma.$$

Then the contraction with H_σ^{ij} shows

$$0 = (n+2)\text{tr } \delta a + H_\sigma^{ij} \delta \bar{h}_{ij}^\sigma$$

since $\text{tr } \delta B = -\text{tr } \delta b = \text{tr } \delta a$. By (3.3), we have

$$\delta \bar{h}_{ij}^\sigma = \delta Q_i^k h_{kj}^\sigma + T_{ij}^\sigma,$$

where

$$T_{ij}^\sigma = X_i(u_j^\sigma) + u_j^\rho \tau_\rho^\sigma(X_i) - \Gamma_{ij}^k u_k^\sigma.$$

Hence

$$(3.7) \quad H_\sigma^{ij} \delta \bar{h}_{ij}^\sigma = \frac{n+1}{2} \text{tr } \delta Q + T,$$

where

$$(3.8) \quad T = H_\sigma^{ij} T_{ij}^\sigma.$$

Since

$$\begin{aligned} \text{tr } \delta b &= -\text{tr } \delta a = \frac{1}{n+2} H_\sigma^{ij} \delta \bar{h}_{ij}^\sigma, \\ \text{tr } \delta Q &= -\text{tr } \delta q = p^\sigma \text{tr } S_\sigma, \end{aligned}$$

we can see

$$(3.9) \quad \delta\theta_t = A\theta \quad \text{where} \\ A = \frac{n+1}{2(n+2)} p^\sigma \text{tr } S_\sigma + \frac{1}{n+2} T.$$

Then we need to compute δvol .

Fix an affine normalized basis $\{X_i, \xi_\sigma\}$ for which $h^\sigma = v^\sigma$ holds. We shortly write \bar{f} for \bar{f}_t and do not write the summation symbol for repeated indices unless ambiguity arises. First note that

$$\bar{f}_* X_i = f_* X_i + t X_i(p^\sigma) \xi_\sigma + t p^\sigma (-S_\sigma X_i + \tau_\sigma^\rho(X_i) \xi_\rho).$$

We put

$$S_\sigma(X_i) = S_{\sigma i}^j X_j \quad \text{and} \quad q_i^j = \delta_i^j - t p^\sigma S_{\sigma i}^j.$$

Then we have

$$\bar{f}_* X_i = q_i^j f_* X_j + t(X_i(p^\sigma) + p^\rho \tau_\rho^\sigma(X_i)) \xi_\sigma.$$

Further we put

$$u_j^\sigma = Q_j^i (p_i^\sigma + p^\rho \tau_\rho^\sigma) \quad \text{where} \quad Q = q^{-1}, \\ Y_j = Q_j^i X_i.$$

Then

$$(3.2) \quad \bar{f}_* Y_j = f_* X_j + t u_j^\sigma \xi_\sigma.$$

The basis $\{Y_j, \xi_\sigma\}$ is unimodular. A simple calculation shows

$$D_{Y_i} \bar{f}_* Y_j = Q_i^k \{f_* \nabla_{X_k} X_j + h^\sigma(X_k, X_j) \xi_\sigma + t X_k(u_j^\sigma) \xi_\sigma \\ + t u_j^\sigma (-f_* S_\sigma X_k + \tau_\sigma^\rho(X_k) \xi_\rho)\}.$$

Letting

$$\nabla_{X_i} X_j = \Gamma_{ij}^k X_k,$$

we see

$$(3.3) \quad \bar{h}_{ij}^\sigma = \bar{h}^\sigma(Y_i, Y_j) \\ = Q_i^k \{h_{kj}^\sigma + t X_k(u_j^\sigma) + t u_j^\rho \tau_\rho^\sigma(X_k) \\ - t \Gamma_{kj}^\ell u_\ell^\sigma + t^2 u_j^\rho S_{\rho k}^\ell u_\ell^\sigma\}.$$

Now we choose an affine normalized basis $\{Z_j, \eta_\rho\}$ for \bar{f} by setting

$$Z_j = \sum a_j^i Y_i, \quad \eta_\rho = \sum b_\rho^\sigma \xi_\sigma + U_\rho$$

for appropriate a , b , and U so that the h^σ relative to $\{Z_j, \eta_\rho\}$ is equal to v^σ and $\det a \det b = 1$. Then the affine volume form θ_t for \bar{f}_t is given by

$$(3.4) \quad \theta_t = (\det b) \theta.$$

it is easy to see that g is the identity matrix and $\{X_i, Z_\sigma\}$ is a unimodular basis. Since $D_{X_i} Z_\sigma = 0$, we see that the affine normal field N is generated by Z_σ and is parallel; $S_\sigma = 0$ and both ∇ and ∇^\perp are flat; hence, $C = 0$. The immersion (2.10) is called the *Veronese immersion*.

Conversely, the property $C = 0$ has the following geometrical meaning.

Theorem 2.7. Let $f : M^n \rightarrow \mathbf{R}^{n+p}$, $p = n(n+1)/2$, be a nondegenerate immersion and assume that the cubic form relative to the affine normal field vanishes everywhere. Then the immersion is locally projectively equivalent to the Veronese immersion.

We give a proof relying on the paper [6], where a projective treatment of nondegenerate immersion $f : M^n \rightarrow \mathbf{P}^{n(n+3)/2}$ is formulated. Note that the definition of the cubic form is given similarly in the projective case and that the vanishing of the cubic form is a projective property. Then the result follows from Theorem 5.1 of [6] which states that the vanishing of the cubic form implies the projective equivalence of the given immersion to the Veronese immersion.

§3. Affine minimal immersion

Let $f : M^n \rightarrow \mathbf{R}^{n+p}$, $p = n(n+1)/2$, be a nondegenerate immersion and θ the affine volume form. We assume that M is oriented so that θ is globally defined. To any relatively compact domain D in M , we associate its volume by

$$vol(D) = \int_D \theta.$$

In this section, we compute a variational formula of this functional relative to a normal variation of f :

$$(3.1) \quad \bar{f}_t(x) = f(x) + t \sum p^\sigma \xi_\sigma,$$

which depends on a real parameter t ; we assume that t is sufficiently small so that \bar{f}_t defines a nondegenerate immersion and that p^σ are compactly supported smooth functions whose supports are contained in D .

The result we prove is the following.

Theorem 3.1. Let $f : M^n \rightarrow \mathbf{R}^{n+p}$, $p = n(n+1)/2$, be a nondegenerate immersion. Then the functional vol is critical if and only if the trace of S_ξ vanishes identically for all ξ .

In the following we use the notation

$$\delta g = \left. \frac{d}{dt} g \right|_{t=0}$$

for the function g depending on t . Let θ_t denote the affine volume form of the immersion \bar{f}_t and write

$$vol(t) = \int_D \theta_t.$$

which implies that the connection ∇^\perp is determined by the cubic form and the connection ∇ . Define an endomorphism $A = (A_{\rho\ell}^{\sigma j})$ belonging to $\text{End}(\mathcal{T}^* \otimes \mathcal{T}, \mathcal{N}^* \otimes \mathcal{N})$ by

$$A_{\rho\ell}^{\sigma j} = \sum H_\rho^{jk} h_{k\ell}^\sigma.$$

This can be considered to be also an element of $\text{End}(\text{End}(\mathcal{N}), \text{End}(\mathcal{T}))$, which we denote by A^\perp to clarify the difference from A . We regard ∇_X and ∇_X^\perp as elements of $\text{End } \mathcal{T}$ and $\text{End } \mathcal{N}$, respectively. For each element X , the quadratic form $C(X, \cdot, \cdot)$, denoted by C_X , is considered to be an element of $\text{End } \mathcal{N}$ through the identification of $S^2\mathcal{T}$ with \mathcal{N} as was exhibited in §1. Then the formula (2.8) and (2.11) can be written shortly as

$$\begin{aligned} \nabla_X^\perp &= C_X + 2A(\nabla_X) \\ \frac{n+2}{2}\nabla_X &= A^\perp(\nabla_X^\perp). \end{aligned}$$

Example 2.6. Let (x^1, \dots, x^n) denote the coordinates of \mathbf{R}^n and $(z^1, \dots, z^n, \dots, z^\sigma, \dots)$ the coordinates of $\mathbf{R}^{n(n+3)/2}$. Define the immersion $f : \mathbf{R}^n \rightarrow \mathbf{R}^{n(n+3)/2}$ by

$$z^i = x^i, \quad z^\sigma = f^\sigma(x),$$

where f^σ are functions of x . We fix the vector fields by

$$X_i = \frac{\partial}{\partial x^i}, \quad Z_i = \frac{\partial}{\partial z^i}, \quad Z_\sigma = \frac{\partial}{\partial z^\sigma}.$$

Then, we see $f_*X_i = Z_i + \sum f_i^\sigma Z_\sigma$ and

$$D_{X_i}f_*X_j = \sum f_{ij}^\sigma Z_\sigma, \quad D_{X_i}Z_\sigma = 0,$$

where $f_i^\sigma = \partial f^\sigma / \partial x^i$ and $f_{ij}^\sigma = \partial^2 f^\sigma / \partial x^i \partial x^j$. Define a $p \times p$ matrix $g = (g_\rho^\sigma)$ by

$$g_\rho^\sigma = \begin{cases} \frac{1}{2}f_{ii}^\sigma & \text{when } \rho = (ii) \\ f_{ij}^\sigma & \text{when } \rho = (ij), i \neq j \end{cases}$$

Then the nondegeneracy of the immersion is equivalent to the condition $\lambda = \det g \neq 0$. Put $\xi_\sigma = \sum g_\sigma^\rho Z_\rho$. Then, we have

$$D_{X_i}f_*X_j = (1 + \delta_{ij})\xi_{(ij)}$$

and $\omega(X_i, \xi_\sigma) = \lambda$. If we assume $\lambda > 0$, then $\{\mu X_i, \nu \xi_\sigma\}$, where $\mu = \lambda^{-1/n(n+2)}$ and $\nu = \lambda^{-2/n(n+2)}$, is a unimodular basis with h^σ the standard tensor v^σ . In particular, consider the case where

$$(2.10) \quad f^\sigma(x) = x^i x^j \quad \text{for } \sigma = (ij);$$

Proof. Fix an affine normalized basis $\{X_i, \xi_\sigma\}$ so that $h^\sigma = v^\sigma$. Put $\nabla_{X_i} X_j = \sum \Gamma_{ij}^p X_p$ and $\nabla_{X_i}^\perp \xi_\rho = \gamma_{i\rho}^\sigma \xi_\sigma$. By definition, we have

$$(2.8) \quad C_{ijk}^\sigma = -\sum \Gamma_{ij}^p h_{pk}^\sigma - \sum \Gamma_{ik}^p h_{jp}^\sigma + \sum h_{jk}^\rho \gamma_{i\rho}^\sigma.$$

Hence the apolarity condition (2.5) implies that the tensor

$$(2.9) \quad \sum H_\sigma^{\ell k} C_{ijk}^\sigma = -\frac{n+2}{2} \Gamma_{ij}^\ell - \frac{1}{2} \delta_{j\ell} \sum \Gamma_{ik}^k + \sum H_\sigma^{\ell k} h_{jk}^\rho \gamma_{i\rho}^\sigma$$

vanishes. Then the contraction relative to (ℓ, j) shows

$$-(n+1) \sum \Gamma_{ik}^k + \sum \gamma_{i\sigma}^\sigma = 0.$$

Since $\{X_i, \xi_\sigma\}$ is unimodular, we have

$$\sum \Gamma_{ik}^k + \sum \gamma_{i\sigma}^\sigma = 0.$$

Therefore, $\sum \Gamma_{ik}^k = \sum \gamma_{i\sigma}^\sigma = 0$. In particular, $\sum \tau_\sigma^\sigma = 0$, namely, by (1.8), $\nabla\theta = 0$, which concludes the proof.

The vanishing of the right-hand side of (2.7) implies

$$(2.10) \quad \frac{n+2}{2} \Gamma_{ij}^\ell = \sum H_\sigma^{\ell k} h_{jk}^\rho \gamma_{i\rho}^\sigma.$$

In other words, we have

Proposition 2.3. Assume that the immersion $f : M^n \rightarrow \mathbf{R}^{n+p}$, $p = n(n+1)/2$, is nondegenerate and N is the affine normal field. Then the connection ∇ is determined by the normal connection ∇^\perp relative to an affine normalized basis by the formula (2.10).

Remark 2.4. The apolarity condition for $n = 2$ when $h^\sigma = v^\sigma$ is

$$C_{ij1}^{(11)} + C_{ij2}^{(12)} = 0 \quad \text{and} \quad C_{ij1}^{(21)} + C_{ij2}^{(22)} = 0.$$

This is seen to be the same as the condition considered in [2]; refer to Theorem 1.2 of the paper. The affine normal space also coincides with that given in [2]. The affine volume form θ is written as follows:

$$\theta(X_1, \dots, X_n) = \epsilon \left| 2^{-n} \omega(X_1, \dots, X_n, D_{X_1} X_1, \dots, D_{X_i} X_j, \dots, D_{X_n} X_n) \right|^{1/(n+2)}$$

where $1 \leq i \leq j \leq n$ and $\epsilon = \pm 1$ depending on the orientation of (X_1, \dots, X_n) .

Remark 2.5. Assume $h^\sigma = v^\sigma$. Then from (2.8) follows the identity

$$(2.11) \quad \gamma_{i\rho}^\sigma = \sum H_\rho^{jk} C_{ijk}^\sigma + 2 \sum H_\rho^{jk} h_{k\ell}^\sigma \Gamma_{ij}^\ell,$$

which, composed again with α^{-1} , yields a map

$$\chi : \mathcal{N} \longrightarrow \mathcal{T}.$$

The component-wise expression of χ is seen to be

$$(2.4) \quad \chi(\xi_\sigma) = \chi_\sigma^\ell X_\ell \quad \text{where} \quad \chi_\sigma^\ell = \sum H_\rho^{\ell k} C_{ijk}^\rho H_\sigma^{ij}.$$

Now compute the χ for the basis $\{Y_j, \eta_\rho\}$, denoted by $\bar{\chi}$; then, we see

$$(2.5) \quad \bar{\chi}_\rho^\ell = \sum A_i^\ell b_\rho^\alpha \{ \chi_\alpha^i + \sum L_{\alpha j}^{\beta i} B_\beta^\sigma g_\sigma^j \},$$

where

$$L_{\alpha j}^{\beta i} = \frac{n+3}{2} \delta_\alpha^\beta \delta_j^i + \sum h_{jk}^\beta H_\alpha^{ik}$$

is an endomorphism L of the space $\text{End}(\mathcal{N}, \mathcal{T})$. Remark that L is invertible; in fact, its inverse is given by

$$\frac{2}{n+3} (\delta_\beta^\alpha \delta_i^j - \frac{1}{n+2} \sum h_{ik}^\alpha H_\beta^{kj})$$

as is seen by the formulas in Lemma A.1. Hence we have proved the following proposition.

Proposition 2.1. Assume that the immersion is nondegenerate. Then there exists a basis $\{X_i, \xi_\sigma\}$ such that $\chi = 0$. The transversal field N is uniquely determined.

The last statement follows also from (2.5); that is, given two such bases, the equation (2.5) implies $g_\rho^\ell = 0$, namely, $Z_\rho = 0$. Moreover, for such bases we see that the formula (2.3) reduces to

$$(2.6) \quad \bar{C}_{pqr}^\rho = \sum a_p^i a_q^j a_r^k B_\sigma^\rho C_{ijk}^\sigma.$$

The condition $\chi = 0$ is equivalent to

$$(2.7) \quad \sum H_\sigma^{\ell k} C_{ijk}^\sigma = 0 \quad \text{for any } i, j, \ell,$$

which is called the *apolarity condition*.

When the apolarity condition is satisfied, we call the transversal field N the *affine normal field*. The induced affine connection ∇ is uniquely defined in view of (1.10). In order to uniquely determine the volume form θ_ξ , we need to know how to choose ξ ; our choice is to follow the convention of choosing a unimodular basis satisfying the apolarity condition and $h^\sigma = v^\sigma$ and call such a basis an *affine normalized basis*. See Remark 2.4. For such a choice of ξ , we call θ_ξ the *affine volume form*. It is globally defined as far as M is oriented and the orientation is fixed.

Proposition 2.2. Assume that the immersion $f : M^n \rightarrow \mathbf{R}^{n+p}$, $p = n(n+1)/2$, is nondegenerate and let N be the affine normal field. Then the immersion is equiaffine, that is, $\nabla\theta = 0$, where ∇ is the induced affine connection and θ is the affine volume form.

Notice that $\det b = (\det a)^{n+1}$ in this case.

§2. Cubic forms

We define the cubic form $C = (C^\sigma)$ by

$$(2.1) \quad C^\sigma(X, Y, Z) = X(h^\sigma(Y, Z)) - h^\sigma(\nabla_X Y, Z) - h^\sigma(Y, \nabla_X Z) + \sum h^\rho(Y, Z)\tau_\rho^\sigma(X).$$

By the equation of Codazzi (1.4), this form is symmetric. Let $\{Y_j, \eta_\rho\}$ be a new basis defined by (1.14). Then, using (1.9)-(1.12), we have

$$(2.2) \quad \begin{aligned} \bar{C}^\sigma(X, Y, Z) = \sum C^\rho(X, Y, Z)B_\rho^\sigma + \sum \{ \bar{h}^\alpha(X, Y)h^\rho(Z_\alpha, Z) \\ + \bar{h}^\alpha(X, Z)h^\rho(Y, Z_\alpha) + \bar{h}^\alpha(Y, Z)h^\rho(X, Z_\alpha) \} B_\rho^\sigma. \end{aligned}$$

Set

$$Z_\rho = \sum g_\rho^\ell X_\ell$$

and define

$$C_{ijk}^\sigma = h^\sigma(X_i, X_j, X_k), \quad \bar{C}_{pqr}^\rho = \bar{h}^\rho(Y_p, Y_q, Y_r).$$

Then (2.2) implies

$$(2.3) \quad \bar{C}_{pqr}^\rho = \sum a_p^i a_q^j a_r^k B_\sigma^\rho \{ C_{ijk}^\sigma + J_{ij,k}^\sigma + J_{jk,i}^\sigma + J_{ki,j}^\sigma \},$$

where

$$J_{ij,k}^\sigma = \sum h_{ij}^\beta B_\beta^\alpha g_\alpha^\ell h_{k\ell}^\sigma.$$

Let us here summarize the meaning of the bilinear form α and the cubic form C : For a choice of a basis $\{X_i, \xi_\sigma\}$, we associate a vector space \mathcal{T} generated by the local vector fields X_1, \dots, X_n and a vector space \mathcal{N} generated by ξ_σ . The symmetric bilinear form α defines a map

$$\alpha : S^2(\mathcal{T}) \longrightarrow \mathcal{N}$$

at each point and the cubic form defines a map

$$C : S^3(\mathcal{T}) \longrightarrow \mathcal{N}.$$

The nondegeneracy assumption implies that the mapping α is bijective. If we identify α with the set $\{h_{ij}^\sigma\}$, then its inverse that we denote by $\{H_\sigma^{ij}\}$ is given by the formula (3) in Appendix. Now we can consider the composition

$$\psi = \alpha^{-1} \circ C : S^3(\mathcal{T}) \longrightarrow S^2(\mathcal{T}).$$

Then the contraction of this mapping defines a map

$$\text{tr } \psi : S^2(\mathcal{T}) \longrightarrow \mathcal{T},$$

where B denotes the inverse of the matrix b . Also, by comparing the expressions

$$D_X \eta_\rho = -\bar{S}_\rho(X) + \sum \bar{\tau}_\rho^\sigma(X) \eta_\sigma$$

and

$$D_X \eta_\rho = \sum X(b_\rho^\sigma) \xi_\sigma + \sum b_\rho^\sigma (-S_\sigma X + \sum \tau_\sigma(X)^\alpha \xi_\alpha) + \nabla_X Z_\rho + \sum h^\sigma(X, Z_\rho) \xi_\sigma,$$

we get

$$(1.11) \quad \bar{S}_\rho(X) = \sum b_\rho^\sigma S_\sigma(X) - \sum \bar{\tau}_\rho^\sigma(X) Z_\sigma + \nabla_X Z_\rho,$$

$$(1.12) \quad \bar{\tau}_\rho^\sigma(X) = \sum X(b_\rho^\alpha) B_\alpha^\sigma + \sum b_\rho^\alpha \tau_\alpha^\beta(X) B_\beta^\sigma + \sum h^\alpha(X, Z_\rho) B_\alpha^\sigma.$$

In particular,

$$(1.13) \quad \sum \bar{\tau}_\rho^\rho(X) = \sum \tau_\rho^\rho(X) + X(\log |\det b|) + \sum h^\alpha(X, Z_\rho) B_\alpha^\rho.$$

Now, in addition to ξ , we choose a basis $\{X_i\}$ of tangent vector fields so that the set $\{X_i, \xi_\sigma\} = \{X_1, \dots, X_n, \xi_{(11)}, \dots, \xi_{(nn)}\}$ has the positive orientation. We call $\{X_i, \xi_\sigma\}$ a *unimodular basis* provided that $\theta_\xi(X_1, \dots, X_n) = 1$. We are interested in the components

$$h_{ij}^\sigma = h^\sigma(X_i, X_j).$$

Let us consider a change of basis from $\{X_i, \xi_\sigma\}$ to $\{Y_j, \eta_\rho\}$ given by

$$(1.14) \quad Y_j = \sum a_j^i X_i, \quad \eta_\rho = \sum b_\rho^\sigma \xi_\sigma + Z_\rho.$$

Then, because of (1.9), we have

$$(1.15) \quad \bar{h}_{ij}^\sigma = \bar{h}^\sigma(Y_i, Y_j) = \sum a_i^k h_{k\ell}^\rho a_j^\ell B_\rho^\sigma.$$

Because of nondegeneracy, we can choose a unimodular basis $\{X_i, \xi_\sigma\}$ so that h^σ takes a special form. We define the matrices v^σ by

$$v^{(ij)} = E_{ij} + E_{ji},$$

where E_{ij} is the $n \times n$ -matrix with 1 in (ij) -th component and 0's in the others. Then we can let $h^\sigma = v^\sigma$ for some unimodular basis. Then (1.15) shows that $\{Y_j, \eta_\rho\}$ also has this property if and only if the matrices a and b satisfy the relations

$$(1.16) \quad \det a \det b = 1, \quad b = a \otimes a;$$

the latter implies that b is the symmetric product of a , i.e.,

$$b_{(rr)}^{(pq)} = a_r^p a_r^q \quad \text{and} \quad b_{(rs)}^{(pq)} = a_r^p a_s^q + a_r^q a_s^p \quad \text{for} \quad r \neq s.$$

Refer to [4].

We introduce

Definition 1.1. The immersion f is said to be *non-degenerate* if the map α is surjective.

In this paper we always assume the non-degeneracy of f . In this case we have $p = n(n+1)/2$.

To see the decompositions (1.1) and (1.2) more closely, we choose a basis of sections of N , i.e., a set of independent transversal vector fields $\xi = \{\xi_\sigma\}$, which we label by using the set of elements $\{\sigma = (ij); 1 \leq i \leq j \leq n\}$. In the following, we shall use the index (ij) also for $i > j$ that we identify with (ji) . Then the equations (1.1) and (1.2) are rewritten as

$$\begin{aligned} D_X f_* Y &= f_*(\nabla_X Y) + \sum_\sigma h^\sigma(X, Y) \xi_\sigma, \\ D_X \xi_\sigma &= -f_*(S_\sigma X) + \sum_\rho \tau_\sigma^\rho(X) \xi_\rho, \end{aligned}$$

where h^σ are symmetric bilinear forms, τ_σ^ρ are 1-forms, and S_σ is the short form of S_{ξ_σ} . To each choice of such a basis ξ , we associate a volume form θ_ξ defined by

$$(1.7) \quad \theta_\xi(X_1, \dots, X_n) = \omega(X_1, \dots, X_n, \xi_{(11)}, \dots, \xi_\sigma, \dots, \xi_{(nn)});$$

Here $\{\xi_\sigma, \sigma = (ij), i \leq j\}$ is arranged by the lexicographic order. We easily see that

$$(1.8) \quad \nabla_X \theta_\xi = \sum \tau_\sigma^\sigma(X) \theta_\xi.$$

Here and in the following we use the summation convention: when an expression contains terms with repeated indices, one an upper index and the other a lower index, a summation over these indices is automatically intended, although we put summation symbols just for making clear where the summation is taken.

We next examine the dependence of these quantities on the choice of transversal field and basis. Let $\{\eta_\rho\}$ be another local basis, which is expressed as

$$\eta_\rho = \sum b_\rho^\sigma \xi_\sigma + Z_\rho,$$

where $b = (b_\rho^\sigma)$ is a nonsingular matrix of size p and Z_ρ is a local tangent vector field. We denote by \bar{h}^σ , $\bar{\nabla}$, \bar{S}_σ , and $\bar{\tau}_\rho^\sigma$ the corresponding quantities relative to η_ρ . Comparing the two expressions

$$D_X Y = \nabla_X Y + \sum h^\sigma(X, Y) \xi_\sigma$$

and

$$D_X Y = \bar{\nabla}_X Y + \sum \bar{h}^\rho(X, Y) \eta_\rho = \bar{\nabla}_X Y + \sum \bar{h}^\rho(X, Y) Z_\rho + \sum \bar{h}^\rho(X, Y) b_\rho^\sigma \xi_\sigma,$$

we have

$$(1.9) \quad \bar{h}^\rho(X, Y) = \sum h^\sigma(X, Y) B_\sigma^\rho,$$

$$(1.10) \quad \bar{\nabla}_X Y = \nabla_X Y - \sum h^\sigma(X, Y) B_\sigma^\rho Z_\rho,$$

Decruyenaere et al. [1] gave a characterization of affine minimal surfaces in \mathbf{R}^5 in terms of affine shape operator. Namely, a nondegenerate surface in \mathbf{R}^5 is minimal if and only if the trace of S_ξ vanishes identically for all ξ . The second aim of this paper is showing that his characterization holds also for general dimension.

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§1. The basic equations

Let M be an n -dimensional manifold and $f : M \rightarrow \mathbf{R}^{n+p}$ an immersion. We denote by ω the usual parallel volume form on \mathbf{R}^{n+p} and by D the standard flat affine connection of \mathbf{R}^{n+p} . The tangent space of M at x is denoted by $T_x M$. We choose around each point a local field N of transversal spaces of dimension p , denoted by $x \mapsto N_x \subset T_{f(x)}(\mathbf{R}^{n+p})$ such that it holds

$$T_{f(x)}(\mathbf{R}^{n+p}) = f_*(T_x M) + N_x.$$

For all vector fields X and Y , we have a decomposition

$$(1.1) \quad D_X f_* Y = f_* \nabla_X Y + \alpha(X, Y),$$

where $\nabla_X Y \in T_x(M)$ and $\alpha(X, Y) \in N_x$ at each point x . It is straightforward to see that ∇ is a torsion-free affine connection on M and α is a symmetric bilinear form with values in N . We call ∇ the induced affine connection and α the affine fundamental form. For a vector field ξ with value in N , we write

$$(1.2) \quad D_X \xi = -f_* S_\xi X + \nabla_X^\perp \xi,$$

where $S_\xi X \in T_x(M)$ and $\nabla_X^\perp \xi \in N_x$ at each point x . Then S_ξ is an endomorphism of $T_x M$ and ∇^\perp is a connection on the space N . We call S_ξ the affine shape operator and ∇^\perp the affine normal connection. Remark that all these notions depend on the choice of N .

In the following we assume that M is oriented; it causes no harm since the considerations are going to be local.

We define the covariant derivation of α and S by the formulas

$$\begin{aligned} (\nabla_X \alpha)(Y, Z) &= \nabla_X^\perp(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z), \\ (\nabla_X S)_\xi(Y) &= \nabla_X(S_\xi Y) - S_\xi(\nabla_X Y) - S_{\nabla_X^\perp \xi} Y. \end{aligned}$$

Then, we have the following equations called the equation of Gauss (1.3), the equations of Codazzi (1.4) and (1.5), and the equation of Ricci (1.6):

$$(1.3) \quad R(X, Y)Z = S_{\alpha(Y, Z)}X - S_{\alpha(X, Z)}Y,$$

$$(1.4) \quad (\nabla_X \alpha)(Y, Z) = (\nabla_Y \alpha)(X, Z),$$

$$(1.5) \quad (\nabla_X S)_\xi(Y) = (\nabla_Y S)_\xi(X),$$

$$(1.6) \quad \alpha(X, S_\xi Y) - \alpha(S_\xi X, Y) = R^\perp(X, Y)\xi.$$

Introduction

One purpose of affine differential geometry is to study the properties of submanifolds M^n of \mathbf{R}^m that are invariant under the group of all unimodular affine transformations. When M^n is a hypersurface, the affine theory is now rather well-understood; a principal idea is introducing an equiaffine structure induced from the ambient space to the hypersurface. An equiaffine structure is by definition a geometric structure consisting of a torsion-free affine connection and a parallel volume form. As for surfaces in \mathbf{R}^4 , Nomizu and Vrancken [5] gave a new formulation showing that there is also a canonical way of introducing an equiaffine structure. This is also the case of surfaces in \mathbf{R}^5 as shown in [2]. The first aim of this paper is generalizing this study to the case of n -dimensional submanifolds in \mathbf{R}^{n+p} , where $p = n(n+1)/2$, by transferring the projective theory of n -dimensional submanifolds in the projective space \mathbf{P}^{n+p} to the affine case, which was treated in [6]. Let M be an n -dimensional manifold and $f : M \rightarrow \mathbf{R}^{n+p}$ an immersion. The immersion f is said to be nondegenerate if the first osculating space at any point is maximal. We show that, for a nondegenerate immersion, there is a canonical way of determining a field of transversal spaces to f , thus inducing uniquely an affine connection and a volume form, both of which define an equiaffine structure.

More precisely, let ω denote the usual parallel volume form of the ambient space \mathbf{R}^m , D the standard flat affine connection, and N the field of transversal spaces. Then, for any vector fields on M , we have a decomposition

$$D_X f_* Y = f_* \nabla_X Y + \alpha(X, Y),$$

where $\nabla_X Y$ is tangent to M and $\alpha(X, Y)$ belongs to N ; ∇ is the induced connection and α is called the affine fundamental form. We will show that there is a canonical way of choosing a set of vector fields with values in N which spans N_x at each point x , say ξ_1, \dots, ξ_p . Then the n -form θ defined by

$$\theta(X_1, \dots, X_n) = \omega(X_1, \dots, X_n, \xi_1, \dots, \xi_p)$$

is the induced affine volume form. For describing the geometry of the immersion f , we need to know the motion of these vector fields; as usual, we have a decomposition

$$D_X \xi = -f_* S_\xi X + \nabla_X^\perp \xi$$

for any local section ξ of N , where $S_\xi X$ is tangent to M and $\nabla_X^\perp \xi$ is belonging to N . Thus, we get an endomorphism S_ξ of the tangent space and the connection of the field N . S_ξ is called the affine shape operator.

Now, relative to the volume form θ , we can define a volume functional for immersed nondegenerate submanifolds and get the notion of critical submanifolds relative to this functional; such submanifolds are said to be affine minimal in this paper. Recently,