THE CLASSIFICATION OF PROJECTIVELY HOMOGENEOUS SURFACES II

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This paper classifies locally projectively homogeneous surfaces in $\mathbb{P}^3$ that are either ruled or degenerate, and thus complements the classification in [6]. A surface M is said to be locally projectively homogeneous in $\mathbb{P}^3$ if for all points p and q of M, there exists a neighborhood U_p of p in M, and a projective transformation A of $\mathbb{P}^3$, such that $A(p) = q$ and $A(U_p) \subset M$. If $U_p = M$ for all p, then M is said to be homogeneous. Projectively homogeneous surfaces which are non-ruled and nondegenerate are classified in [6].

Theorem. *Let M^2 be a locally projectively homogeneous surface in $\mathbb{P}^3$ which is non-ruled and nondegenerate. Then M^2 is is projectively equivalent to*

$\|0\|$ *nondegenerate quadratic surfaces*;

$\|1\|$ $(Y - X^2)^3 = k(Z + 2X^3 - 3XY)^2, \quad k \neq -\frac{1}{4}$

$\|2\|$ $Z = Y^2 + \epsilon X^k, \quad k \neq 0, 1, 2;$

$\|3\|$ $Z = Y^2 + \epsilon e^X;$

$\|4\|$ $Z = Y^2 + \epsilon X \log X;$

$\|5\|$ $Y^2 + \epsilon Z^2 = X^{1-2/\lambda}$ *where* $\lambda > 2;$

$\|7\|$ $Y^2 - \epsilon Z^2 = X^{1-2/\lambda}$ *where* $0 < \lambda < 2;$

$\|6\|$ $X = \exp(\epsilon Y^2 - Z);$

$\|8\|$ $Y^2 - \epsilon Z^2 = e^X;$

$\|9\|$ $(Y^2 - \epsilon Z^2) = (X^2 + 1)\exp(-\frac{4}{\lambda}\arctan X);$

$\|10\|$ $Z = X^k Y^\ell;$

$\|11\|$ $Z = \log X + k \log Y;$

$\|12\|$ $Z = (X^2 + Y^2)^{k/2} \exp\big(\ell \arctan \frac{X}{Y}\big);$

$\|13\|$ $Z = \frac{1}{2}\log(X^2 + Y^2) + \ell \arctan \frac{X}{Y};$

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||14|| $$\arctan Z = \frac{k}{2} \log \frac{X^2+Y^2}{1+Z^2} + \ell \arctan \frac{X}{Y}.$$

Similar questions, dealing with different groups such as the group of all equiaffine transformations, centroaffine transformations and general affine transformations, have been treated amongst others in [5], [8], [1], [4] and [2].

Main Theorem. *Let M^2 be a locally projectively homogeneous surface in $\mathbb{P}^3$ which is ruled or degenerate. Then either M^2 is a plane or a conic, or M^2 is projectively equivalent to one of the following nondegenerate ruled surfaces*

||15|| $\arctan Z = k \arctan(Y/X), \qquad k > 1;$

||16|| $Z = \arctan(Y/X);$

||17|| $\arctan Z = k \log(Y/X), \qquad k > 0;$

||18|| $Z = Ye^X;$

||19|| $Z = (Y/X)^k, \qquad 0 < k < 1;$

||20|| $\log \dfrac{X^2+Y^2}{1+Z^2} = k\left(\arctan(Y/X) - \arctan Z\right), \qquad k > 0;$

||21|| $Z = XY + \frac{1}{3}X^3;$

||22|| $Z = XY + X \log X;$

||23|| $Z = XY + (1+X^2)\arctan X;$

||24|| $Z = XY + X^k;$

||25|| $Z = XY + X^2 \log X;$

||26|| $Z = XY + e^X;$

||27|| $\log \dfrac{X+YZ}{1+Y^2} = k \arctan Y,$

or to a cylinder (or rather a cone) over one of the following planar curves

||28|| $Y = X^m$ *where* $m \neq \pm 1;$

||29|| $X = e^{\lambda s}\cos \mu s \quad Y = e^{\lambda s} \sin \mu s \quad \mu \neq 0,$ *called logarithmic spiral*;

||30|| $Y = Xe^X,$

or to the tangent-developable surface over the cubic curve

||31|| $$\alpha(u) = [1, u, u^2, u^3].$$

We remark that a degenerate surface is automatically ruled. The methods used in order to obtain the Main Theorem are based on the construction of a suitable frame. Unfortunately, in the cases considered here, there never exists a unique choice of frame. Therefore, a careful analysis on how the fundamental equations change under a change of structure is necessary in order to construct data which remain invariant under such a change. Since M is locally projectively homogeneous, it then follows that these data have to be constant.

1. Fundamental Equations

We recall the notion of projective structure and fix notations used in this paper. Two torsion-free affine connections D and $\overline{D}$ are said to be projectively equivalent if there is a 1-form ϕ such that

$$\overline{D}_X Y = D_X Y + \phi(X)Y + \phi(Y)X \tag{1.1}$$

for any vector fields X and Y. A projective structure on a manifold is a union of locally defined torsion-free affine connections patched together by this equivalence relation. We will tacitly assume that each affine connection is torsion-free and Ricci symmetric, i.e. admits a parallel volume element. When two such equiaffine connections are projectively equivalent, it follows that $d\phi = 0$ in (1.1).

Let $\widetilde{M}$ be an $(n+1)$-dimensional differentiable manifold with a projective structure P and f an immersion of a manifold M into $\widetilde{M}$. We fix for the moment a torsion-free affine connection D in the structure P defined on an open set U and consider M immersed in U. Let ξ be an arbitrary vector field transversal to M. For any vector fields X, Y on M we write

$$D_X Y = \nabla_X Y + h(X,Y)\xi, \tag{1.2}$$

$$D_X \xi = -SX + \tau(X)\xi, \tag{1.3}$$

thus defining an affine connection ∇, a symmetric tensor h of type $(0,2)$, a tensor S of type $(1,1)$, called the shape operator, and a 1-form τ on M. Let R^D and R denote the curvature tensors of D and ∇ respectively and let Ric^D and Ric be the Ricci tensors. The normalized Ricci tensors γ^D and γ are defined by

$$\gamma^D(U,V) = \frac{1}{n}\mathrm{Ric}^D(U,V) = \frac{1}{n}\mathrm{tr}\{W \longrightarrow R^D(W,U)V\}$$
$$\gamma(X,Y) = \frac{1}{(n-1)}\mathrm{Ric}(X,Y) = \frac{1}{(n-1)}\mathrm{tr}\{Z \longrightarrow R(Z,X)Y\}.$$

Assuming that D is projectively flat, we have the following fundamental equations:

(1.4) -Gauss- $R(X,Y)Z = \gamma^D(Y,Z)X - \gamma^D(X,Z)Y + h(Y,Z)SX - h(X,Z)SY$

(1.5) -Codazzi- $(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z)$

(1.6) -Codazzi- $(\nabla_Y S)(X) - \tau(Y)SX - \gamma^D(Y,\xi)X = (\nabla_X S)(Y) - \tau(X)SY - \gamma^D(X,\xi)Y$

(1.7) -Ricci- $h(SX,Y) + (\nabla_X \tau)Y = h(X,SY) + (\nabla_Y \tau)X.$

From the first identity we have the relation

$$\gamma(X,Y) = \gamma^D(X,Y) + \frac{1}{(n-1)}\{\mathrm{tr}S \cdot h(X,Y) - h(SX,Y)\}. \tag{1.8}$$

Referring to these identities we define the following invariants:

(1.9) -cubic form- $C(X,Y,Z) = (\nabla_X h)(Y,Z) + \tau(X)h(Y,Z)$

(1.10) -shape form- $S(X,Y) = h(SX,Y) - \gamma^D(X,Y)$

Definitions and formulas above all depend on the choice of a transversal vector field ξ and a connection D; we summarize this dependence in the next two sets of formulas:
By a change of ξ to $\xi' = (U + \xi)/\lambda$, several quantities will change by the following rules:

$$\begin{aligned}
&\nabla'_X Y = \nabla_X Y - h(X,Y)U,\\
&h'(X,Y) = \lambda h(X,Y),\\
&\tau'(X) = \tau(X) + \eta(X) - X(\log\lambda),\\
&\lambda S'X = SX - \nabla_X U + (\tau(X) + \eta(X))U,\\
&C'(X,Y,Z) = \lambda C(X,Y,Z) + \lambda \mathrm{Cyc}_{X,Y,Z}(h(X,Y)\eta(Z)),\\
&\mathcal{S}'(X,Y) = \mathcal{S}(X,Y) + C(U,X,Y) - (\nabla_X \eta)(Y) + \eta(X)\eta(Y),
\end{aligned} \tag{1.11}$$

where $\eta(\cdot) = h(U,\cdot)$ and $\mathrm{Cyc}_{X,Y,Z}$ means cyclic summation over X, Y and Z.
By a change of the connection D to a connection $\overline{D}$ related by (1.1) we have the formulas:

$$\begin{aligned}
&\overline{\nabla}_X Y = \nabla_X Y + \phi(X)Y + \phi(Y)X,\\
&\overline{h}(X,Y) = h(X,Y),\\
&\overline{\tau}(X) = \tau(X) + \phi(X),\\
&\overline{S} = S - \phi(\xi)I,\\
&\gamma^{\overline{D}}(X,Y) = \gamma^{D}(X,Y) - \{(D_X\phi)(Y) - \phi(X)\phi(Y)\},\\
&\overline{\mathcal{S}}(X,Y) = \mathcal{S}(X,Y) - 2\phi(\xi)h(X,Y) + \{(\nabla_X\phi)(Y) - \phi(X)\phi(Y)\},\\
&\overline{C}(X,Y,Z) = C(X,Y,Z) - \mathrm{Cyc}_{X,Y,Z}\,\{\phi(X)h(Y,Z)\}.
\end{aligned} \tag{1.12}$$

Part I. Nondegenerate ruled projectively homogeneous surfaces

In Part I, we assume that the fundamental form h is nondegenerate and classify nondegenerate ruled surfaces that are locally projectively homogeneous. In Section 2 we recall from [6] how to attach a frame to the immersion for general dimension that has a special form. In Section 3, we further normalize the choice of frames suitable for nondegenerate ruled surfaces assuming projective homogeneity. Then in Section 4 we give the concrete forms of the immersions corresponding to such distinguished frames, hereby completing the classification for the present part.

2. Special Choice of (D,ω,ξ)

The triple (D,ω,ξ) consists of a connection D from a given projective structure that is throughout assumed to be projectively flat, a parallel volume element ω, i.e.

$$D\omega = 0, \tag{N.1}$$

and a transversal vector field ξ along f. As proved in [6], there always exists a triple (D,ω,ξ) satisfying the following three conditions when h is nondegenerate:

$$\tau = 0, \tag{N.2}$$

$$\mathrm{tr}(K_X) = 0, \tag{N.3}$$

$$\mathrm{tr}_h\,\mathcal{S} = 0 \tag{N.4}$$

Here, K_X is the tensor defined by

$$h(K_X Y, Z) = -\tfrac{1}{2} C(X, Y, Z),$$

and tr_h means the trace relative to h. Such triples, say, (D, ω, ξ) and $(\overline{D}, \overline{\omega}, \overline{\xi})$ are related as follows.

$$\begin{aligned} \overline{D}_X Y &= D_X Y + \phi(X) Y + \phi(Y) X, \\ \overline{\omega} &= \sigma\omega, \\ \overline{\xi} &= \frac{1}{\lambda}(U + \xi), \end{aligned} \tag{2.1}$$

where U is a tangent vector field, σ is a non-vanishing scalar function, and $\phi = \frac{1}{n+2} d \log \sigma$ is a 1-form; they need to satisfy the identities

$$h(U, X) = \phi(X), \qquad d \log \lambda = 2\phi|_M, \qquad \phi(U + 2\xi) = 0. \tag{2.2}$$

This says that such triples are determined along the immersion up to scalar functions σ. Certain combinations of formulas in (1.11) and (1.12) show

$$\overline{\mathcal{S}}(X, Y) = \mathcal{S}(X, Y) + C(U, X, Y), \tag{2.3}$$

$$\lambda \gamma^{\overline{D}}(X, \overline{\xi}) = \gamma^D(X, \xi) - \mathcal{S}(U, X) - \frac{1}{2} C(U, U, X). \tag{2.4}$$

For more details, we refer to Section 1 of [6].

Remark. Lemma 2.5 of the paper [6] claims that we can choose a triple satisfying (N.1) to (N.4) and one additional condition (requiring the Fubini-Pick invariant to be a nonzero constant) uniquely as long as the Fubini-Pick invariant does not vanish. However, this claim is too strong because we can argue the behavior of the connection D and the volume form ω only along the immersion. To be precise, we can claim as follows: If the Fubini-Pick invariant does not vanish, we can choose a triple so that the induced objects ∇, h, S, $\mathcal{S}$ and the tensor γ^D restricted to the immersion are unique, and moreover, the 1-form $\gamma^D(\cdot, \xi)$ is also unique.

3. Distinguished frames for ruled surfaces

We now suppose $n = 2$ and consider immersed surfaces in $\widetilde{M} = \mathbb{P}^3$ with the canonical projectively flat structure D. We assume that h is nondegenerate. The Fubini-Pick invariant J is by definition

$$J = \tfrac{1}{2} h(K, K). \tag{3.1}$$

Since the classification of locally projectively homogeneous surfaces for the case $J \neq 0$ was done in [6], we consider the case $J = 0$. Any surface with $J = 0$ is known to be ruled (see [7], p.90), while the surface with $K = 0$, namely with $C = 0$, is a quadratic surface. Hence we can assume that $C \neq 0$ and h is indefinite.

In this section we choose a special frame $\{X_1, X_2, \xi\}$ and a scalar function λ so that several geometric quantities have simple forms. We look for frames with the property that $\omega(X_1, X_2, \xi) = 1$; such a frame is said to be unimodular.

We first choose $\{X_1, X_2\}$ by requiring

$$h(X_1, X_1) = 0, \quad h(X_1, X_2) = 1, \quad h(X_2, X_2) = 0. \tag{3.2}$$

Then the condition (N.3) is equivalent to $C(X_1, X_2, \cdot) = 0$ and the Fubini-Pick invariant is $J = C(X_1, X_1, X_1)C(X_2, X_2, X_2)/8$. Hence we can assume

$$C(X_2, \cdot\,, \cdot\,) = 0 \tag{3.3}$$

on a certain open set; we put

$$C(X_1, X_1, X_1) = -2c, \tag{3.4}$$

where $c \neq 0$ by the assumption $C \neq 0$.

Lemma 3.1. *For an appropriate frame $\{X_1, X_2\}$, the connection ∇ is written as follows:*

$$\nabla_{X_1} X_1 = aX_1 + cX_2, \quad \nabla_{X_1} X_2 = -aX_2, \quad \nabla_{X_2} X_1 = \nabla_{X_2} X_2 = 0. \tag{3.5}$$

Proof. Because of (3.3), the differentiation of (3.2) implies

$$h(\nabla_{X_2} X_2, X_2) = h(\nabla_{X_2} X_1, X_2) + h(X_1, \nabla_{X_2} X_2) = h(\nabla_{X_2} X_1, X_1) = 0.$$

Hence, we can put $\nabla_{X_2} X_2 = bX_2$ and $\nabla_{X_2} X_1 = -bX_1$ for some function b. Let now $Y_1 = \mu X_1$ and $Y_2 = \mu^{-1} X_2$ for some unknown function μ. Then

$$\nabla_{Y_2} Y_1 = (X_2(\log \mu) - b)X_1 \quad \text{and} \quad \nabla_{Y_2} Y_2 = \mu^{-2}(b - X_2(\log \mu))X_2.$$

Hence, choosing μ so that $X_2(\log \mu) = b$, we see that $\{Y_1, Y_2\}$ satisfies the last condition. By renaming, we assume that $\{X_1, X_2\}$ already satisfies the condition. Then the differentiation of (3.2) relative to X_1 implies

$$h(\nabla_{X_1} X_1, X_1) = c, \quad h(\nabla_{X_1} X_1, X_2) + h(X_1, \nabla_{X_1} X_2) = 0, \quad h(\nabla_{X_1} X_2, X_2) = 0.$$

Hence, we have proved the lemma. □

Lemma 3.2. *There exists a frame such that $c = 1$.*

Proof. Fix $\{X_1, X_2\}$ satisfying the condition above and choose ξ so that $\{X_1, X_2, \xi\}$ is a unimodular frame; then define a new unimodular frame $\{Y_1, Y_2, \overline{\xi}\}$ by the equations

$$Y_1 = \mu X_1, \quad Y_2 = \nu X_2, \quad \overline{\xi} = \frac{1}{\lambda}(\xi + U) \quad \text{where} \quad \lambda\mu\nu = 1.$$

Then, relative to the change of the connection D to $\overline{D}$, we have by (1.11) and (1.12)

$$\begin{aligned}\overline{\nabla}_{Y_2} Y_1 &= \nabla_{Y_2} Y_1 + \phi(Y_1)Y_2 + \phi(Y_2)Y_1 - h(Y_1, Y_2)U \\ &= \nu X_2(\mu)X_1 + \mu\nu\nabla_{X_2} X_1 + \mu\nu(\phi(X_1)X_2 + \phi(X_2)X_1 - U)\end{aligned}$$

the last equality follows from (3.5) and (2.2). Similarly,

$$\begin{aligned}\overline{\nabla}_{Y_2} Y_2 &= \nabla_{Y_2} Y_2 + 2\phi(Y_2)Y_2 - h(Y_2, Y_2)U \\ &= \nu X_2(\nu)X_2 + 2\nu^2\phi(X_2)X_2 \\ &= -\nu^2 X_2(\log\mu)X_2\end{aligned}$$

because of (2.2) and $\lambda\mu\nu = 1$. On the other hand, the cubic form $\overline{C}$ for $\{Y_1, Y_2\}$ has the property (3.3) and $\overline{C}(Y_1, Y_1, Y_1) = -2\lambda\mu^3 c$. Hence, $\lambda = 1/c$, $\mu = 1$, and $\nu = c$ suffice to get a required frame. □

The computation above shows that the frames with the properties above are determined up to a scalar μ satisfying $X_2(\mu) = 0$:

$$Y_1 = \mu X_1, \quad Y_2 = \mu^2 X_2, \quad \overline{\xi} = \mu^{-3}\xi. \tag{3.6}$$

Lemma 3.3. *Relative to the frame chosen above, we have*

$$\gamma = -a_2 h, \quad tr\, S = a_2, \quad \mathcal{S} = 0, \quad \gamma^D(X, Y) = h(SX, Y), \tag{3.7}$$

where $a_2 = X_2(a)$.

Proof. It is easy to see that

$$R(X_1, X_2)X_1 = -a_2 X_1 \quad \text{and} \quad R(X_1, X_2)X_2 = a_2 X_2,$$

from which follow $\gamma(X_1, X_1) = \gamma(X_2, X_2) = 0$ and $\gamma(X_1, X_2) = -a_2$; hence $\gamma = -a_2 h$. Then $\mathcal{S}(X, Y) = (\mathrm{tr}\, S + a_2)h(X, Y)$ by (1.8) and (1.10). Now refer to (N.4): we see $\mathrm{tr}\, S = -a_2$ and $\mathcal{S} = 0$. The identity (1.10) implies the last claim.

We put

$$SX_1 = pX_1 + qX_2, \qquad SX_2 = rX_1 + sX_2. \tag{3.8}$$

Because of (1.7), we have $s = p$ and, because of (3.7),

$$2p + a_2 = 0. \tag{3.9}$$

Remark that

$$\gamma^D(X_1, X_1) = q, \quad \gamma^D(X_1, X_2) = p, \quad \gamma^D(X_2, X_2) = r. \tag{3.10}$$

We compute the left-hand side of the Ricci identity (1.6):

$$\mathrm{LHS} = (r_1 - p_2 + 2ar)X_1 + (p_1 - q_2 + r)X_2,$$

where $r_1 = X_1(r)$, $p_2 = X_2(p)$, and so on. This implies

$$\gamma^D(X_1, \xi) = p_1 - q_2 + r, \qquad \gamma^D(X_2, \xi) = p_2 - r_1 - 2ar. \tag{3.11}$$

The projective flatness of D implies the following:

(3.12)

In fact, the projective flatness of D implies the vanishing of

$$L(X, Y, Z) = (D_X \gamma^D)(Y, Z) - (D_Y \gamma^D)(X, Z).$$

By the data in (3.10) and (3.11), we see

$$\begin{aligned} L(X_1, X_2, X_1) &= 2(p_1 - q_2), \quad L(X_1, X_2, X_2) = -2\gamma^D(X_2, \xi), \\ L(X_1, X_2, \xi) &= -X_2(\gamma^D(X_2, \xi)). \end{aligned}$$

This implies (3.12). In particular,

$$\gamma^D(X_1, \xi) = r, \qquad \gamma^D(X_2, \xi) = 0. \qquad \square \tag{3.13}$$

Now let us assume that the surface is locally projectively homogeneous. By checking the transformation rules we can further diminish ambiguity of choosing frames. In the following, for a quantity Q, its transformed quantity is denoted by $\overline{Q}$. First note that $U = -\frac{3}{2}X_1(\log \mu)X_2$. Then, (2.3) implies

$$\gamma^{\overline{D}}(Y_1, \overline{\xi}) = \mu^4 \quad \gamma^D(X_1, \xi) = \mu^4 r.$$

Hence, whether $r = 0$ or not is a projective invariant property. Since $X_2(r) = 0$ and $X_2(\mu) = 0$, we can assume $|r| = 1$ in case $r \neq 0$, by letting $\mu = |r|^{-1/4}$. In this case, the frame $\{X_1, X_2, \xi\}$ is uniquely determined and, hence, a, b, p and q are constant by homogeneity. Moreover, (3.9) and (3.12) show that

$$a = p = 0.$$

Next suppose that $r \equiv 0$. Since $U = -\frac{3}{2}X_1(\log \mu)X_2$ and $X_2(\mu) = 0$, we see $\phi(U) = 0$; then $\phi(\xi) = 0$ because of (2.2). We also see $\nabla_{X_2}U = 0$. Hence, the transformation formulas in Section 1 shows that $\overline{S}Y_2 = \mu^5 X_2$; namely,

$$\overline{p} = \mu^3 p.$$

If $p \neq 0$, then $\mu = |p|^{-1/3}$ gives $|\overline{p}| = 1$. Since $p_2 = 0$ by (3.12), we can assume that $p = 1$ and in this way the frame is uniquely determined. Since a is a constant, we have $a_2 = 0$; this leads to $p = 0$ by (3.9), which is however a contradiction. Therefore $p \equiv 0$. As the final step of the present argument, we pay attention to the connection ∇. The formulas in Section 1 imply that

$$\overline{\nabla}_{Y_1} Y_1 = \mu^2(a - 2X_1(\log \mu))X_1 + \mu^2 X_2.$$

Since $X_2(a) = 0$, we can solve $a = 2X_1(\log \mu)$ and may assume that $a = 0$ and μ is a constant. Then, the identity $\overline{S}Y_1 = \mu^4 SX_1$ implies

$$\overline{q} = \mu^2 q. \tag{3.15}$$

If q is constant, then we can assume $q = 0$, 1, or -1. If q is not constant, then (3.15) shows

$$|\overline{q}|^{-3/2}Y_1(|\overline{q}|) = |q|^{-3/2}X_1(|q|)$$

which is a nonzero constant also by homogeneity. Since $[X_1, X_2] = aX_2 = 0$, we can choose a coordinate system $\{u, v\}$ so that $X_1 = \partial/\partial u$ and $X_2 = \partial/\partial v$. Because of $X_2(q) = 0$, we have

$$q = \alpha u^{-2} \tag{3.16}$$

for a constant α.

By the whole argument above we have seen that any locally projectively homogeneous surface has one of special frames listed below:

(II.a) $a = p = 0, \quad r = \pm 1, \quad q$ is any constant;

(II.b) $a = p = q = r = 0$;

(II.c) $a = p = r = 0, \qquad q = \pm 1$;

(II.d) $a = p = r = 0, \qquad q = \alpha u^{-2}, \quad \alpha \neq 0$.

4. The associated system of differential equations and its integration

We give in this section explicit forms of the surfaces with the distinguished frames that are fixed in the previous section.

In Section 1, we considered an immersion M^n into $\widetilde{M}^{n+1}$. Assume now $\widetilde{M} = \mathbb{P}^{n+1}$ with projectively flat connection D. To any hypersurface in $\mathbb{P}^{n+1}$ we associate its (local) lift, an immersion of codimension 2, into $\mathbb{R}^{n+2} - \{0\} \subset \mathbb{R}^{n+2}$, which is denoted by

$$z : M^n \longrightarrow \mathbb{R}^{n+2}.$$

In [6], we saw that such a z is determined by solving the system of differential equations:

$$\begin{aligned} \mathcal{D}_X z_* Y &= -\gamma^D(X, Y)\eta + z_*(D_X Y), \\ \mathcal{D}_X \xi &= -\gamma^D(X, \xi)\eta + z_*(D_X \xi), \\ \mathcal{D}_X \eta &= z_* X, \end{aligned} \tag{4.1}$$

where η denotes the position vector of the mapping z and $\mathcal{D}$ the ordinary flat connection on $\mathbb{R}^{n+2}$.

Since the relevant quantities are all determined in the previous section, we can write down the systems corresponding to the distinguished frames. Let us choose a coordinate system $\{u, v\}$ so that $X_1 = \partial/\partial u$ and $X_2 = \partial/\partial v$; this is possible because $[X_1, X_2] = 0$ in all cases. Thus we get the systems as follows.

(II.a)
$$\begin{aligned} &z_{uu} = -qz + z_v, \quad z_{uv} = \xi, \quad z_{vv} = -rz, \\ &\xi_u = rz - qz_v, \quad \xi_v = -rz_u, \quad \text{where} \quad r = \pm 1; \end{aligned}$$

(II.b)
$$\begin{aligned} &z_{uu} = z_v, \quad z_{uv} = \xi, \quad z_{vv} = 0, \\ &\xi_u = 0, \quad \xi_v = 0; \end{aligned}$$

(II.c)
$$\begin{aligned} &z_{uu} = -qz + z_v, \quad z_{uv} = \xi, \quad z_{vv} = 0, \\ &\xi_u = -qz_v, \quad \xi_v = 0, \quad \text{where} \quad q = \pm 1; \end{aligned}$$

(II.d)
$$\begin{aligned} &z_{uu} = -\alpha u^{-2} z + z_v, \quad z_{uv} = \xi, \quad z_{vv} = 0, \\ &\xi_u = -\alpha u^{-2} z_v, \quad \xi_v = 0 \end{aligned}$$

The actual integration of these systems shall be sketched. We treat only with the immersion z; notice that the affine normal ξ is given by z_{uv} for all cases.

For the case (II.a), assume first $r = -1$. Depending on the value of q, we have five cases: $q > 1$, $q = 1$, $1 > q > -1$, $q = -1$, and $q < -1$. The respective sets of independent solutions in this order are

$$\{e^{-v}\cos\lambda u,\ e^{-v}\sin\lambda u,\ e^{v}\cos\mu u,\ e^{v}\sin\mu u\} \quad \text{where} \quad \lambda^2 = 1+q,\ \mu^2 = q-1,$$
$$\{e^{v},\ ue^{v},\ e^{-v}\cos\sqrt{2}u,\ e^{-v}\sin\sqrt{2}u\},$$
$$\{e^{-v}\cos\lambda u,\ e^{-v}\sin\lambda u,\ e^{\mu u+v},\ e^{-\mu u+v}\} \quad \text{where} \quad \lambda^2 = 1+q,\ \mu^2 = 1-q,$$
$$\{e^{-v},\ ue^{-v},\ e^{\sqrt{2}u+v},\ e^{-\sqrt{2}u+v}\},$$
$$\{e^{\lambda u-v},\ e^{-\lambda u-v},\ e^{\mu u+v},\ e^{-\mu u+v}\} \quad \text{where} \quad \lambda^2 = -1-q,\ \mu^2 = 1-q.$$

Each set defines an immersion into $\mathbb{P}^3$ with inhomogeneous coordinates $[1, X, Y, Z]$. The image is (included in) the surface, which we list in the following with numbers following the list of surfaces in [6]:

$\|15\|$ $\quad \arctan Z = k\arctan(Y/X), \qquad k > 1,$

$\|16\|$ $\quad Z = \arctan(Y/X),$

$\|17\|$ $\quad \arctan Z = k\log(Y/X), \qquad k > 0,$

$\|18\|$ $\quad Z = Ye^{X},$

$\|19\|$ $\quad Z = (Y/X)^k, \qquad 0 < k < 1.$

The parameter k is determined by q.

Assume next $r = 1$. Then the set of independent solutions is

$$\{e^{\lambda u}\cos\left(\frac{u}{2\lambda}+v\right),\ e^{\lambda u}\sin\left(\frac{u}{2\lambda}+v\right),\ e^{-\lambda u}\cos\left(-\frac{u}{2\lambda}+v\right),\ e^{-\lambda u}\sin\left(-\frac{u}{2\lambda}+v\right)\}$$

where $2\lambda^2 = -q + \sqrt{1+q^2}$. The resulting surface is

$\|20\|$ $$\log\frac{X^2+Y^2}{1+Z^2} = k\left(\arctan(Y/X) - \arctan Z\right), \qquad k > 0.$$

The system (II.b) has a solutions given in the form

$$z = (6Au + 2B)v + Au^3 + Bu^2 + Cu + D,$$

where A, B, C, and D are independent constant vectors. The surface is equivalent to the Cayley surface

$\|21\|$ $$Z = XY + \frac{1}{3}X^3.$$

The set of independent solutions for the case (II.c) when $q = -1$ is

$$\{e^{-u},\ e^{u},\ (v+2u)e^{u},\ (v-2u)e^{-u}\}$$

and that when $q = 1$ is

$$\{\cos u,\ \sin u,\ u \sin u + 2v \cos u,\ u \cos u - 2v \sin u\}.$$

The surfaces are respectively

$\|22\|$ $$Z = XY + X \log X,$$
$\|23\|$ $$Z = XY + (1 + X^2) \arctan X.$$

To solve (II.d), it is better to transform the variables $\{u, v, z\}$ to new variables $\{w, v, \zeta\}$ by putting $w = e^u$ and $\zeta = e^{-w/2} z$. Then the transformed system is

$$\zeta_{ww} = \left(\frac{1}{4} - \alpha\right) \zeta + e^{2w} \zeta_v \quad \text{and} \quad \zeta_{vv} = 0.$$

If $1/4 - \alpha > 0$, then put $\lambda = \sqrt{1/4 - \alpha}$. In this case, if $\lambda \neq 1$ (i.e., $\alpha \neq -3/4$), a set of independent solutions is

$$\{e^{-\lambda w},\ e^{\lambda w},\ ve^{-\lambda w} + \frac{1}{4(1-\lambda)} e^{(2-\lambda)w},\ ve^{\lambda w} + \frac{1}{4(1+\lambda)} e^{(2+\lambda)w}\};$$

the surface is projectively equivalent to

$\|24\|$ $$Z = XY + X^k.$$

Here $k = 1 + 1/\lambda > 1$ and we exclude the case $k \neq 2$ since $\lambda \neq 1$. A subcase when $k = 3$ (i.e., $\alpha = 0$) is already counted above in $\|21\|$. If $\lambda = 1$, then a set of independent solutions is

$$\{e^{-w}, e^w, ve^{-w} + we^w, ve^w + \frac{1}{8} e^{3w}\},$$

which defines the surface

$\|25\|$ $$Z = XY + X^2 \log X.$$

When $1/4 - \alpha = 0$, a set of independent solutions is

$$\{1, w, v + \frac{1}{4} e^{2w}, vw + \frac{1}{4} e^{2w}(w - 1)\};$$

the corresponding surface is projectively equivalent to

$\|26\|$ $$Z = XY + e^X.$$

For the last case when $1/4 - \alpha = -\lambda^2 < 0$, a set of independent solutions is given by

$$\{\cos \lambda w, \sin \lambda w, v \cos \lambda w + \frac{1}{4(1+\lambda^2)} e^{2w} (\cos \lambda w + \lambda \sin \lambda w),$$
$$v \sin \lambda w + \frac{1}{4(1+\lambda^2)} e^{2w} (-\lambda \cos \lambda w + \sin \lambda w)\}.$$

Now the surface is equivalent to

$$\|27\| \qquad \log \frac{X+YZ}{1+Y^2} = k \arctan Y,$$

where $k = 2/\lambda > 0$.

Thus we end the listing of surfaces. As is easily seen, all surfaces are locally projectively homogeneous.

Part II. Degenerate projectively homogeneous surfaces

In the second part, we assume that $\operatorname{rank} h = 1$ and thus complete the classification of locally projectively homogeneous surfaces. In Section 5, we first prove the existence of a structure $\{D, \omega, \xi\}$ such that $C = 0$ and of a frame $\{X_1, X_2\}$ such that $\tau(X_2) = 0$. In Section 6, we then deal with the case that $\gamma^D(X_1, X_2) = 0$ and show that such projectively homogeneous surfaces are congruent to cylinders on projectively homogeneous plane curves. Finally, in Section 7, we conclude the classification by assuming that $\gamma^D(X_1, X_2) \neq 0$. The surface obtained in that section turns out to be a tangent-developable surface over a cubic curve.

§5. Specialized choice of $\{D, \omega, \xi\}$ for degenerate surfaces

Throughout this part, we will assume $\operatorname{rank} h = 1$; then the surface is said to be 1-degenerate. In the following lemmas, we will gradually improve our choice of structure $\{D, \omega, \xi\}$ and of a frame $\{X_1, X_2\}$ specially adapted to the problem. At each step, we will also investigate the invariance of such a frame.

Lemma 5.1. *Given a structure $\{D, \omega, \xi\}$, there exists a frame $\{X_1, X_2\}$ such that*

$$h(X_1, X_1) = 1, \quad h(X_1, X_2) = h(X_2, X_2) = 0, \quad \omega(X_1, X_2, \xi) = 1.$$

Moreover, for such a frame, we have that $C(X_2, X_2, \cdot) = 0$, and that $\nabla_{X_2} X_2$ only has a component in the X_2-direction.

Proof. The first statement is obvious. In order to prove the second, we notice that

$$C(V, X_2, X_2) = V h(X_2, X_2) - 2h(\nabla_V X_2, X_2) = 0,$$

for any vector field V. Applying Codazzi's equation now completes the proof. □

Since h is degenerate, we cannot use the normalization in Section 2. We start with a triple $\{D, \omega, \xi\}$ satisfying the condition $D\omega = 0$ and make special choices to know what kinds of invariants we get in the following. Let us remark that two structures $\{D, \omega, \xi\}$ and $\{\overline{D}, \overline{\omega}, \overline{\xi}\}$ have relations as in (2.1), where we have the relation $\phi = \frac{1}{4} d \log \sigma$, because of equiaffineness.

Lemma 5.2. *There exist a structure $\{D, \omega, \xi\}$ and a frame $\{X_1, X_2\}$, satisfying the conditions of Lemma 5.1, such that the cubic form C vanishes identically. Moreover, if $\{D, \omega, \xi\}$ and $\{\overline{D}, \overline{\omega}, \overline{\xi}\}$ determine two such structures, with corresponding frames X_1, X_2 and $\overline{X}_1, \overline{X}_2$, then they are related by*

$$(*) \qquad \begin{aligned} \overline{D}_X Y &= D_X Y + \phi(X) Y + \phi(Y) X, \\ \overline{\xi} &= \tfrac{1}{\lambda}(U + \xi), \\ \overline{X}_2 &= \mu X_2, \\ \overline{X} &= \mu X + \rho X \end{aligned}$$

where $U = \frac{1}{4}X_1(\log\sigma)X_1 + bX_2$, $\lambda = \nu^{-2}$, $\mu = \nu^{-3}\sigma^{-1}$, $\phi(X_i) = \eta(X_i) = h(U, X_i)$ *and* $X_2(\sigma) = X_2(\nu) = 0$.

Proof. We consider a change of structure given by

$$\overline{D}_X Y = D_X Y + \phi(X)Y + \phi(Y)X,$$
$$\overline{\omega} = \sigma\omega, \quad \overline{\xi} = \tfrac{1}{\lambda}(U + \xi),$$

where $\phi = \frac{1}{4}d\log\sigma$. Under such a change, the frame $\{X_1, X_2\}$ can be varied in the following way:

$$\overline{X}_1 = \nu X_1 + cX_2, \quad \overline{X}_2 = \mu X_2.$$

Since $\overline{h}(\overline{X}_1, \overline{X}_1) = \lambda h(\overline{X}_1, \overline{X}_1) = \lambda\nu^2$, we deduce that $\lambda = \nu^{-2}$. Similarly, by computing $\overline{\omega}(\overline{X}_1, \overline{X}_2, \overline{\xi})$, we deduce that $\mu = \frac{\lambda}{\nu\sigma} = \nu^{-3}\sigma^{-1}$. Also, we find

$$\begin{aligned}\overline{C}(X_2, X_1, X_1) &= C'(X_2, X_1, X_1) - \phi(X_2)h'(X_1, X_1) - 2\phi(X_1)h'(X_1, X_2)\\ &= \lambda C(X_2, X_1, X_1) + \lambda(2h(X_1, X_2)\eta(X_1) + h(X_1, X_1)\eta(X_2))\\ &\quad - \phi(X_2)h'(X_1, X_1) - 2\phi(X_1)h'(X_1, X_2)\\ &= \lambda C(X_2, X_1, X_1) - \lambda\phi(X_2)h(X_1, X_1).\end{aligned}$$

Hence, using Lemma 5.1, we also get

$$\overline{C}(\overline{X}_2, \overline{X}_1, \overline{X}_1) = \mu\nu^2\lambda(C(X_2, X_1, X_1) - \phi(X_2)). \tag{5.1}$$

Hence, by choosing ϕ (i.e., by choosing σ) appropriately, we may assume $C(X_2, X_1, X_1) = 0$. As follows from (5.1), this restrict the possible freedom of changing the structure and the frame by imposing the condition that $\phi(X_2) = 0$, (i.e., $X_2(\sigma) = 0$).

Next, we compute

$$\begin{aligned}\overline{C}(X_1, X_1, X_1) &= C'(X_1, X_1, X_1) - 3\phi(X_1)h'(X_1, X_1)\\ &= \lambda C(X_1, X_1, X_1) + 3\lambda h(U, X_1) - 3\lambda\phi(X_1).\end{aligned}$$

Taking into account that we are restricting ourselves to changes for which $C(X_2, X_1, X_1)$ remains zero, we see

$$\overline{C}(\overline{X}_1, \overline{X}_1, \overline{X}_1) = \nu^3\lambda(C(X_1, X_1, X_1) + 3h(U, X_1) - 3\phi(X_1)).$$

Hence, by choosing U appropriately, we may assume $C(X_1, X_1, X_1) = 0$. By restricting now to changes of structures which preserve this property we find that $\eta(X_1) := h(U, X_1) = \phi(X_1)$. Since, we already know $\eta(X_2) = 0 = \phi(X_2)$, we have $\phi|_M = \eta$.

Finally, we have

$$\begin{aligned}\overline{\tau}(\overline{X}_2) &= \tau'(\overline{X}_2) + \phi(\overline{X}_2)\\ &= \tau(\overline{X}_2) + \eta(\overline{X}_2) - \overline{X}_2(\log\lambda),\\ &= \mu(\tau(X_2) - X_2(\log\lambda)).\end{aligned}$$

Hence, we may assume $\tau(X_2) = 0$ and restrict ourselves to changes for which $X_2\lambda = 0$. This completes the proof of the lemma. □

Let us now take a structure $\{D, \omega, \xi\}$ and a frame X_1, X_2 satisfying the previous lemmas. We write

$$\nabla_{X_1}X_1 = a_0X_1 + a_1X_2, \quad \nabla_{X_1}X_2 = a_4X_1 + a_2X_2,$$
$$\nabla_{X_2}X_1 = a_5X_1 + a_3X_2, \quad \nabla_{X_2}X_2 = a_6X_2.$$

Then, we have the following lemma:

Lemma 5.3. *We have*

(1) $a_4 = a_5 = a_6 = 0$,
(2) $\gamma^D(X_2, X_2) = 0 = \mathcal{S}(X_2, X_2)$,
(3) $\gamma^D(X_1, X_2) = X_2(a_2)$,
(4) $\tau(X_1) = 2a_0$,
(5) $X_2(\gamma^D(X_1, X_2)) = 0$,
(6) $X_2(\tau(X_1)) = -h(X_1, SX_2)$.

Proof. Since the cubic form C vanishes identically from the previous lemma, we get

$$\begin{aligned} 0 &= C(X_2, X_1, X_1) = \tau(X_2) - 2h(\nabla_{X_2} X_1, X_1) = -2a_5, \\ 0 &= C(X_1, X_2, X_1) = -h(\nabla_{X_1} X_2, X_1) = -a_4, \\ 0 &= C(X_1, X_1, X_1) = \tau(X_1) - 2h(\nabla_{X_1} X_1, X_1) = \tau(X_1) - 2a_0. \end{aligned}$$

Using the previous equations together with the fact that $D\omega = 0$, we obtain

$$\begin{aligned} (D_{X_2}\omega)(X_1, X_2, \xi) &= -\omega(D_{X_2}X_1, X_2, \xi) - \omega(X_1, D_{X_2}X_2, \xi) - \omega(X_1, X_2, D_{X_2}\xi) \\ &= -\omega(X_1, \nabla_{X_2}X_2, \xi) = -a_6. \end{aligned}$$

Hence $\nabla_{X_2} X_2 = 0$ and this proves (1) and (4).

Next, we use the Gauss equation. On one side we have

$$R(X_1, X_2)X_2 = \gamma^D(X_2, X_2)X_1 - \gamma^D(X_1, X_2)X_2,$$

but on the other side, by an immediate computation we find

$$R(X_1, X_2)X_2 = \nabla_{X_1}\nabla_{X_2}X_2 - \nabla_{X_2}\nabla_{X_1}X_2 - \nabla_{[X_1,X_2]}X_2 = -X_2(a_2)X_2.$$

Thus by comparing components, we obtain (2) and (3).

Since the structure determined by D and ω is projectively flat, we obtain

$$\begin{aligned} X_2(\gamma^D(X_1, X_2)) &- \gamma^D(\nabla_{X_2}X_1, X_2) - \gamma^D(X_1, \nabla_{X_2}X_2) \\ &= X_1(\gamma^D(X_2, X_2)) - 2\gamma^D(\nabla_{X_1}X_2, X_2). \end{aligned}$$

Hence, using also (1) and (2), we obtain (5).

Finally, we are going to use the Ricci equation. We have

$$\begin{aligned} (\nabla_{X_2}\tau)(X_1) &= X_2(\tau(X_1)) - \tau(\nabla_{X_2}X_1) = X_2(\tau(X_1)), \\ (\nabla_{X_1}\tau)(X_2) &= X_1(\tau(X_2)) - \tau(\nabla_{X_1}X_2) = 0. \end{aligned}$$

So it follows from the Ricci equation that

$$X_2(\tau(X_1)) = h(SX_1, X_2) - h(X_1, SX_2) = -h(X_1, SX_2). \qquad \square$$

To conclude this section, we will now investigate how some of the invariants appearing in the previous lemma may change under a change of structure as given by Lemma 5.2.

Lemma 5.4. *Let $\{D, \omega, \xi\}$ with frame $\{X_1, X_2\}$ and $\{\overline{D}, \overline{\omega}, \overline{\xi}\}$ with frame $\{\overline{X}_1, \overline{X}_2\}$ be two structures satisfying the conditions of Lemma 5.2 (and related as in Lemma 5.2). If we denote the invariants with respect to the second structure by adding a $^{-}$, we have*

(1) $\overline{\tau}(\overline{X}_1) = \nu(\tau(X_1) + X_1(\frac{1}{2}\log\sigma - \log\lambda))$,

(2) $\overline{S}\,\overline{X}_2 = \frac{\mu}{\lambda}(SX_2 - \frac{1}{4}X_1(\log\sigma)\nabla_{X_2}X_1 + X_2(b)X_2 - \phi(\xi + U)X_2)$,

(3) $\overline{h}(\overline{S}\,\overline{X}_2, \overline{X}_1) = \mu\nu h(SX_2, X_1)$,

(4) $\overline{\mathcal{S}}(X, Y) = \mathcal{S}(X, Y) - \phi(U + 2\xi)h(X, Y)$,

(5) $\overline{\mathcal{S}}(\overline{X}_1, \overline{X}_2) = \mu\nu\mathcal{S}(X_1, X_2)$,

(6) $\gamma^{\overline{D}}(\overline{X}_1, \overline{X}_2) = \mu\nu\gamma^D(X_1, X_2)$.

Proof. We have

$$\begin{aligned}
\overline{\tau}(\overline{X}_1) &= \tau'(\overline{X}_1) + \phi(\overline{X}_1) \\
&= \tau(\overline{X}_1) + \eta(\overline{X}_1) - \overline{X}_1(\log\lambda) + \phi(\overline{X}_1) \\
&= \tau(\overline{X}_1) + \overline{X}_1(\tfrac{1}{2}\log\sigma - \log\lambda) \\
&= \tau(\nu X_1 + cX_2) + (\nu X_1 + cX_2)(\tfrac{1}{2}\log\sigma - \log\lambda) \\
&= \nu(\tau(X_1) + X_1(\tfrac{1}{2}\log\sigma - \log\lambda)).
\end{aligned}$$

This proves (1). The proof of (2) follows by a similar computation. In order to obtain (3), we use (2). We have

$$\begin{aligned}
\overline{h}(\overline{S}\,\overline{X}_2, \overline{X}_1) &= \lambda h(\overline{S}\,\overline{X}_2, \nu X_1) \\
&= \mu\nu(h(SX_2, X_1) - \tfrac{1}{4}X_1(\log\sigma)h(\nabla_{X_2}X_1, X_2)) \\
&= \mu\nu h(SX_2, X_1).
\end{aligned}$$

The proof of (4), (5) and (6) is now done in a similar way. □

§6. Cylinder over projectively homogeneous plane curves

We now consider the case where $\gamma^D(X_1, X_2) = 0$ and prove that any 1-degenerate projectively homogeneous surface with this property is a cylinder over a projectively homogeneous plane curve. First, notice that the fact $\gamma^D(X_1, X_2) = 0$ is independent of the choice of frame and structure satisfying the conditions of Lemma 5.2, because of Lemma 5.4. We put

$$SX_1 = pX_1 + qX_2, \quad SX_2 = rX_1 + tX_2.$$

From Lemma 5.3, it now follows that $X_2(a_2) = 0$ and from the Gauss equation it follows that

$$\begin{aligned}
&\gamma^D(X_1, X_2)X_1 - \gamma^D(X_1, X_1)X_2 - SX_2 \\
&\qquad = \nabla_{X_1}\nabla_{X_2}X_1 - \nabla_{X_2}\nabla_{X_1}X_1 - \nabla_{[X_1, X_2]}X_2.
\end{aligned}$$

Hence

(6.1) $$-\gamma^D(X_1, X_1)X_2 - SX_2$$

Hence by comparing the X_1-components, we obtain

$$r = X_2(a_0). \tag{6.2}$$

Since D and ω determine an equiaffine structure, we also have

$$\begin{aligned} 0 &= (D_{X_1}\omega)(X_1, X_2, \xi) \\ &= -\omega(\nabla_{X_1}X_1, X_2, \xi) - \omega(X_1, \nabla_{X_1}X_2, \xi) - \omega(X_1, X_2, D_{X_1}\xi) \\ &= -a_0 - a_2 - \tau(X_1). \end{aligned} \tag{6.3}$$

Comparing this with Lemma 5.3, it follows that $a_2 = -3a_0$. This in turn implies

$$0 = X_2(a_2) = -3X_2(a_0) = -3r.$$

Hence $r = 0$ and $X_2(\tau(X_1)) = 0$.

Lemma 6.1. *Let $\{D, \omega, \xi\}$ be a structure and $\{X_1, X_2\}$ a frame such that the conditions of Lemmas 5.1 and 5.2 are satisfied. Assume that $\gamma^D(X_1, X_2) = 0$. Then there exist a structure $\{\overline{D}, \overline{\omega}, \overline{\xi}\}$ and a frame $\{\overline{X}_1, \overline{X}_2\}$ which also satisfies the conditions of Lemmas 5.1 and 5.2, and has $\overline{\tau}(\overline{X}_1) = 0$.*
Moreover, any two structures $\{D, \omega, \xi\}$ with corresponding frame $\{X_1, X_2\}$ and $\{\overline{D}, \overline{\omega}, \overline{\xi}\}$ with corresponding frame $\{\overline{X}_1, \overline{X}_2\}$ satisfying the first part of the lemma are related by $(*)$, where $U = \frac{1}{4}X_1(\log\sigma)X_1 + bX_2$, $\lambda = \nu^{-2}$, $\mu = \nu^{-3}\sigma^{-1}$, $X_2(\sigma) = X_2(\nu) = 0$, $\phi(X_i) = \eta(X_i)$ and $X_1(\log \sigma\nu^4) = 0$.

Proof. It follows from Lemma 5.4 that $\overline{\tau}(\overline{X}_1) = \nu(\tau(X_1) + X_1(\frac{1}{2}\log\sigma - \log\lambda))$. We now consider the following system of differential equations for a function α. We put

$$X_1(\alpha) = \tau(X_1), \quad X_2(\alpha) = 0.$$

The integrability condition for this system is satisfied since

$$X_1(X_2(\alpha)) - X_2(X_1(\alpha)) = (\nabla_{X_1}X_2 - \nabla_{X_2}X_1)\alpha = 0.$$

The required structure is now obtained by putting $\alpha = \log\sigma\nu^4$.

If we now have two structures (with corresponding frames) satisfying these conditions, it follows once more from Lemma 5.4 that $X_1(\log\sigma\nu^4) = 0$. □

By using now the Gauss equation, the Codazzi equation for S, and the projective flatness, a straightforward computation yields the following lemma.

Lemma 6.2. *Let $\{D, \omega, \xi\}$ be a structure with corresponding frame satisfying the conditions of Lemma 6.1. Then, we have*

$$\begin{aligned} &-\gamma^D(X_1, X_1) - t = X_1(a_3) - X_2(a_1), \\ &X_1(t) - \gamma^D(X_1, \xi) = pa_3 + X_2(q) - a_3t, \\ &0 = X_2(p) - \gamma^D(X_2, \xi), \\ &X_2(\gamma^D(X_1, X_1)) = -\gamma^D(X_2, \xi). \end{aligned}$$

Lemma 6.3. *Let $\{D,\omega,\xi\}$ be a structure and $\{X_1,X_2\}$ a frame such that the conditions of Lemma 6.1 is satisfied. Assume that $\gamma^D(X_1,X_2)=0$. Then there exist a structure $\{\overline{D},\overline{\omega},\overline{\xi}\}$ and a frame $\{\overline{X}_1,\overline{X}_2\}$ which also satisfies the conditions of Lemma 6.1 and has $\overline{\mathcal{S}}(\overline{X}_1,\overline{X}_1)=0$, $\overline{\nabla}_{\overline{X}_2}\overline{X}_1=0$ and $\overline{\nabla}_{\overline{X}_1}\overline{X}_1=0$.*
Moreover, any two structures $\{D,\omega,\xi\}$ with corresponding frame $\{X_1,X_2\}$ and and $\{\overline{D},\overline{\omega},\overline{\xi}\}$ with corresponding frame $\{\overline{X}_1,\overline{X}_2\}$ satisfying the first part of the lemma are related by $(*)$ where $U=\frac{1}{4}X_1(\log\sigma)X_1+bX_2$, $\lambda=\nu^{-2}$, $\mu=\nu^{-3}\sigma^{-1}$, $X_2(\sigma)=X_2(\nu)=0$, $\phi(U+2\xi)=0$, $X_2(c)=-\nu\phi(X_1)$, $b=X_1(c/\nu)$, $\phi(X_i)=\eta(X_i)$ and $X_1(\log\sigma\nu^4)=0$.

Proof. We have

$$\overline{\mathcal{S}}(\overline{X}_1,X_1)=\nu^2(\mathcal{S}(X_1,X_1)-\phi(U+2\xi)).$$

Hence, by choosing $\phi(\xi)$ appropriately, we may assume that $\overline{\mathcal{S}}(\overline{X}_1,\overline{X}_1)=0$ and if we have two structures with this property, then it follows that $\phi(U+2\xi)=0$.

Next, we compute $\overline{\nabla}_{\overline{X}_2}\overline{X}_1$. Then we see

$$\overline{a}_3=\nu(a_3+\phi(X_1)+\tfrac{X_2(c)}{\nu}).$$

Hence, by choosing c appropriately, we may assume that $\overline{a}_3=0$ and it follows that if we have two structures with this property, then $X_2(c)=-\nu\phi(X_1)=X_1(\nu)$.

Finally, we compute $\nabla_{\overline{X}_1}\overline{X}_1$. We get

$$\mu\overline{a}_1=\nu^2a_1+\nu X_1(c)+cX_2(c)-\nu^2b+2\nu c\phi(X_1),$$

from which we deduce that by choosing b appropriately, we may assume $\overline{a}_1=0$. Furthermore, any two structures with this property are related by

$$\begin{aligned}\nu^2b&=\nu X_1(c)+cX_2(c)+2\nu c\phi(X_1)\\&=\nu X_1(c)+\nu c\phi(X_1)=\nu X_1(c)-cX_1(\nu).\end{aligned}$$ □

Let us now assume that we have two structures $\{D,\omega,\xi\}$ with corresponding frame $\{X_1,X_2\}$ and $\{\overline{D},\overline{\omega},\overline{\xi}\}$ with corresponding frame $\{\overline{X}_1,\overline{X}_2\}$ satisfying both the previous lemmas. Then, we obtain

$$X_2(b)=X_2(X_1(\tfrac{c}{\nu}))=X_1(X_2(\tfrac{c}{\nu}))=X_1(\tfrac{X_2(c)}{\nu})=-X_1(\phi(X_1)).$$

By a straightforward computation, we also get

$$\lambda\overline{p}=p-X_1(\phi(X_1))+\tfrac{1}{2}\phi(X_1)^2 \tag{6.4}$$

Since $\mathcal{S}(X_1,X_1)=0$, we have $p=\gamma^D(X_1,X_1)$. We also have by Lemma 6.2

$$X_2(p)=\gamma^D(X_2,\xi)=-X_2(\gamma^D(X_1,X_1))=-X_2(p).$$

Hence $X_2(p)=0$. Expressing then the identity

$$(D_{X_1}\gamma^D)(X_2,\xi)=(D_{X_2}\gamma^D)(X_1,\xi),$$

we have

$$X_1(\gamma^D(X_2,\xi))=X_2(\gamma^D(X_1,\xi))$$

which is equivalent to

$$-X_1(X_2(p)) = X_2(-X_1(p) - X_2(q)).$$

Hence, we find that $X_2(X_2(q)) = 0$.

Now, since $X_2(p) = 0$, it follows that we can choose $\phi(X_1)$ in such a way that $\overline{p} = 0$ in view of (6.4). Therefore, let us now assume that we work with structures with this additional property. Then, we already know

$$\begin{aligned}
&\nabla_{X_i} X_j = 0, \qquad i, j \in \{1, 2\},\\
&h(X_1, X_1) = 1, \quad h(X_1, X_2) = h(X_2, X_2) = 0,\\
&\omega(X_1, X_2, \xi) = 1, \quad \mathcal{S}(X_i, X_j) = 0,\\
&SX_1 = qX_2, \quad SX_2 = 0,\\
&\gamma^D(X_1, X_2) = \gamma^D(X_2, X_2) = \gamma^D(X_1, X_1) = 0,\\
&\gamma^D(X_1, \xi) = -X_2(q), \quad \gamma^D(X_2, \xi) = 0,
\end{aligned}$$

with $X_2(X_2(q)) = 0$. Moreover, we still have the following degree of freedom:

$$\begin{aligned}
&\overline{D}_X Y = D_X Y + \phi(X)Y + \phi(Y)X,\\
&\overline{\xi} = \tfrac{1}{\lambda}(U + \xi),\\
&\overline{X}_2 = \mu X_2, \quad \overline{X}_1 = \nu X_1 + cX_2,\\
&U = \phi(X_1)X_1 + bX_2,
\end{aligned}$$

where

$$\begin{aligned}
&X_2(\sigma) = X_2(\nu) = 0 = X_2(\lambda) = X_2(\mu), \quad X_1(\sigma\nu^4) = 0,\\
&\phi(X_1)^2 + 2\phi(\xi) = 0, \quad \phi(X_i) = \eta(X_i),\\
&X_1(\nu) = X_2(c), \quad b = X_1(\tfrac{c}{\nu}), \quad X_1(\phi(X_1)) = \tfrac{1}{2}\phi(X_1)^2.
\end{aligned}$$

Since $\nabla_{X_i} X_j = 0$, we can choose coordinates u and v such that $X_1 = \partial/\partial u$ and $X_2 = \partial/\partial v$. Then we use again the local lift from an immersion of M into $\mathbb{P}^3$ to an immersion $z : M^2 \to \mathbb{R}^4$, as recalled in Section 4. Using the obtained information, we get

$$\begin{aligned}
&z_{uu} = \xi, \quad z_{uv} = 0, \quad z_{vv} = 0,\\
&\xi_u = q_2 z - v q_2 z_v, \quad \xi_v = 0,
\end{aligned} \tag{6.5}$$

where $q_2 = X_2(q)$, a function only of u. Solving the system (6.5), we obtain

$$z(u, v) = A(u) + Bv,$$

where B is a constant vector and A satisfies:

$$\begin{aligned}
&\xi = A''\\
&A''' = \xi' = q_2(A + vB) - vq_2 B = q_2 A
\end{aligned} \tag{6.6}$$

from which we deduce that A is a planar curve and that B is transversal to this plane. Our aim in the following is to show that the plane curve is projectively homogeneous. For this aim, we look after the invariants above more carefully.

Since the functions σ, ν, λ and μ only depend on the variable u, we can find functions f and g of u and a non-vanishing constant α such that

$$\nu = e^g, \quad \sigma = \alpha e^{-4g}, \quad \mu = \tfrac{1}{\alpha}e^g, \quad \lambda = e^{-2g},$$
$$\phi(X_1) = -g', \quad c = e^g(f + vg'), \quad b = f' + vg'',$$

where g is a solution of the differential equation:

$$g'' + \tfrac{1}{2}(g')^2 = 0.$$

We then obtain

$$\overline{q} = \tfrac{\nu}{\lambda\mu}(q - f'' - g'f'),$$

and since $X_2(X_2(q)) = 0$, we can write $q = q_1(u) + vq_2(u)$. Since

$$\overline{X}_2(\overline{q}) = \mu\overline{X}_2(q) = \tfrac{\nu}{\lambda}X_2(q) = \tfrac{\nu}{\lambda}q_2, \tag{6.7}$$

we see that whether q_2 vanishes or not is independent of the choice of basis. Let us assume first $q_2 \neq 0$. Then, we have

$$\begin{aligned}
\overline{X}_1(\tfrac{\overline{q}}{\overline{X}_2(\overline{q})}) &= (\nu X_1 + cX_2)(\tfrac{\overline{q}}{\nu q_2/\lambda}) \\
&= (\nu X_1 + cX_2)(\tfrac{q - f'' - g'f'}{\mu q_2}) \\
&= -\tfrac{\nu}{\mu^2}X_1(\mu)(\tfrac{q_1}{q_2} + v - \tfrac{f''}{q_2} - \tfrac{g'f'}{q_2}) \\
&\quad - \tfrac{\nu}{\mu}(\tfrac{q_1}{q_2} - \tfrac{f''}{q_2} - \tfrac{g'f'}{q_2})' + \tfrac{c}{\mu} \\
&= -\alpha g'(\tfrac{q_1}{q_2} - \tfrac{f''}{q_2} - \tfrac{g'f'}{q_2}) - \alpha f + \alpha(\tfrac{q_1}{q_2} - \tfrac{f''}{q_2} - \tfrac{g'f'}{q_2})'.
\end{aligned}$$

Hence, by choosing f appropriately, we may assume

$$0 = X_1(\tfrac{q}{X_2(q)}) = \tfrac{\partial}{\partial u}(\tfrac{q_1}{q_2}).$$

So, there exists a constant β such that $q_1 = \beta q_2$. Therefore, by translating the v-coordinate, if necessary, we may assume $q_1 = 0$. We now put $\overline{q}_2 = \overline{X}_2(\overline{q})$ and $q_2 = X_2(q)$. By (6.7), we know

$$\overline{q}_2 = e^{3g}q_2, \tag{6.8}$$

and thus

$$\begin{aligned}
&\overline{X}_1(\log\overline{q}_2) = 3g'e^g + e^gX_1(\log q_2), \\
&\overline{X}_1\overline{X}_1(\log\overline{q}_2) = 3(g'' + (g')^2)e^{2g} + g'e^{2g}X_1(\log q_2) + e^{2g}X_1X_1(\log q_2), \\
&\tfrac{1}{6}(\overline{X}_1(\log\overline{q}_2))^2 = \tfrac{3}{2}(g')^2e^{2g} + e^{2g}g'X_1(\log q_2) + e^{2g}(X_1(\log q_2))^2.
\end{aligned}$$

Hence, since $g'' = -\frac{1}{2}(g')^2$, we deduce that

$$\overline{X}_1\overline{X}_1(\log\overline{q}_2) - \tfrac{1}{6}(\overline{X}_1(\log\overline{q}_2))^2 = e^{2g}(X_1X_1(\log q_2) - \tfrac{1}{6}(X_1(\log q_2))^2) \tag{6.9}$$

Now, we use the fact that M is homogeneous. It follows from (6.8) and (6.9) that there exists a constant κ such that

$$\kappa = |q_2|^{-2/3}(X_1X_1(\log q_2) - \tfrac{1}{6}(X_1(\log q_2))^2) = |q_2|^{-2/3}(\tfrac{q_2''}{q_2} - \tfrac{7}{6}(\tfrac{q_2'}{q_2})^2). \tag{6.10}$$

We now introduce a variable s and a curve C by,

$$\begin{aligned} \tfrac{ds}{du} &= -q_2^{1/3}, \\ A(u) &= q_2^{-1/3}C(s). \end{aligned}$$

Then,

$$\begin{aligned} A' &= -\tfrac{1}{3}q_2^{-4/3}q_2'C - C_s, \\ A'' &= (\tfrac{4}{9}q_2^{-7/3}(q_2')^2 - \tfrac{1}{3}q_2^{-4/3}q_2'')C + \tfrac{1}{3}q_2^{-1}q_2' + q_2^{1/3}C_{ss}, \\ A''' &= (-\tfrac{28}{27}q_2^{-10/3}(q_2')^3 + \tfrac{4}{3}q_2^{-7/3}q_2'q_2'' - \tfrac{1}{3}q_2^{-4/3}q_2''')C \\ &\qquad + (-\tfrac{7}{9}q_2^{-2}(q_2')^2 + \tfrac{2}{3}q_2^{-1}q_2'')C_s - q_2^{2/3}C_{sss}. \end{aligned}$$

Hence, using (6.6) and (6.10) we deduce that

$$\begin{aligned} 0 &= A''' - q_2A, \\ &= -q_2^{2/3}(C_{sss} - \tfrac{2}{3}\kappa C_s + (1 - \tfrac{28}{27}q_2^{-4}(q_2')^3 + \tfrac{4}{3}q_2^{-3}q_2'q_2'' - \tfrac{1}{3}q_2^{-2}q_2''')C), \\ &= -q_2^{2/3}(C_{sss} - \tfrac{2}{3}\kappa C_s + C) \end{aligned}$$

because κ is constant. Applying now the formulas for projectively homogeneous curves, see [9], which are also recalled in Appendix, we deduce that C (and hence also A) is a projective planar curve which is projectively homogeneous.

If $q_2 = 0$, then the curve A is a quadratic curve, which is also projectively homogeneous. This completes the proof in the case where $\gamma^D(X_1, X_2) = 0$.

§7. Tangent-developable surfaces of projectively homogeneous space curves

In this section, we consider the case where $\gamma^D(X_1, X_2) \neq 0$. We will show that any projectively homogeneous surface with this property is a tangent-developable surface over a cubic curve.

We recall from Lemma 5.4 that

$$\gamma^{\overline{D}}(\overline{X}_1, \overline{X}_2) = \mu\nu\gamma^D(X_1, X_2).$$

Hence there exist a structure $\{D, \omega, \xi\}$ and frame $\{X_1, X_2\}$ such that

$$\gamma^D(X_1, X_2) = 1.$$

The proof of the following lemma is straightforward from Lemma 5.2 and is therefore omitted.

Lemma 7.1. *There exists a structure $\{D, \omega, \xi\}$ with corresponding frame $\{X_1, X_2\}$ such that $[X_1, X_2] = 0$, i.e. $a_2 = a_3$. Moreover, any two such structures with corresponding frames have to be related by $(*)$ where $U = \eta(X_1)X_1 + bX_2$, $\lambda = \nu^{-2}$, $\mu = \nu^{-1}$, $\sigma = \nu^{-2}$, $X_2(\nu) = 0$, $\phi(X_2) = 0$, $\phi(X_1) = \eta(X_1) = -\frac{1}{2}X_1(\log \nu)$ and $X_2(c) = -X_1(\nu)$.*

We now further improve our choice of structure and frame. Since $[X_1, X_2] = 0$, we can introduce coordinates u and v such that $X_1 = \partial/\partial u$ and $X_2 = \partial/\partial v$. Using the above formulas, we can introduce local functions α and β, depending only on the variable u, such that

$$\nu = \alpha(u), \tag{7.1}$$

$$c = \beta(u) - v\alpha'(u). \tag{7.2}$$

Since $\overline{a}_2 = \nu a_2 - \frac{3}{2}X_1(\nu)$, we get

$$\begin{aligned} \overline{X}_1(\overline{a}_2) &= (\nu X_1 + cX_2)(\nu a_2 - \tfrac{3}{2}X_1(\nu)) \\ &= \nu X_1(\nu)a_2 + \nu c X_2(a_2) - \tfrac{3}{2}\nu X_1 X_1(\nu). \end{aligned} \tag{7.3}$$

From Lemma 5.3(3), it follows that we can write $a_2 = v + f(u)$. Hence, by using also (7.1) and (7.2), equation (7.3) reduces to

$$\begin{aligned} \overline{X}_1(\overline{a}_2) &= \alpha\alpha' v + \alpha\alpha' f + \alpha(\beta - v\alpha') - \tfrac{3}{2}\alpha\alpha'' \\ &= \alpha\alpha' f + \alpha\beta - \tfrac{3}{2}\alpha\alpha''. \end{aligned}$$

Hence, by choosing β appropriately, we may assume $X_1(a_2) = 0$, and by a translation of the v-coordinate, if necessary, we may therefore assume $a_2 = v$.

We have now seen that for an appropriate choice of the structure $\{D, \omega, \xi\}$ and the frame $\{X_1, X_2\}$ it holds

$$\begin{aligned} &\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = vX_2, \quad \nabla_{X_2} X_2 = 0, \\ &h(X_1, X_1) = 1, \quad h(X_1, X_2) = h(X_2, X_2) = 0, \\ &\gamma^D(X_1, X_2) = 1, \quad \gamma^D(X_2, X_2) = 0. \end{aligned}$$

Hence, for a local lift z of the immersion, we get the equation

$$z_{uv} = -z + vz_v, \quad z_{vv} = 0.$$

Solving the above system of differential equations, we first get

$$z(u, v) = A(u)v + B(u).$$

Then

$$A'(u) = z_{uv} = -A(u)v - B(u) + vA(u) = -B(u).$$

Hence

$$z(u, v) = A(u)v - A'(u).$$

This implies that the surface in question is a tangent-developable surface over a space curve $A(u)$.

On the other hand, we can prove the following proposition

Proposition 7.2. *Let $z(u,v) = f(u)+vf'(u)$ be a 1-degenerate tangent-developable surface where f is a space curve. Then the surface is projectively homogeneous if and only if the space curve is a cubic curve.*

Proof. The 1-degenerate condition is $f''' \not\equiv 0 \pmod{f, f', f''}$, i.e., f is not planar. Assume that the surface z is projectively homogeneous and let P be a projective transformation. Since P transforms any line to a line, we may assume that P transforms tangent lines to tangent lines as far as P is near to the identity; this implies that P acts as

$$P(f(u) + vf'(u)) = (av + b)f(\varphi(u)) + (cv + d)f'(\varphi(u)),$$

where φ is a local diffeomorphism of the u-space and a, b, c, and d are functions only of u. Then we have

$$P(f(u)) = bf(\varphi(u)) + d\,f'(\varphi(u)),$$
$$P(f'(u)) = af(\varphi(u)) + cf'(\varphi(u)).$$

Since f is not planar, it is easy to see $a = b'$, $c = b\varphi'$, $d = 0$. Hence $P(f(u)) = bf(\varphi(u))$, which shows that the curve f is projectively homogeneous. Then we get

$$P(f + vf') = bf(\varphi(u)) + v(bf(\varphi(u)))'.$$

This identity means that the transformation φ determines the transformation P. Therefore, in order that the surface be projectively homogeneous, the linear isotropy of the projective automorphism group of the space curve f must be at least 1-dimensional. This is the case only for cubic curve, as will be shown in Proposition A2 of the appendix.

Conversely, given a cubic curve in the form $A(u) = [1, u, u^2, u^3]$, the corresponding surface is

$$z(u, v) = [1, u - v, u^2 - 2uv, u^3 - 3u^2v].$$

It is now easy to see that the surface is projectively homogeneous and the coordinates $[1, X, Y, Z]$ satisfy

$$(Y - X^2)^3 = v^6 = -\tfrac{1}{4}(Z + 2X^3 - 3XY)^2. \tag{7.4}$$

This corresponds to the degenerate surface of the Enriques family of surfaces that was excluded in the list $\|1\|$.

Appendix : Projectively homogeneous plane curves and a characterization of cubic curve

In this appendix, we give a sketch on how to classify projectively homogeneous plane curves by modifying the discussion in [9], and complete the proof of Proposition 7.2.

1. Plane curves

We consider nondegenerate plane curves. We regard the curve as the immersion of a t-space into $\mathbb{P}^2$:

By taking appropriate homogeneous coordinates, each coordinate of the immersion z is a solution of an ordinary differential equation of third order, which we can write as

$$z''' + 3P_2 z' + P_3 z = 0. \tag{a.1}$$

A set of linear independent solutions $\{z^1, z^2, z^3\}$ defines an immersion $t \to [z^1(t), z^2(t), z^3(t)] \in \mathbb{P}^2$. Put $P = P_3 - (3/2)P_2'$, then the cubic form Pdt^3 is an invariant called the Laguerre-Forsyth invariant; it is independent of representation of the curve ([9, III,§1]). When $P \equiv 0$, the curve is equivalent to a quadratic curve. When $P \neq 0$, the curve has a parametrization s for which $Pdt^3 = ds^3$ and this parameter is called the projective length parameter. Relative to s, the curve is defined by the equation

$$z''' + 2kz' + (1 + k')z = 0. \tag{a.2}$$

The coefficient k is called the projective curvature.

Now assume that the curve is projectively homogeneous. Then, if the curve is not a quadratic curve, the corresponding equation is

$$z''' + 2kz' + z = 0, \tag{a.3}$$

where k is a constant. By integrating this equation, we can see that such a curve is equivalent to one of the following curves represented by inhomogeneous coordinates $[1, X, Y]$:

[P1] $Y = X^m$ where $m \neq \pm 1$;

[P2] $X = e^{\lambda s} \cos \mu s \quad Y = e^{\lambda s} \sin \mu s \quad \mu \neq 0,$ called logarithmic spiral;

[P3] $Y = Xe^X$.

The quadratic curve is included in [P1]. Since these curves are all projectively homogeneous, we end the classification:

Proposition A1. *Any nondegenerate projectively homogeneous plane curve is projectively equivalent to either a quadratic curve or one of the listed curves above.*

2. Space curves

We next consider space curves, curves in the projective 3-space, which are non degenerate in the sense that they are not included in any lower dimensional linear subspace. It is an immersion of a t-space into $\mathbb{P}^3$:

$$z : (t) \longrightarrow \mathbb{P}^3.$$

Each coordinate of the immersion z is a solution of an ordinary differential equation of fourth order, which we can write as

$$z'''' + 4p_1 z''' + 6p_2 z'' + 4p_3 z' + p_4 z = 0. \tag{a.4}$$

By a change of z to $w = \lambda z$ for an appropriate function λ, we can make w to satisfy the equation

$$w'''' + 6q_2 w'' + 4q_3 w' + q_4 w = 0. \tag{a.5}$$

where q_i are defined by the formulas:

$$
\begin{aligned}
q_2 &= p_2 - p_1' - p_1^2, \\
q_3 &= p_3 - p_1'' - 3p_1p_2 + 2p_1^3, \\
q_4 &= p_4 - 4p_1p_3 + 6p_2p_1^2 - 6p_2p_1' - 3p_1^4 + 6p_1^2p_1' + 3(p_1')^2 - p_1''',
\end{aligned}
\tag{a.6}
$$

Then we define two forms θ_3 and θ_4 by

$$
\begin{aligned}
\theta_3 &= \left(q_3 - \frac{3}{2}q_2'\right) dt^3 \\
\theta_4 &= \left(q_4 - 2q_3' - \frac{81}{25}q_2^2 + \frac{6}{5}q_2''\right) dt^4.
\end{aligned}
\tag{a.7}
$$

They are invariants to the space curve: given a transformation of coordinates and the parameter such as

$$
z \longmapsto u = \mu z, \qquad t \longmapsto s = \varphi(t),
\tag{a.8}
$$

define two forms τ_3 and τ_4 to the curve u by the same procedure. Then we have

$$
\tau_3 = \varphi^*\theta_3 \quad \text{and} \quad \tau_4 = \varphi^*\theta_4.
$$

These two invariants are fundamental for the study of space curves; for example, if $\theta_3 = \theta_4 = 0$, then the curve is necessarily a cubic curve. Refer to [9], Chapter VIII.

Now it is easy to prove the next proposition.

Proposition A2. *For a projectively homogeneous space curve in the projective space $\mathbb{P}^3$, assume that the linear isotropy of the automorphism group is at least 1-dimensional. Then the curve is a cubic curve.*

Proof. We have seen that a space curve is determined by an ordinary differential equation

$$
f'''' + 6pf'' + 4qf' + rf = 0.
\tag{a.9}
$$

Suppose there is a $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) = 0$ and

$$
b(t)f(\varphi(t)) = P(f(t))
\tag{a.10}
$$

for some projective transformation P. Then two immersion f and $bf(\varphi)$ are projectively equivalent. Hence, by the argument above, we have

$$
\varphi^*\theta_3 = \theta_3 \quad \text{and} \quad \varphi^*\theta_4 = \theta_4,
$$

where θ_3 and θ_4 are the invariants to $(a.9)$. Since $\varphi(0) = 0$, we can see $\varphi'(0) = 1$ if $\theta_3(0) \neq 0$ and $\varphi'(0) = \pm 1$ if $\theta_4(0) \neq 0$. Hence, both θ_3 and θ_4 vanish simultaneously at 0, because the linear isotropy has to be infinite. Then by the assumption that the curve is projectively homogeneous, this holds everywhere and therefore the curve is cubic. □

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