Special surfaces defined by Appell's F_4 and F_2

Takeshi SASAKI
Kobe University

Abstracts. Appell's systems F_4 and F_2 of hypergeometric differential equations define surfaces in the projective 3-space. This paper will show the variety of such surfaces in view of projective differential geometry and especially treats surfaces called isothermally asymptotic surfaces, cubic surfaces, Roman surfaces of Steiner, and projectively minimal surfaces. While the surfaces of F_4 are always isothermally asymptotic, both F_4 and F_2 have only a few examples of cubic surfaces and projectively minimal surfaces.

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§1. Special classes of projective surfaces

1-1. We recall some basic notions on surfaces in the projective 3-space by using systems of differential equations such as

$$(E1)\qquad \begin{aligned} z_{xx} &= l z_{xy} + a z_x + b z_y + p z, \\ z_{yy} &= m z_{xy} + c z_x + d z_y + q z, \end{aligned}$$

where (x, y) are independent variables and z is the unknown function. The system has four independent solutions when $lm \neq 1$; see, *e.g.*, [SY1]. We assume $lm \neq 1$ throughout the paper. In this case, the mapping

$$(x, y) \longrightarrow [z^1, z^2, z^3, z^4] \in \mathbf{P}^3$$

defines an immersion of a surface, where z^1, z^2, z^3, and z^4 are independent solutions; in the following, we call this surface the (projective) surface defined by the system. It is uniquely defined up to projective transformation, independently of the choice of solutions. The quadratic form

$$\varphi_2 = l dx^2 + 2 dx dy + m dy^2$$

is uniquely associated to the surface up to conformal factor; namely, it defines a conformal structure on the surface. Because of the assumption $lm \neq 1$, we can always find coordinates (u, v) so that this quadratic form simplifies to $dudv$; such coordinates are called *asymptotic coordinates*. Relative to an asymptotic coordinate system (u, v), the equations are reduced to the form

$$\begin{aligned} z_{uu} &= a z_u + b z_v + p z, \\ z_{vv} &= c z_u + d z_v + q z. \end{aligned}$$

Then, the cubic form called the Fubini-Pick form takes the form

$$\varphi_3 = b\, du^3 + c\, dv^3,$$

which is intrinsically defined up to a conformal factor. It vanishes if and only if the surface is quadratic. By checking the integrability condition of the system, it is easy to see that $a_v = d_u$ holds. Hence, we can reduce the system further by multiplying a suitable factor to the unknown so that it has the form

$$(E2) \qquad \begin{aligned} z_{uu} &= bz_v + pz, \\ z_{vv} &= cz_u + qz, \end{aligned}$$

which is called a normal form of $(E1)$. The integrability condition of this system is seen to be

$$\begin{aligned} 2p_v &= (b\,c)_u + b\,c_u - b_{vv}, \\ 2q_u &= (b\,c)_v + c\,b_v - c_{uu}, \\ p_{vv} + (b\,q)_v + q\,b_v &= q_{uu} + (c\,p)_u + p\,c_u. \end{aligned}$$

Let z be a 4-vector consisting of independent solutions and put

$$w = z \wedge z_x \wedge z_y,$$

which belongs to $\wedge^3\mathbf{C}^4$, the dual space of $\mathbf{C}^4$. Then, w defines an immersion called the dual immersion, or the dual surface. By referring to the system $(E2)$, w satisfies the system

$$\begin{aligned} w_{uu} &= -bw_v + (p + b_v)w, \\ w_{vv} &= -cw_u + (q + c_u)w, \end{aligned}$$

which is called the dual system.

1-2. Now let us recall the definition of certain classes of surfaces used in this paper.

The integrability condition tells us that the coefficients b and c cannot decide the coefficients p and q and that we have possibility of deforming equation while keeping the two forms φ_2 and φ_3 invariant; in other words, we have nonequivalent surfaces with the same φ_2 and φ_3. Remark that this is not the case for $n \geq 3$. Such surfaces are said to be *projectively applicable.*

To be more precise, let us replace p and q with $p + f(u)$ and $q + g(v)$ not changing b and c. Then, the integrability of the new system is reduced to

$$2gb_v + bg_v = 2fc_u + cf_u.$$

This is a linear system and, if the system is once satisfied, then $p + \lambda f$ and $q + \lambda g$ define a new integrable system. Namely, the original system has a continuous deformation.

Next class of surfaces, called *isothermally asymptotic surfaces*, is defined by the condition

$$(IA) \qquad (\log b/c)_{uv} = 0.$$

For such surfaces, we can always find asymptotic coordinates so that $b = c$ and then φ_3

is conformal to $du^3 + dv^3$. We have the following remarkable subclasses.

a: quadratic surfaces
b: cubic surfaces
c: Roman surface of Steiner

Refer to [F]; relying on this paper, we recall the condition that the surface belongs to each subclass. The subclass a) is defined by the condition $b = c = 0$, as we remarked above. The subclass b) is defined by the condition

$$(C) \qquad \begin{aligned} p &= -\frac{1}{4}\frac{\partial^2 b}{\partial u^2} + \frac{5}{16}\left(\frac{\partial \log b}{\partial u}\right)^2 + \frac{3}{4}\frac{\partial b}{\partial v}, \\ q &= -\frac{1}{4}\frac{\partial^2 b}{\partial v^2} + \frac{5}{16}\left(\frac{\partial \log b}{\partial v}\right)^2 + \frac{3}{4}\frac{\partial b}{\partial u}. \end{aligned}$$

See [L, p.477]. The subclass c) is defined by

$$(S1) \qquad 9\frac{\partial^2 \log b}{\partial u \partial v} - 4b^2 = 0,$$

$$(S2) \qquad p = -\frac{1}{2}\frac{\partial^2 \log b}{\partial u^2} + \frac{1}{16}\left(\frac{\partial \log b}{\partial u}\right)^2 - \frac{7}{4}\frac{\partial b}{\partial v},$$

$$(S3) \qquad q = -\frac{1}{2}\frac{\partial^2 \log b}{\partial t^2} + \frac{1}{16}\left(\frac{\partial \log b}{\partial v}\right)^2 - \frac{7}{4}\frac{\partial b}{\partial u}.$$

Refer to [F] or [B, p.149–150].

As for the third class of surfaces in this paper, we recall the notion of projectively minimal surfaces. The integral

$$\int bc\, dudv$$

is a well-defined functional for projective surfaces and any critical surface relative to this functional is called a *projectively minimal surface*. For its intrinsic treatment, refer to [B] and [S]. The condition for the surface to be projectively minimal is

$$(P) \qquad \begin{aligned} b_{vvv} &= 4b_v q + 2bq_v + 2b_v c_u + bc_{uv}, \\ c_{uuu} &= 4c_u p + 2cp_u + 2b_v c_u + cb_{uv}. \end{aligned}$$

The last class of surfaces we mention is Demoulin surface. To any given surface, we can generally associate four surfaces that are defined as envelope surfaces of the family of Lie quadrics. Each surface is called a Demoulin transform of the original surface. The theory of Demoulin transformation is remarkable in connection with projectively minimal surfaces; see, *e.g.*, [S]. By definition, a projective surface is called a *Demoulin surface* if all Demoulin transforms fall into one surface. It is characterized by the condition

$$(D) \qquad \begin{aligned} p &= \frac{1}{2}\frac{\partial^2 \log c}{\partial^2 u} + \frac{1}{4}\left(\frac{\partial \log c}{\partial u}\right)^2 - \frac{1}{2}\frac{\partial b}{\partial v}, \\ q &= \frac{1}{2}\frac{\partial^2 \log b}{\partial^2 v} + \frac{1}{4}\left(\frac{\partial \log b}{\partial v}\right)^2 - \frac{1}{2}\frac{\partial c}{\partial u}. \end{aligned}$$

§2. Surfaces defined by the system F_4

2-1. Appell's system F_4 is
(F_4)

$$\begin{aligned} z_{xx} &= \frac{2y}{1-x-y}z_{xy} + \frac{(a+b+1)x - c_1(1-y)}{x(1-x-y)}z_x + \frac{(a+b+1-c_2)y}{x(1-x-y)}z_y + \frac{ab}{x(1-x-y)}z \\ z_{yy} &= \frac{2x}{1-x-y}z_{xy} + \frac{(a+b+1-c_1)x}{y(1-x-y)}z_x + \frac{(a+b+1)y - c_2(1-x)}{y(1-x-y)}z_y + \frac{ab}{y(1-x-y)}z. \end{aligned}$$

When we need to refer to complex parameters a, b, c_1, and c_2 explicitly, we denote the system by $F_4(a,b,c_1,c_2)$. It has an apparent symmetry relative to the exchange of a and b. It is also symmetric by exchanging c_1 and c_2, and simultaneously x and y. The dual system of $F_4(a,b,c_1,c_2)$ is $F_4(1-a,1-b,2-c_1,2-c_2)$; Refer to [SY2]. The Appell's function F_4 is defined by the series

$$F_4(a,b,c_1,c_2;x,y) = \sum_{m,n=0}^{\infty} \frac{(a,m+n)(b,m+n)}{(c_1,m)(c_2,n)\,m!\,n!}x^m y^n,$$

where $(a,m) = a(a+1)\cdots(a+m-1)$ is the Pochhammer symbol. When $c_1 \neq 1$ and $c_2 \neq 1$, the system $F_4(a,b,c_1,c_2)$ has four independent solutions of the form

$$\begin{aligned} &F_4(a,b,c_1,c_2;x,y), \\ &x^{1-c_1}F_4(a+1-c_1,b+1-c_1,2-c_1,c_2;x,y), \\ &y^{1-c_2}F_4(a+1-c_2,b+1-c_2,c_1,2-c_2;x,y), \\ &x^{1-c_1}y^{1-c_2}F_4(a+2-c_1-c_2,b+2-c_1-c_2,2-c_1,2-c_2;x,y); \end{aligned}$$

Refer to [AKF]. From this representation, we see that the systems $F_4(a,b,c_1,c_2)$, $F_4(a+1-c_1,b+1-c_1,2-c_1,c_2)$, $F_4(a+1-c_2,b+1-c_2,c_1,2-c_2)$, and $F_4(a+2-c_1-c_2,b+2-c_1-c_2,2-c_1,2-c_2)$ are equivalent in the sense that the corresponding projective surfaces are equivalent to each other by projective transformations, as far as $c_1 \neq 1$ and $c_2 \neq 1$.

By looking at the quadratic form φ_2 that is conformal to

$$ydx^2 + (1-x-y)dxdy + xdy^2,$$

we can see that an asymptotic coordinate system (u,v) is given, say, by

$$(x,y) = (u(1-v), v(1-u)). \tag{1}$$

The mapping $(u,v) \to (x,y)$ is two-fold and branched along the divisor $\{(1-x-y)^2-4xy = 0\}$, where the form φ_2 degenerates. Referring to the coordinates (u,v), the system is written as

$$\begin{aligned} z_{uu} &= \left(\frac{c_2}{1-u} - \frac{c_1}{u} - \frac{e}{1-u-v}\right)z_u - \frac{ev(1-v)}{u(1-u)(1-u-v)}z_v + \frac{ab}{u(1-u)}z, \\ z_{vv} &= -\frac{eu(1-u)}{v(1-v)(1-u-v)}z_u + \left(\frac{c_1}{1-v} - \frac{c_2}{v} - \frac{e}{1-u-v}\right)z_v + \frac{ab}{v(1-v)}z, \end{aligned} \tag{2}$$

where

$$e = c_1 + c_2 - a - b - 1.$$

Hence, the cubic form is

$$\varphi_3 = -e\left(\frac{v(1-v)}{u(1-u)(1-u-v)}du^3 + \frac{u(1-u)}{v(1-v)(1-u-v)}dv^3\right)$$

and the projective surface is quadratic if and only if $e = 0$; see [SY1]. By multiplying a suitable factor to the unknown, the system is reduced to

$$\begin{aligned} z_{uu} &= -\frac{ev(1-v)}{u(1-u)(1-u-v)}z_v + pz, \\ z_{vv} &= -\frac{eu(1-u)}{v(1-v)(1-u-v)}z_u + qz, \end{aligned} \tag{3}$$

where

$$\begin{aligned} p &= \frac{ab}{u(1-u)} + \frac{1}{4}\left(\frac{c_2}{1-u} - \frac{c_1}{u} - \frac{e}{1-u-v}\right)^2 \\ &\quad + \frac{1}{2}\left(\frac{e(1-v)}{1-u} - \frac{ev}{u} - \frac{e}{1-u-v}\right)\left(\frac{c_1}{1-v} - \frac{c_2}{v} - \frac{e}{1-u-v}\right) \\ &\quad - \frac{1}{2}\frac{c_2}{(1-u)^2} - \frac{1}{2}\frac{c_1}{u^2} + \frac{1}{2}\frac{e}{(1-u-v)^2}, \\ q &= \frac{ab}{v(1-v)} + \frac{1}{4}\left(\frac{c_1}{1-v} - \frac{c_2}{v} - \frac{e}{1-u-v}\right)^2 \\ &\quad + \frac{1}{2}\left(\frac{e(1-u)}{1-v} - \frac{eu}{v} - \frac{e}{1-u-v}\right)\left(\frac{c_2}{1-u} - \frac{c_1}{u} - \frac{e}{1-u-v}\right) \\ &\quad - \frac{1}{2}\frac{c_1}{(1-v)^2} - \frac{1}{2}\frac{c_2}{v^2} + \frac{1}{2}\frac{e}{(1-u-v)^2}. \end{aligned}$$

Here is an important fact on the system that the cubic form depends only on the value e, which means that we have a three-parameter family of surfaces that are applicable to each other. In fact, by defining

$$\begin{aligned} p_0 &= \frac{e}{2(1-u-v)^2} + \frac{e^2}{4}\left(\frac{2}{u(1-v)} + \frac{2}{v(1-u)} - \frac{2(1-2v)}{v(1-v)(1-u-v)} + \frac{3}{(1-u-v)^2}\right), \\ q_0 &= \frac{e}{2(1-u-v)^2} + \frac{e^2}{4}\left(\frac{2}{u(1-v)} + \frac{2}{v(1-u)} - \frac{2(1-2u)}{u(1-u)(1-u-v)} + \frac{3}{(1-u-v)^2}\right), \end{aligned}$$

the coefficients

$$\begin{aligned} p_1 &= p_0 + \frac{a_1}{u(1-u)} + \frac{a_2}{u^2} + \frac{a_3}{(1-u)^2}, \\ q_1 &= q_0 + \frac{a_1}{v(1-v)} + \frac{a_3}{v^2} + \frac{a_2}{(1-v)^2}, \end{aligned}$$

together with the cubic coefficients b and c above define an integrable system where a_1, a_2, and a_3 are arbitrary constant. The relation with the F_4-parameters is

$$a_1 = (2ab - c_1c_2 + (a+b+1)e)/2, \quad a_2 = c_1(c_1-2)/4, \quad a_3 = c_2(c_2-2)/4.$$

It is known that the quadratic form $2bc\,dudv = 2e^2/(1-u-v)^2\,dudv$ is an absolute invariant called the projective metric. The Gaussian curvature $-(\partial^2(2bc)/\partial u\partial v)/(2bc)$ is always constant $-1/e^2$. Remark that a projective surface with a 3-parameter family of projective applicability is isothermally asymptotic and the Gauss curvature of the projective metric is constant; refer to [B, §119]. The system F_4 thus is an example of this phenomenon.

We next introduce another coordinates (s,t) so that the cubic form is conformal to ds^3+dt^3. It is enough to define (s,t) by integrating the identities

$$\frac{ds}{du} = (u(1-u))^{-2/3}, \qquad \frac{dt}{dv} = (v(1-v))^{-2/3}.$$

With respect to these coordinates, the system can be written in the form

$$z_{ss} = Bz_t + Pz \qquad z_{tt} = Bz_s + Qz, \tag{4}$$

where

$$\begin{aligned} B &= -\frac{e(uv(1-u)(1-v))^{1/3}}{1-u-v}, \\ P &= (u(1-u))^{4/3}P_0, \\ Q &= (v(1-v))^{4/3}Q_0, \end{aligned}$$

and

$$\begin{aligned} P_0 &= \frac{18ab+4-12e-9(e-c_1)(e-c_2)}{18}\left(\frac{1}{1-u}+\frac{1}{u}\right) \\ &\quad +\frac{e(6+9e)}{18}\left(\frac{1}{v(1-u)}+\frac{1}{u(1-v)}\right) \\ &\quad +\frac{8-18c_2+9c_2^2}{36(1-u)^2}+\frac{8-18c_1+9c_1^2}{36u^2} \\ &\quad -\frac{e(2+3e)(1-2v)}{6v(1-v)(1-u-v)}+\frac{e(2+3e)}{4(1-u-v)^2}, \\ Q_0 &= \frac{18ab+4-12e-9(e-c_1)(e-c_2)}{18}\left(\frac{1}{1-v}+\frac{1}{v}\right) \\ &\quad +\frac{e(6+9e)}{18}\left(\frac{1}{v(1-u)}+\frac{1}{u(1-v)}\right) \\ &\quad +\frac{8-18c_1+9c_1^2}{36(1-v)^2}+\frac{8-18c_2+9c_2^2}{36v^2} \\ &\quad -\frac{e(2+3e)(1-2u)}{6u(1-u)(1-u-v)}+\frac{e(2+3e)}{4(1-u-v)^2}. \end{aligned}$$

2-2. With these preparations, let us consider the cases where the projective surface defined by F_4 belongs to special classes of surfaces we recalled in §1-2. Since any surface of F_4 is isothermally asymptotic, we can apply the condition (C) to the system (4). By a direct computation, it is seen that the surface is a cubic surface only when

$$\begin{aligned}(a,b,c_1,c_2)=\ &(1,1/2,1/2,1/2), && (1,3/2,1/2,3/2), && (1,3/2,3/2,1/2),\\ &(2,3/2,3/2,3/2), && (1/3,-1/6,1/2,1/2), && (1/3,5/6,1/2,3/2),\\ &(1/3,5/6,3/2,1/2), && (4/3,5/6,3/2,3/2), && (1/2,1,1/2,1/2),\\ &(3/2,1,1/2,3/2), && (3/2,1,3/2,1/2), && (3/2,2,3/2,3/2),\\ &(5/6,1/3,1/2,3/2), && (5/6,1/3,3/2,1/2), && (5/6,4/3,3/2,3/2),\\ &(-1/6,1/3,1/2,1/2).\end{aligned}$$

By the apparent symmetry of F_4 mentioned above and by the equivalence explained in §2-1, the cases we should consider reduce to

$$(a,b,c_1,c_2)=(1,1/2,1/2,1/2),\quad (1/3,-1/6,1/2,1/2). \tag{5}$$

In the first case, the monodromy group is reducible and finite (see [K1]); independent solutions are

$$\begin{aligned}z_1&=F_4(1,1/2,1/2,1/2;x,y)=\frac{1-x-y}{(1-x-y)^2-4xy},\\ z_2&=x^{1/2}F_4(3/2,1,3/2,1/2;x,y)=\frac{\sqrt{x}(1-x+y)}{(1-x-y)}z_1,\\ z_3&=y^{1/2}F_4(3/2,1,1/2,3/2;x,y)=\frac{\sqrt{y}(1+x-y)}{(1-x-y)}z_1,\\ z_4&=x^{1/2}y^{1/2}F_4(2,3/2,3/2,3/2;x,y)=\frac{\sqrt{xy}}{1-x-y}z_1.\end{aligned}$$

Then, computation shows the identity

$$z_1^2z_4-z_1z_2z_3+z_2^2z_4+z_3^2z_4-4z_4^3=0.$$

In terms of inhomogeneous coordinates (X,Y,Z) of $\mathbf{P}^3$, it is

$$X^2+Y^2+Z^2=XYZ+4. \tag{6}$$

For the second case, the monodromy group is finite and irreducible; this case belongs to Example 8.3 of [K2]. We put

$$z_1=F_4(1/3,-1/6,1/2,1/2;x,y),\quad z_2=\frac{x^{1/2}}{3}F_4(5/6,1/3,3/2,1/2;x,y),$$

$$z_3=\frac{y^{1/2}}{3}F_4(5/6,1/3,1/2,3/2;x,y),\quad z_4=\frac{x^{1/2}y^{1/2}}{9}F_4(4/3,5/6,3/2,3/2;x,y).$$

These solutions satisfy the identity

$$3z_1^2z_4-3z_1z_2z_3+3z_2^2z_4+3z_3^2z_4+4z_4^3=0.$$

In terms of inhomogeneous coordinates

$$X^2+Y^2+Z^2=XYZ-\frac{4}{3}. \tag{7}$$

We next check the condition (S) so that the surface is a Roman surface of Steiner. Since the identity $(S1)$ means that the curvature is equal to -4/9, we see that $e=3/2$ or $e=-3/2$ is necessary. When $e=3/2$, the conditions $(S2)$ and $(S3)$ imply

$$\begin{aligned}(a,b,c_1)= \quad &(1/2,0,3/2), \quad (-1/2,0,3/2), \quad (0,-1/2,3/2), \quad (0,1/2,3/2),\\ &(-1/2,0,1/2), \quad (-1/2,-1,1/2), \quad (0,-1/2,1/2), \quad (-1,-1/2,1/2),\end{aligned}$$

When $e=-3/2$, we have no solution. Hence, taking into account the symmetry, we need to consider only the case where

$$(a,b,c_1,c_2)=(-1/2,-1,1/2,1/2).$$

Since the independent solutions of $F_4(-1/2,-1,1/2,1/2)$ are

$$\begin{aligned} z_1 &= F_4(-1/2,-1,1/2,1/2)=1+x+y,\\ z_2 &= x^{1/2}F_4(0,-1/2,3/2,1/2)=x^{1/2},\\ z_3 &= y^{1/2}F_4(0,-1/2,1/2,3/2)=y^{1/2},\\ z_4 &= x^{1/2}y^{1/2}F_4(1/2,0,3/2,3/2)=x^{1/2}y^{1/2},\end{aligned}$$

this case is rather trivial. Anyway, it satisfies the quartic relation

$$-z_1z_2z_3z_4+z_2^2z_3^2+z_2^2z_4^2+z_3^2z_4^2=0. \tag{8}$$

Remarks. **1**. The dual of the system $F_4(1/2,0,3/2,3/2)$ is equivalent to the system $F_4(1,1/2,1/2,1/2)$, one of cubic cases.

2. In [F], Ferapontov summarized the condition so that the surface belongs to Kummer's quartics when $b=c$:

$$\begin{aligned} &9\frac{\partial^2\log b}{\partial u\partial v}-4b^2=0,\\ p &= \frac{11}{16}\frac{\partial^2\log b}{\partial u^2}+\left(\frac{\partial\log b}{\partial u}\right)^2-\frac{1}{2}\frac{\partial b}{\partial v},\\ q &= \frac{11}{16}\frac{\partial^2\log b}{\partial v^2}+\left(\frac{\partial\log b}{\partial v}\right)^2-\frac{1}{2}\frac{\partial b}{\partial u}.\end{aligned} \tag{K}$$

By computation, it is seen that, for the system F_4, the surface becomes Kummer's quartics only when it is the Roman surface above.

3. The system F_4 can define several quartic surfaces. In [K2], it was shown that $(a,b,c_1,c_2)=(3/8,-1/8,1/2,1/2)$ and $(1/4,-1/12,1/2,1/2)$ are such cases. The problem to find all quartic surfaces represented by F_4 is open.

We next check the condition for projective minimality. By computation, the condition (P) leads to the two cases

$$(a, b, c_1, c_2) = (1, 1, 1, 1), \quad (0, 0, 1, 1).$$

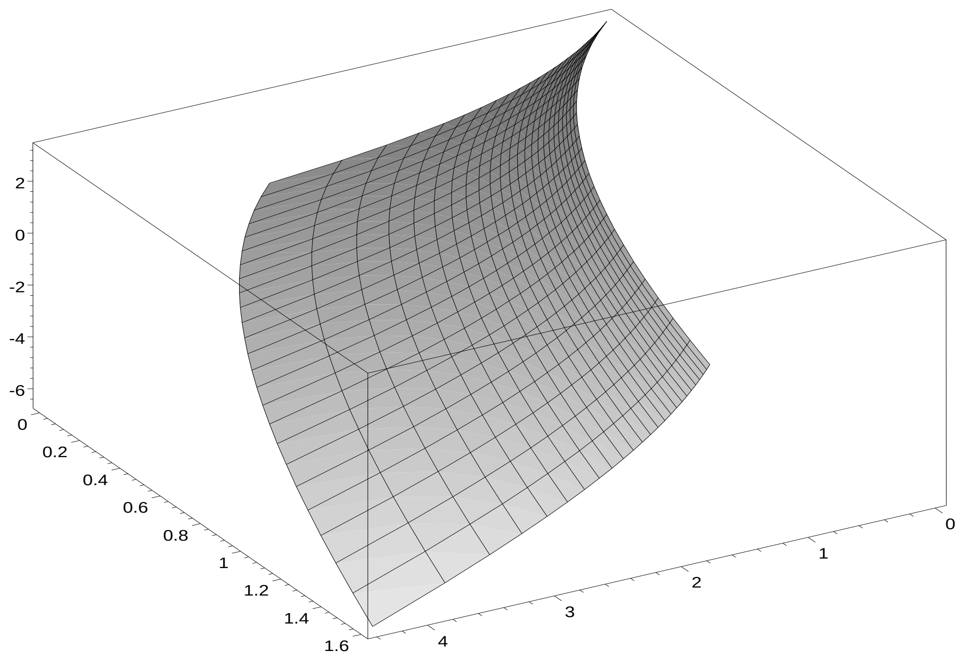
Let us derive the explicit form of the surface for the first case. The independent solutions of $F_4(1, 1, 1, 1)$ are

$$\begin{aligned} z^1 &= 1/(1-u-v) = F_4(1,1,1,1;x,y) \\ z^2 &= (\log(1-u) - \log(v))/(1-u-v) \\ z^3 &= (\log(1-v) - \log(u))/(1-u-v) \\ z^4 &= (2L(u) + 2L(v) + \log(1-u)\log(1-v) - \log(u)\log(v))/(1-u-v), \end{aligned}$$

where

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} + \frac{1}{2}\log(x)\log(1-x)$$

is the dilog function. The surface looks like as in the picture.

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plot3d([z2/z1,z3/z1,z4/z1],u=1/100..1/5,v=1/5..4/5
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For the case $F_4(0, 0, 1, 1)$, it is enough to note that the system is dual to $F_4(1, 1, 1, 1)$. One choice of independent solutions are

$$w^1 = 1,$$

$$
\begin{aligned}
w^2 &= \log(1-u)+\log(v),\\
w^3 &= \log(u)+\log(1-v).\\
w^4 &= 2L(u)+2L(v)+\log(u)\log(v)-\log(1-u)\log(1-v),
\end{aligned}
$$

Remark **5**. Both surfaces above are improper affine hyperspheres as well as Demoulin surfaces. The function $z^4/z^1 = 2L(u)+2L(v)+\log(1-u)\log(1-v)-\log(u)\log(v)$ as the function of $X = z^2/z^1 = \log(1-u)-\log(v)$ and $Y = z^3/z^1 = \log(1-v)-\log(u)$ is the Legendre transformation of the function w^4 as the function of $U = w^2 = \log(1-u)+\log(v)$ and $V = w^3 = \log(1-v)+\log(u)$, and *vice visa*.

§3. Surfaces defined by the system F_2

3-1. Appell's system F_2 is

$$
(F_2)\qquad
\begin{aligned}
z_{ss} &= \frac{t}{1-s}z_{st}+\frac{(a+b_1+1)s-c_1)}{s(1-s)}z_s+\frac{b_1 t}{s(1-s)}z_t+\frac{ab_1}{s(1-s)}z,\\
z_{tt} &= \frac{s}{1-t}z_{st}+\frac{b_2 s}{t(1-t)}z_s+\frac{(a+b_2+1)t-c_2}{t(1-t)}z_t+\frac{ab_2}{r(1-t)}z,
\end{aligned}
$$

in the coordinates (s,t). We denote the system by $F_2(a,b_1,b_2,c_1,c_2)$. It is symmetric by exchanging b_1 and b_2, c_1 and c_2, and simultaneously s and t. The dual system of $F_2(a,b_1,b_2,c_1,c_2)$ is $F_2(1-a,1-b_1,1-b_2,2-c_1,2-c_2)$; Refer to [SY2]. The Appell's function F_2 is defined by the series

$$
F_2(a,b_1,b_2,c_1,c_2;s,t)=\sum_{m,n=0}^{\infty}\frac{(a,m+n)(b_1,m)(b_2,n)}{(c_1,m)(c_2,n)\,m!\,n!}s^m t^n.
$$

When $c_1\neq 1$ and $c_2\neq 1$, the system $F_2(a,b_1,b_2,c_1,c_2)$ has four independent solutions of the form

$$
\begin{aligned}
&F_2(a,b_1,b_2,c_1,c_2;s,t),\\
&s^{1-c_1}F_2(a+1-c_1,b_1+1-c_1,b_2,2-c_1,c_2;s,t),\\
&t^{1-c_2}F_2(a+1-c_2,b_1,b_2+1-c_2,c_1,2-c_2;s,t),\\
&s^{1-c_1}t^{1-c_2}F_2(a+2-c_1-c_2,b_1+1-c_1,b_2+1-c_2,2-c_1,2-c_2;s,t).
\end{aligned}
$$

Refer to [AKF]. From this representaion, we see that the systems $F_2(a,b_1,b_2,c_1,c_2)$, $F_2(a+1-c_1,b_1+1-c_1,b_2,2-c_1,c_2)$, $F_2(a+1-c_2,b_1,b_2+1-c_2,c_1,2-c_2)$, and $F_4(a+2-c_1-c_2,b_1+1-c_1,b_2+1-c_2,2-c_1,2-c_2)$ are equivalent.

Change the coordinates (s,t) to (x,y) by

$$
x=\left(\frac{s}{s+t-2}\right)^2,\qquad y=\left(\frac{t}{s+t-2}\right)^2.
$$

Then, the quadratic form transforms to that of F_4: Hence, we can use again the coordinate system (u,v), given in (1), as an asymptotic coordinate system of F_2. The cubic form

$\varphi_3 = Bdu^3 + Cdv^3$ has the following coefficients.

$$\begin{aligned} B &= -e_1 \frac{v(1-v)^2(uv-v-\eta\xi+\eta)}{2u(1-u)(1-u-v)\xi\eta(1-\xi-\eta)} - e_2 \frac{v^2(1-v)(uv-u-\eta\xi+\xi)}{2u(1-u)(1-u-v)\xi\eta(1-\xi-\eta)} \\ &\quad + e_3 \frac{v(1-v)(\xi uv+\eta uv-\xi v-\eta u+\eta\xi)}{2u(1-u)(1-u-v)\xi\eta(1-\xi-\eta)}, \\ C &= -e_1 \frac{u^2(1-u)(uv-v-\eta\xi+\eta)}{2v(1-v)(1-u-v)\xi\eta(1-\xi-\eta)} - e_2 \frac{u(1-u)^2(uv-u-\eta\xi+\xi)}{2v(1-v)(1-u-v)\xi\eta(1-\xi-\eta)} \\ &\quad + e_3 \frac{(1-u)u(\xi uv+\eta uv-\xi v-\eta u+\eta\xi)}{2v(1-v)(1-u-v)\xi\eta(1-\xi-\eta)}, \end{aligned}$$

where

$$e_1 = 2b_1 - c_1, \quad e_2 = 2b_2 - c_2, \quad e_3 = 2a + 1 - c_1 - c_2,$$
$$\xi = (u(1-v))^{1/2}, \qquad \eta = (v(1-u))^{1/2}.$$

This form of the cubic form says that the systems F_2 have the same fundamental forms φ_2 and φ_3 as long as the values of e_1, e_2, and e_3 remain invariant; that is, F_2 is also projectively applicable with two-parameter family of deformations.

3-2. We sketch the computations for F_2 similar to those computations done for F_4. From the expression of the cubic form, the surface is quadratic if and only if $e_1 = e_2 = e_3 = 0$. The isothermally asymptotic condition (IA) holds if and only if

$$\begin{array}{lll} (1)\ e_1 = e_2 = 0 & (2)\ e_1 = e_3 = 0 & (3)\ e_2 = e_3 = 0 \\ (4)\ e_1 = 0, e_3 = \pm e_2 & (5)\ e_3 = 0, e_2 = \pm e_1 & (6)\ e_2 = 0, e_3 = \pm e_1. \end{array}$$

Here is a remark that the system F_2 reduces to F_4 when the condition (1) holds; see p. 242 of [E] or [SY1]. Another remark is that the formula (31) of p. 25 of [AKF]

$$F_2(a, b_1, b_2, c_1, c_2; s, t) = (1-t)^{-a} F_2(a, b_1, c_2 - b_2, c_1, c_2; s/(1-t), t/(t-1))$$

says that the system $F_2(a, b_1, b_2, c_1, c_2)$ is equivalent to the system $F_2(a, b_1, c_2 - b_2, c_1, c_2)$, which means the symmetry $(e_1, e_2, e_3) \leftrightarrow (-e_1, e_2, e_3)$. The formula

$$F_2(a, b_1, b_2, c_1, c_2; s, t) = (1-s)^{-a} F_2(a, c_1 - b_1, b_2, c_1, c_2; s/(s-1), t/(1-s))$$

implies the symmetry $(e_1, e_2, e_3) \leftrightarrow (e_1, -e_2, e_3)$. Further, we have an apparent symmetry $(e_1, e_2, e_3) \leftrightarrow (e_2, e_1, e_3)$. Hence, up to symmetries above, the isothermally asymptotic conditions that are proper to F_2 reduce to the three cases

$$\text{(i) } e_3 = 0, e_1 = e_2, \quad \text{(ii) } e_2 = e_3 = 0, \quad \text{(iii) } e_1 = 0, e_2 = e_3.$$

How about the Gaussian curvature of the projective metric? Computation shows that the curvature can be constant only when the surface is isothermally asymptotic. Its value is equal to $-1/e_2^2$ for (i) and (iii) and to $-4/e_1^2$ for (ii).

Now let us check the cubic condition (C): For (i) and (ii), we get no cubic surfaces. For (iii), the solutions up to symmetry consist of two:

$$(a, b_1, b_2, c_1, c_2; e_2) = (1)\ \ (1, 0, 3/2, 0, 3/2; 3/2) \qquad (2)\ \ (1/3, 0, 5/6, 0, 3/2, 1/6).$$

The independent solutions for the case (1) is

$$z_1 = F_2(1,0,3/2,0,3/2), \quad z_2 = sF_2(2,1,3/2,2,3/2),$$

$$z_3 = t^{-1/2}F_2(1/2,0,1,0,1/2), \quad z_4 = st^{-1/2}F_2(3/2,1,1,2,1/2)$$

and the corresponding cubic surface is

$$z_1z_3z_4 + z_1z_4^2 - z_2z_3^2 - z_1^2z_2 - z_1z_2^2 = 0.$$

For the case (2), by taking

$$z_1 = F_2(1/3,0,5/6,0,3/2), \quad z_2 = sF_2(4/3,1,5/6,2,3/2)/3,$$

$$z_3 = -3t^{-1/2}F_2(-1/6,0,1/3,0,1/2), \quad z_4 = st^{-1/2}F_2(5/6,1,1/3,2,1/2)$$

as independent solutions, we get the cubic surface

$$4z_1z_3z_4 + z_1z_4^2 + 4z_2z_3^2 + 4z_2z_3z_4 + 4z_1^2z_2 + 4z_1z_2^2 + \frac{4}{3}z_2^3 + z_2z_4^2 = 0.$$

The final computation is on the projective minimality. Apply the condition (P); the computation, fairly long, shows that the surface $F_2(a,b_1,b_2,c_1,c_2)$ is projectively minimal if and only if it holds

$$(a,b_1,b_2,c_1,c_2) = \begin{array}{ll} (1)\ (0,1/2,0,1,1) & \qquad (2)\ (1,1/2,1,1,1) \\ (3)\ (1,1/2,2,1,2) & \qquad (4)\ (1/2,1,1,1,1). \end{array}$$

We here rely on the special identity in [AKF, p. 27]

$$F_4(\alpha,\alpha+1/2-\beta,\gamma,\beta+1/2;x,y) = \\ (1+\sqrt{y})^{-2\alpha}F_2(\alpha,\alpha+1/2-\beta,\beta,\gamma,2\beta;\frac{x}{(1+\sqrt{y})^2},\frac{4\sqrt{y}}{(1+\sqrt{y})^2}),$$

which implies, when $\alpha = 0$, $\beta = 1/2$, and $\gamma = 1$, the equivalence of $F_2(0,1/2,0,1,1)$ with $F_4(0,0,1,1)$ and, when $\alpha = 1$, $\beta = 1/2$, and $\gamma = 1$, its dual $F_2(1,1/2,1,1,1)$ with $F_4(1,1,1,1)$. Hence, the cases (3) and (4) are to be considered. However, it is not easy to give exact solutions of the system; we just mention that the monodromy group of the system corresponding to these cases is reducible according to the criterion in [K3] and, in fact, the subsystem is described by Appell's F_1.

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