HYPERBOLIC SCHWARZ MAP FOR THE HYPERGEOMETRIC DIFFERENTIAL EQUATION

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Abstract. The Schwarz map of the hypergeometric differential equation is studied since the beginning of the last century. Its target is the complex projective line, the 2-sphere. This paper introduces the *hyperbolic Schwarz map*, whose target is the hyperbolic 3-space. This map can be considered to be a lifting to the 3-space of the Schwarz map. This paper studies the singularities of this map, and visualize its image when the monodromy group is a finite group or a typical Fuchsian group. General cases will be treated in a forthcoming paper.

Contents

1. Introduction

Consider the *hypergeometric differential equation*

$$E(a,b,c) \qquad x(1-x)u'' + \{c-(a+b+1)x\}u' - abu = 0,$$

and define its *Schwarz map* by

$$s : X = \boldsymbol{C} - \{0,1\} \ni x \longmapsto u_0(x) : u_1(x) \in Z \cong \boldsymbol{P}^1, \tag{1.1}$$

where u_0 and u_1 are linearly independent solutions of $E(a,b,c)$ and $\boldsymbol{P}^1$ is the complex projective line. The Schwarz map of the hypergeometric differential equation is studied by Schwarz when the parameters (a,b,c) are real.

In general, for a system $dU = U\Omega$ of linear differential equations of rank 2 defined on X, where Ω is a 2×2-matrix 1-form holomorphic in X, we define its *hyperbolic Schwarz map*, denoted by $\mathscr{S}$, as the composition of the (multi-valued) map

$$X \ni x \longmapsto H = U(x)\,{}^t\overline{U}(x) \in \mathrm{Her}^+(2) \tag{1.2}$$

and the natural projection $\mathrm{Her}^+(2) \to \boldsymbol{H}^3 := \mathrm{Her}^+(2)/\boldsymbol{R}^+$, where $U(x)$ is a fundamental solution of the system, $\mathrm{Her}^+(2)$ the space of positive-definite Hermitian matrices of size 2, and $\boldsymbol{R}^+$ the multiplicative group of positive real numbers; the space $\boldsymbol{H}^3$ is called the *hyperbolic* 3-*space*. Note that the target of the hyperbolic Schwarz map is $\boldsymbol{H}^3$, whose boundary is $\boldsymbol{P}^1$, which is the target of the Schwarz

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map. In this sense, our hyperbolic Schwarz map is a lift-to-the-air of the Schwarz map. Note also that the monodromy group of the system acts naturally on $\boldsymbol{H}^3$.

There are no standard way to transform our equation $E(a,b,c)$ into a matrix system. In this paper, we transform $E(a,b,c)$ into the so-called SL-form:

$$E^{\mathrm{SL}}(a,b,c) \qquad\qquad u'' - q(x)u = 0,$$

and define the matrix equation by

$$d(u,u') = (u,u')\Omega, \quad \Omega = \begin{pmatrix} 0 & q(x) \\ 1 & 0 \end{pmatrix}. \tag{1.3}$$

The image surface (of X under $\mathscr{S}$) in $\boldsymbol{H}^3$ is one of the *flat fronts*, which is a flat surface with a certain kind of singularities, studied in [Kokubu et al. 2003, Kokubu et al. 2005]. The classical Schwarz map is considered as the *hyperbolic Gauss map* of the hyperbolic Schwarz map as a flat front.

Note that the symmetry of the equation $E(a,b,c)$, which descends to the Schwarz map, does not necessarily descends to the hyperbolic Schwarz map; this is a cost for the lift to the 3-space.

We study the hyperbolic Schwarz map $\mathscr{S}$ of the equation $E(a,b,c)$ when the parameters (a,b,c) are real, especially when its monodromy group is a finite (polyhedral) group or a Fuchsian group. In general, generic singularities of flat fronts are cuspidal edges and swallowtails, see Section 4. In our special case, we find that, in each case, there is a simple closed curve C in X around ∞, and two points

$$P^{\pm} \in X^{\pm} \cap C, \quad X^{\pm} = \{x \in X \mid \pm\Im x > 0\},$$

such that the image surface has cuspidal edges only along $\mathscr{S}(C - \{P^+, P^-\})$, and has swallowtails only at $\mathscr{S}(P^{\pm})$. We made our best to visualize the image surfaces; we often show part of the surfaces, several copies of the images of $X^{\pm}$, since each of the images of the three intervals $(-\infty,0),(0,1)$ and $(1,+\infty)$ lies on a totally geodesic surface in $\boldsymbol{H}^3$. Basic ingredients of the hypergeometric function and its Schwarz map can be found in [Iwasaki et al. 1991] and [Yoshida 1997].

When the parameters are not real and the monodromy group is a Schottky group or a Kleinian group, the hyperbolic Schwarz map of $E(a,b,c)$ is studied in a forthcoming paper.

2. Preliminaries

2.1. Models of the hyperbolic 3-space. The hyperbolic 3-space $\boldsymbol{H}^3 = \mathrm{Her}^+(2)/\boldsymbol{R}^+$ can be identified with the upper half-space $\boldsymbol{C} \times \boldsymbol{R}^+$ as

$$\boldsymbol{C} \times \boldsymbol{R}^+ \ni (z,t) \longmapsto \begin{pmatrix} t^2 + |z|^2 & \bar{z} \\ z & 1 \end{pmatrix} \in \mathrm{Her}^+(2),$$

$$\begin{pmatrix} h & \bar{w} \\ w & k \end{pmatrix} \in \mathrm{Her}^+(2) \longmapsto \boldsymbol{C} \times \boldsymbol{R}^+ \ni \frac{1}{k}\Big(w, \sqrt{hk - |w|^2}\Big).$$

It can be also identified with a subvariety

$$L_1 = \{x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1\}$$

of the Lorentz-Minkowski 4-space

$$L(+,-,-,-) = \big\{(x_0,x_1,x_2,x_3) \in \boldsymbol{R}^4 \mid x_0^2 - x_1^2 - x_2^2 - x_3^2 > 0, x_0 > 0\big\}$$

by

$$\mathrm{Her}^+(2) \ni \begin{pmatrix} h & \bar{w} \\ w & k \end{pmatrix} \longmapsto \frac{1}{2\sqrt{hk - |w|^2}}\left(h + k, w + \bar{w}, \frac{w - \bar{w}}{i}, h - k\right) \in L_1$$

and with the Poincaré ball

$$B_3 = \{(x_1, x_2, x_3) \in \boldsymbol{R}^3 \mid x_1^2 + x_2^2 + x_3^2 < 1\},$$

by

$$L_1 \ni (x_0, x_1, x_2, x_3) \longmapsto \frac{1}{1+x_0}(x_1, x_2, x_3) \in B_3.$$

We use these models according to convenience.

2.2. **Local exponents and transformation into the SL-form.** The local exponents of the equation $E(a,b,c)$ at $0, 1$ and ∞ are given as $\{0, 1-c\}, \{0, 1-a-b\}$ and $\{a, b\}$, respectively. Denote the differences of the local exponents by

$$\mu_0 = 1 - c, \quad \mu_1 = c - a - b, \quad \mu_\infty = b - a. \tag{2.1}$$

The equation $E(a,b,c)$ transforms into the SL-form $E^{\mathrm{SL}}(a,b,c)$ with

$$q = -\frac{1}{4}\left\{\frac{1-\mu_0^2}{x^2} + \frac{1-\mu_1^2}{(1-x)^2} + \frac{1+\mu_\infty^2 - \mu_0^2 - \mu_1^2}{x(1-x)}\right\},$$

by the projective change of the unknown

$$u \longmapsto \sqrt{x^c(1-x)^{a+b+1-c}}\; u.$$

Unless otherwise stated, we always take a pair (u_0, u_1) of linearly independent solutions of $E^{\mathrm{SL}}(a,b,c)$ satisfying $u_0u_1' - u_0'u_1 = 1$, and set

$$U = \begin{pmatrix} u_0 & u_0' \\ u_1 & u_1' \end{pmatrix}.$$

2.3. **Monodromy group.** The group of isometries of $\boldsymbol{H}^3$ is generated by the orientation preserving ones

$$H \longmapsto PH{}^t\overline{P}, \qquad H \in \boldsymbol{H}^3, \quad P \in GL_2(\boldsymbol{C}),$$

and the reversing one $H \to {}^tH$.

Let $\{u_0, u_1\}$ be a pair of linearly independent solutions of $E^{\mathrm{SL}}(a,b,c)$ and $\{v_0, v_1\}$ another such pair. Put

$$U = \begin{pmatrix} u_0 & u_0' \\ u_1 & u_1' \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} v_0 & v_0' \\ v_1 & v_1' \end{pmatrix}.$$

Then there is a non-singular matrix, say P, such that $U = PV$ and so that

$$U{}^t\overline{U} = PV{}^t\overline{V}{}^t\overline{P}.$$

Thus the hyperbolic Schwarz map

$$\mathscr{S}\colon X \ni x \longmapsto H(x) = U(x){}^t\overline{U}(x) = \begin{pmatrix} |u_0|^2 + |u_0'|^2 & u_1\bar{u}_0 + u_1'\bar{u}_0' \\ \bar{u}_1u_0 + \bar{u}_1'u_0' & |u_1|^2 + |u_1'|^2 \end{pmatrix} \in \boldsymbol{H}^3 \tag{2.2}$$

is determined by the system up to orientation preserving automorphisms. The monodromy group $\mathrm{Mon}(a,b,c)$ with respect to U acts naturally on $\boldsymbol{H}^3$ by

$$H \longmapsto MH{}^t\overline{M}, \quad M \in \mathrm{Mon}(a,b,c).$$

Note that the hyperbolic Schwarz map to the upper half-space model is given by

$$X \ni x \longmapsto \frac{\left(u_0(x)\bar{u}_1(x) + u_0'(x)\bar{u}_1'(x),\ 1\right)}{|u_1(x)|^2 + |u_1'(x)|^2} \in \boldsymbol{C} \times \boldsymbol{R}^+.$$

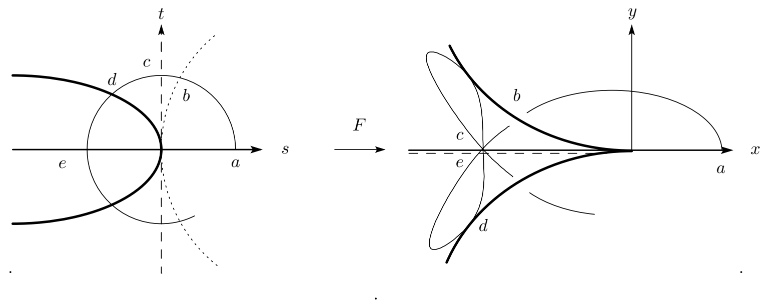


FIGURE 2.1. Image under the map F

2.4. **Singularities of fronts.** A smooth map f from a domain $U \subset \boldsymbol{R}^2$ to a Riemannian 3-manifold N^3 is called a *front* if there exists a unit vector field $\nu\colon U \to T_1N$ along the map f such that df and ν are perpendicular and the map $\nu\colon U \to T_1N$ is an *immersion*, where T_1N is the unit tangent bundle of N. We call ν the *unit normal vector field* of f. Not that, if we identify T_1N with the unit cotangent bundle T_1N^*, the condition $df \perp \nu$ is equivalent to the corresponding map $L\colon U \to T_1^*N$ to be Legendrian with respect to the canonical contact structure T_1^*N. A point $x \in U$ is called a *singular point* of f if the rank df is less than 2 at x. It is well-known that generic singularities of fronts are *cuspidal edges* and *swallowtails* [Arnold et al. 1985]. In this section, we roughly review these types of singularities. General criteria for fronts to be cuspidal edges or swallowtails are given in [Kokubu et al. 2005].

$(2,3)$-*cusp and cuspidal edges.* Recall that the cubic equation $t^3 + xt - y = 0$ in t with real parameters (x, y) has three distinct real roots if and only if its discriminant $27y^2 + 4x^3$ is negative. Consider the map

$$F : \boldsymbol{R}^2 \ni (s,t) \longmapsto (x,y) = (s - t^2, st) \in \boldsymbol{R}^2,$$

whose Jacobian is equal to $s+2t^2$. The image of the (smooth) curve $C : s+2t^2 = 0$ under F is a curve with a cusp of $(2,3)$-type, and is given by $F(C) : 27y^2+4x^3 = 0$. Note that F folds the t-axis to the negative half of the x-axis, and that the inverse image of $F(C)$ consists of C and a curve tangent to C at the origin, indeed we have

$$27y^2 + 4x^3|_{x=s-t^2,\ y=st} = (s + 2t^2)^2(4s - t^2).$$

The hemicircle centered at the origin in the (s, t)-space is mapped by F as is shown in Figure 2.1.

When a $(2,3)$ cusp traveling along a curve transversal to $\boldsymbol{R}^2 \subset \boldsymbol{R}^3$, the locus of the singularity consists of *cuspidal edges*. Precisely speaking, $p \in U$ is a *cuspidal edge* of a front $f\colon U \to \boldsymbol{R}^3$ if there exists local diffeomorphisms ψ and Ψ of (U, p) and $(\boldsymbol{R}^3, f(p))$ such that $\Psi \circ f \circ \psi(u, v) = (u^2, u^3, v) =: f_c$. In other words, the germ of the map f at p is locally A-*equivalent* to f_c.

Swallowtails. Consider the map

$$\widetilde{F} : \boldsymbol{R}^2 \ni (s,t) \longmapsto (x,y,z) = (s - t^2, st, s^2 - 4st^2) \in \boldsymbol{R}^3.$$

This map is singular (rank of the differential is not full) along the curve C, and the image of the point $(-2t^2, t) \in C$ is given as $(-3t^2, -2t^3, 12t^4)$. The hemicircle centered at the origin in the (s, t)-space is mapped by $\widetilde{F}$ as is shown in Figure 2.2. The image surface has three kinds of singularities

(1) *Cuspidal edges* along $\tilde{F}(C) - \{(0,0,0)\}$,
(2) A *swallowtail* at $\{(0,0,0)\}$,

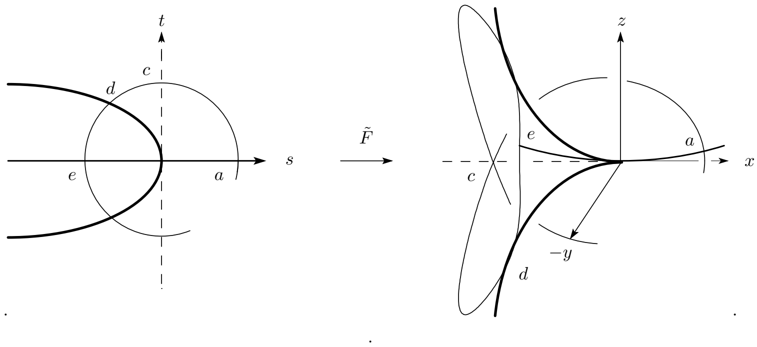


FIGURE 2.2. Swallowtail: Image under the map $\tilde{F}$

(3) *Self-intersection* along the image of the t-axis.

Here, by definition, a swallowtail is a singular point of a differential map $f\colon U \to \boldsymbol{R}^3$, which is A-equivalent to $\widetilde{F}(s,t)$. Another canonical form of the swallowtail is

$$f_s(u,v) = (3u^4 + u^2 v, 4u^3 + 2uv, v),$$

which is A-equivalent to $\widetilde{F}$ as $f_s(u,v) = \Psi \circ \widetilde{F} \circ \psi(u,v)$, where

$$\psi(u,v) = (2v + 4u^2, 2u), \quad \Psi(x,y,z) = \left(\frac{-z+4x^2}{16}, \frac{y}{2}, \frac{x}{2}\right).$$

3. USE OF THE SCHWARZ MAP

Let u and v be solutions of the equation $E^{\mathrm{SL}}(a,b,c)$ such that $uv' - vu' = 1$. The Schwarz map is defined as $X \ni x \mapsto z = u(x)/v(x) \in Z$, which is the *hyperbolic Gauss map* (see Section 3) of the hyperbolic Schwarz map $\mathscr{S}$ as in (2.2). It is convenient to study the hyperbolic Schwarz map (2.2) by regarding z as variable.

Especially when the inverse of the Schwarz map is single-valued globally, this choice of variable is very useful, because the inverse map is often given explicitly as an automorphic function for the monodromy group acting properly discontinuously on the image of the Schwarz map. In particular, the equation (1.3) is written as

$$\frac{dU}{dz} = U\begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix}, \qquad \text{where} \quad \theta = q\frac{dx}{dz} \quad \omega = \frac{dx}{dz}.$$

Then by the representation formula in [Kokubu et al. 2003], the solution U is written by ω, the hyperbolic Gauss map (i.e., the Schwarz map) z and their derivatives:

$$U = i\frac{1}{\sqrt{\dot{x}}}\begin{pmatrix} z\dot{x} & 1 + \dfrac{z}{2}\dfrac{\ddot{x}}{\dot{x}} \\ \dot{x} & \dfrac{1}{2}\dfrac{\ddot{x}}{\dot{x}} \end{pmatrix}, \tag{3.1}$$

where $\dot{} = d/dz$. Here, we summarize the way to show the formula: Since $z'(:= dz/dx) = -1/v^2$ and $\ddot{x} = d^2x/dz^2$, we have

$$v = i\sqrt{\frac{1}{z'}} = i\sqrt{\dot{x}}, \quad u = vz,$$

and

$$v' = \frac{dv}{dx} = \frac{dv}{dz}\frac{dz}{dx} = \frac{i}{2}(\dot{x})^{-3/2}\ddot{x}, \qquad u' = i\frac{1}{\sqrt{\dot{x}}} + z\frac{i}{2}(\dot{x})^{-3/2}\ddot{x}.$$

So, we have (3.1) and

$$H = U^t\overline{U} = \frac{1}{|\dot{x}|}\begin{pmatrix} |z|^2|\dot{x}|^2 + \left|1+\frac{z}{2}\frac{\ddot{x}}{\dot{x}}\right|^2 & z|\dot{x}|^2 + \frac{1}{2}\left(1+\frac{z}{2}\frac{\ddot{x}}{\dot{x}}\right)\frac{\overline{\ddot{x}}}{\overline{\dot{x}}} \\ \bar{z}|\dot{x}|^2 + \frac{1}{2}\left(1+\frac{\bar{z}}{2}\frac{\overline{\ddot{x}}}{\overline{\dot{x}}}\right)\frac{\ddot{x}}{\dot{x}} & |\dot{x}|^2 + \frac{1}{4}\left|\frac{\ddot{x}}{\dot{x}}\right|^2 \end{pmatrix}. \tag{3.2}$$

When the (projective) monodromy group of the equation $E(a,b,c)$ is a polyhedral group or a Fuchsian triangle group, there is a set of real parameters $(\bar{a},\bar{b},\bar{c})$ such that $\bar{a}-a, \bar{b}-b, \bar{c}-c \in \boldsymbol{Z}$, and that the Schwarz map of $E(\bar{a},\bar{b},\bar{c})$ has the single-valued inverse. Such equations are said to be *standard*. The equation $E(a,b,c)$ is standard if $a,b,c\in\boldsymbol{R}$ satisfy

$$k_0 := \frac{1}{|\mu_0|},\quad k_1 := \frac{1}{|\mu_1|},\quad k_\infty := \frac{1}{|\mu_\infty|}\quad \in \quad \{2,3,\dots,\infty\}.$$

Though it is a challenging problem to study transformations of the hyperbolic Schwarz maps of standard equations to the general equations, we study only standard ones in this paper.

4. Singularities of hyperbolic Schwarz maps

Since the equation $E^{\mathrm{SL}}(a,b,c)$ has singularities at 0, 1 and ∞, the corresponding hyperbolic Schwarz map H has singularities at these points. In terms of flat fronts in $\boldsymbol{H}^3$, they are considered as *ends* of the surface. On the other hand, the map H may not be an immersion at $x\in X$, even if x is not a singular point of $E^{\mathrm{SL}}(a,b,c)$. In other words, x is a singular point of the front $H\colon X\to\boldsymbol{H}^3$.

In this section, we analyze properties of these singular points of the hyperbolic Schwarz maps.

4.1. **Singularities on X.** As we have seen in the introduction, the hyperbolic Schwarz map $H\colon X=\boldsymbol{C}-\{0,1\}\to\boldsymbol{H}^3$ can be considered as a flat front in the sense of [Kokubu et al. 2003, Kokubu et al. 2005]. Thus, as a corollary of Theorem 1.1 in [Kokubu et al. 2005], we have

Lemma 4.1. (1) *A point $p\in X$ is a singular point of the hyperbolic Schwarz map H if and only if $|q(p)|=1$,*

(2) *a singular point $x\in X$ of H is A-equivalent to the cuspidal edge if and only if*

$$q'(x)\neq 0 \qquad \textit{and} \qquad q^3(x)-\bar{q}'(x)q'(x)\neq 0,$$

(3) *and singular point $x\in X$ of H is A-equivalent to the swallowtail if and only if*

$$q'(x)\neq 0 \qquad q^3(x)-\bar{q}'(x)q'(x)=0,$$
$$\textit{and}\quad \Re\left\{\frac{1}{q}\left(\left(\frac{q'(x)}{q(x)}\right)'-\frac{1}{2}\left(\frac{q'(x)}{q(x)}\right)^2\right)\right\}\neq 0.$$

We apply Lemma 4.1 to the hypergeometric equation. Using μ_0, μ_1 and μ_∞ as in (2.1), the coefficient of the hypergeometric equation $E^{\mathrm{SL}}(a,b,c)$ is written as

$$q = -\frac{1}{4}\left(\frac{1-\mu_0^2}{x^2}+\frac{1-\mu_1^2}{(1-x)^2}+\frac{1+\mu_\infty^2-\mu_0^2-\mu_1^2}{x(1-x)}\right) =: \frac{-Q}{4x^2(1-x)^2}, \tag{4.1}$$

where

$$Q = 1-\mu_0^2+(\mu_\infty^2+\mu_0^2-\mu_1^2-1)x+(1-\mu_\infty^2)x^2. \tag{4.2}$$

Hence $x\in X$ is a singular point if and only if

$$|Q| = 4|x^2(1-x)^2|. \tag{4.3}$$

Define R by

$$q' = -\frac{Q'x(1-x) - 2Q(1-2x)}{4x^3(1-x)^3} =: \frac{-R}{4x^3(1-x)^3}. \tag{4.4}$$

Then we have

$$q^3(x)\bar{q}'(x) - q'(x) = \frac{Q^3}{4^3x^6(1-x)^6} \cdot \frac{\overline{R}}{4\bar{x}^3(1-\bar{x})^3} + \frac{R}{4x^3(1-x)^3}.$$

So, the condition $q^3(x)\bar{q}'(x) - q'(x) = 0$ is equivalent to '$Q^3\overline{R}^2$ is real non-positive' under the condition (4.3).Therefore, a singular point x is a cuspidal edge if and only if (4.3) and

$$Q^3\overline{R}^2 \quad \text{is not a non-positive real number.} \tag{4.5}$$

Moreover, a singular point x is a swallowtail if and only if

$$Q^3\overline{R}^2 \text{ is real non-positive, and } \Re\left(2|R|^4 - x(1-x)(2R'Q - RQ')\overline{R}^2\right) \neq 0 \tag{4.6}$$

hold, where $' = d/dx$. In fact, since $(q'/q) = R/\big(x(1-x)Q\big)$, we have

$$\begin{aligned}
&\frac{1}{q}\left(\left(\frac{q'}{q}\right)' - \frac{1}{2}\left(\frac{q'}{q}\right)^2\right) \\
&= -\frac{2}{Q^3}\big(2(1-2x)RQ + 2x(1-x)(R'Q - RQ') - R^2\big) \\
&= -\frac{2}{Q^3}(-2R^2 + x(1-x)(2R'Q - RQ')) \\
&= \frac{2R^2\overline{Q}^3}{|R^2\overline{Q}^3|^2}\left(2|R|^4 - x(1-x)(2R'Q - RQ')\overline{R}^2\right).
\end{aligned}$$

4.2. **At a singular point of the equation** $E(a,b,c)$**.** In this subsection, we assume the parameters a, b and c are real. Since q has poles of order 2 at 0, 1 and ∞, $|q| \neq 1$ in a neighborhood of the singularities of the equation $E(a,b,c)$. Here, we study the behavior of X around these points. If X were single-valued on a neighborhood of the end, the following calculations are essentially similar to those in [Gálvez et al. 2000], in which asymptotic behavior of the end of flat fronts is investigated.

Around, for example, $x = 0$, the Schwarz map has the expression $z = x^{|1-c|}(1 + O(x))$. So we may assume that the inverse map has the expression $x = z^\alpha\big(1+O(z)\big)$ for some *real* constant $\alpha(>0)$. Since

$$\dot{x} = \alpha z^{\alpha-1}(1 + O(z)), \quad \frac{\ddot{x}}{\dot{x}} = \frac{\alpha-1}{z}(1 + O(z)),$$

the principal part of the matrix U is given by

$$E_1 := \frac{i}{\sqrt{\alpha z^{\alpha+1}}}\begin{pmatrix} \alpha z^{\alpha+1} & \left(1 + \frac{\alpha-1}{2}\right)z \\ \alpha z^\alpha & \frac{\alpha-1}{2} \end{pmatrix}.$$

We have

$$E_1{}^t\overline{E}_1 := \frac{1}{|\alpha z^{\alpha+1}|}\begin{pmatrix} ** & \left(\frac{\alpha^2-1}{4}\right)z + |\alpha z^\alpha|^2 z \\ ** & \left(\frac{\alpha-1}{2}\right)^2 + |\alpha z^\alpha|^2 \end{pmatrix}.$$

Dihedral: $R_1 : z \mapsto \bar{z}$, $R_2 : z \mapsto e^{2\pi i/n}\bar{z}$, $R_3 : z \mapsto \dfrac{1}{\bar{z}}$.
Tetrahedral: $R_1 : z \mapsto \bar{z}$, $R_2 : z \mapsto -\bar{z}$, $R_3 = R(-\frac{1+i}{\sqrt{2}}, \sqrt{2})$.
Octahedral: $R_1 : z \mapsto \bar{z}$, $R_2 : z \mapsto i\bar{z}$, $R_3 = R(-1, \sqrt{2})$.
Icosahedral: $R_1 : z \mapsto \bar{z}$, $R_2 : z \mapsto \epsilon^2\bar{z}$,
$R_3 = R\left(2\cos\frac{\pi}{5}, \sqrt{1+4\cos^2\frac{\pi}{5}}\right) = \dfrac{-(\epsilon-\epsilon^4)\bar{z}+(\epsilon^2-\epsilon^3)}{(\epsilon^2-\epsilon^3)\bar{z}+(\epsilon-\epsilon^4)}$, $\quad \epsilon = e^{2\pi i/5}$.

TABLE 1

Thus the hyperbolic Schwarz map $\mathscr{S}$ extends to the point $z = 0$ and to the boundary of $\boldsymbol{H}^3$. Its image is nonsingular at $\mathscr{S}(0)$, and is tangent to the boundary at this point.

5. Hyperbolic Schwarz maps

When the monodromy group of the equation $E(a, b, c)$ is a finite group or a typical Fuchsian group, we study the singularities of the hyperbolic Schwarz map, and visualize the image surface.

5.1. **Finite (polyhedral) monodromy groups.**

We first recall fundamental facts about the polyhedral groups and their invariants basically following [Klein 1884].

5.1.1. *Basic data.* Let the triple (k_0, k_1, k_∞) be one of

$$(2,2,n)\ (n = 1,2,\dots), \quad (2,3,3), \quad (2,3,4), \quad (2,3,5),$$

in which case, the projective monodromy group is of finite order N:

$$N = \quad 2n, \quad 12, \quad 24, \quad 60,$$

respectively. Note that

$$\frac{2}{N} = \frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_\infty} - 1.$$

For each case, we give a triplet $\{R_1, R_2, R_3\}$ of reflections whose mirrors bound a Schwarz triangle. These are tabulated in Table 1, where

$$R(c, r) : z \longmapsto \frac{c\bar{z} + r^2 - |c|^2}{\bar{z} - \bar{c}}$$

is the reflection with respect to the circle of radius $r > 0$ centered at c. The monodromy group Mon (a polyhedral group) is the group of even words of these three reflections.

The (single-valued) inverse map

$$s^{-1} : Z \ni z \longmapsto x \in \bar{X} \cong \boldsymbol{P}^1,$$

invariant under the action of Mon, is given as follows. Let $f_0(z), f_1(z)$ and $f_\infty(z)$ be the monic polynomials in z with simple zeros exactly at the images $s(0), s(1)$ and $s(\infty)$, respectively. If $\infty \in Z$ is not in these images, then the degrees of these polynomials are $N/k_0, N/k_1$ and N/k_∞, respectively; if for instance $\infty \in s(0)$, then the degree of f_0 is $N/k_0 - 1$. Now the inverse map s^{-1} is given by

$$x = A_0\,\frac{f_0(z)^{k_0}}{f_\infty(z)^{k_\infty}},$$

where A_0 is a constant; we also have

$$1 - x = A_1\,\frac{f_1(z)^{k_1}}{f_\infty(z)^{k_\infty}}, \quad \frac{dx}{dz} = A\frac{f_0(z)^{k_0-1}f_1(z)^{k_1-1}}{f_\infty(z)^{k_\infty+1}},$$

for some constants A_1 and A. See Table 2.

Dihedral: $(k_0, k_1, k_\infty) = (2, 2, n)$, $N = 2n$.

$$A_0 = \frac{1}{4}, \quad A_1 = -\frac{1}{4}, \quad A = \frac{n}{4},$$
$$f_0 = z^n + 1, \quad f_1 = z^n - 1, \quad f_\infty = z.$$

f_∞ is of degree $1 = 2n/n - 1$, since $\infty \in s(\infty)$, that is, $x(\infty) = \infty$.

Tetrahedral: $(k_0, k_1, k_\infty) = (2, 3, 3)$, $N = 12$.

$$A_0 = -12\sqrt{3}, \quad A_1 = 1, \quad A = 24\sqrt{3},$$
$$\begin{aligned} f_0 &= z(z^4 + 1), \\ f_1 &= z^4 + 2\sqrt{3}z^2 - 1 = (z^2 - 2 + \sqrt{3})(z^2 + 2 + \sqrt{3}), \\ f_\infty &= z^4 - 2\sqrt{3}z^2 - 1 = (z^2 - 2 - \sqrt{3})(z^2 + 2 - \sqrt{3}). \end{aligned}$$

f_0 is of degree $5 = 12/2 - 1$, since $\infty \in s(0)$, that is, $x(\infty) = 0$.

Octahedral: $(k_0, k_1, k_\infty) = (3, 2, 4)$, $N = 24$.

$$A_0 = \frac{1}{108}, \quad A_1 = \frac{-1}{108}, \quad A = \frac{1}{27},$$
$$\begin{aligned} f_0 &= z^8 + 14z^4 + 1 = (z^4 + 2z^3 + 2z^2 - 2z + 1)(z^4 - 2z^3 + 2z^2 + 2z + 1), \\ f_1 &= z^{12} - 33z^8 - 33z^4 + 1 = (z^4 + 1)(z^2 + 2z - 1)(z^2 - 2z - 1)(z^4 + 6z^2 + 1), \\ f_\infty &= z(z^4 - 1) = z(z^2 + 1)(z^2 - 1). \end{aligned}$$

f_∞ is of degree $5 = 24/4 - 1$, since $\infty \in s(\infty)$, that is, $x(\infty) = \infty$.

Icosahedral: $(k_0, k_1, k_\infty) = (3, 2, 5)$, $N = 60$.

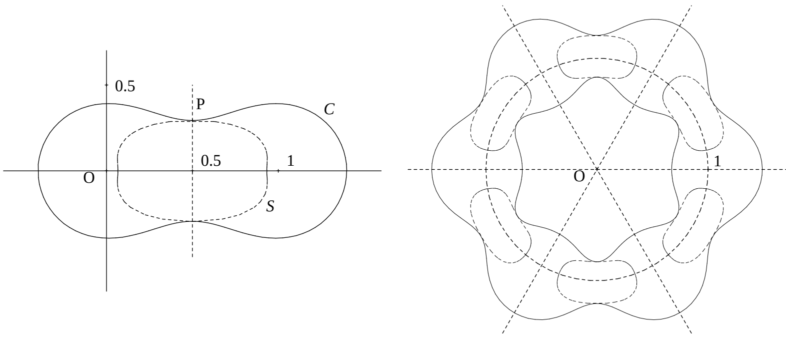
$$A_0 = \frac{-1}{1728}, \quad A_1 = \frac{1}{1728}, \quad A = \frac{-5}{1728},$$
$$\begin{aligned} f_0 &= z^{20} - 228z^{15} + 494z^{10} + 228z^5 + 1 \\ &= (z^4 - 3z^3 - z^2 + 3z + 1)(z^8 - z^7 + 7z^6 + 7z^5 - 7z^3 + 7z^2 + z + 1) \\ &\quad \times (z^8 + 4z^7 + 7z^6 + 2z^5 + 15z^4 - 2z^3 + 7z^2 - 4z + 1), \\ f_1 &= z^{30} + 522z^{25} - 10005z^{20} - 10005z^{10} - 522z^5 + 1 \\ &= (z^2 + 1)(z^8 - z^6 + z^4 - z^2 + 1)(z^4 + 2z^3 - 6z^2 - 2z + 1) \\ &\quad \times (z^8 + 4z^7 + 17z^6 + 22z^5 + 5z^4 - 22z^3 + 17z^2 - 4z + 1) \\ &\quad \times (z^8 - 6z^7 + 17z^6 - 18z^5 + 25z^4 + 18z^3 + 17z^2 + 6z + 1), \\ f_\infty &= z(z^{10} + 11z^5 - 1) \\ &= z(z^2 + z - 1)(z^4 + 2z^3 + 4z^2 + 3z + 1)(z^4 - 3z^3 + 4z^2 - 2z + 1). \end{aligned}$$

f_∞ is of degree $11 = 60/5 - 1$, since $\infty \in s(\infty)$, that is, $x(\infty) = \infty$.

TABLE 2

5.1.2. *Dihedral cases.* We consider a dihedral case: $(k_0, k_1, k_\infty) = (2, 2, n)$, $n = 3$. The curve C in the x-plane defined by (4.3): $|Q| = 4|x(1 - x)|^2$ is symmetric with respect to the line $\Re(x) = 1/2$ and has a shape of a cocoon (see Figure 5.1 (left)). We next study the condition (4.6). The curve $\Im(Q^3\overline{R}^2) = 0$ consists of the line $\Re x = 1/2$, the real axis, and a curve of degree 8. We can prove that, on the upper half x-plane, there is a unique point satisfying the conditions (4.3) and (4.6), (this point is the intersection P of the curve C and the line $\Re x = 1/2$,) and that the image surface has a swallowtail at this point, and has cuspidal edges along $\mathscr{S}(C)$ outside $\mathscr{S}(P)$. We omit the proof since the computation is analogous to the case $(k_0, k_1, k_\infty) = (\infty, \infty, \infty)$; see Section 5.2.1.

The curve which gives the self-intersection is tangent to C at P, and crosses the real axis perpendicularly; this is the dotted curve in Figure 5.1 (left), and is made as follows. Since the curve is symmetric with respect to the line $\Re x = 1/2$, on each level line $\Im x = t$, we take two points x_1 and x_2 ($\Re(x_1 + x_2) = 1$), compute the distance between their images $\mathscr{S}(x_1)$ and $\mathscr{S}(x_2)$, and find the points that the two image points coincide.



in the x-plane   in the z-plane

FIGURE 5.1. The curve $C : |Q| = 4|x(1-x)|^2$, when $(k_0, k_1, k_\infty) = (2, 2, 3)$

We substitute the inverse of the Schwarz map (cf. Table 2):

$$x = \frac{1}{4}\,\frac{(z^n+1)^2}{z^n}, \qquad n = 3$$

into the expression (3.2) of the hyperbolic Schwarz map, and visualize the image surface in the Poincaré ball model explained in Section 2.1. The upper half x-space corresponds to a fan in the z-plane bounded by the lines with argument $0, \pi/3, 2\pi/3$, and the unit circle (see Figure 5.1(right).). The image $s(C)$ consists of two curves; the dotted curves in the figures form the pre-image of the self-intersection.

Let Φ denote the hyperbolic Schwarz map in z-variable:

$$\Phi := \mathscr{S} \circ s^{-1} : Z \ni z \longmapsto H(z) \in \boldsymbol{H}^3.$$

We visualize the image of the hyperbolic Schwarz map when $n = 3$. Figure 5.2(upper left) is a view of the image of one fan in the z-plane under Φ (equivalently, the image of upper/lower half x-plane under $\mathscr{S}$), the upper right figure is the antipode of the left. Figure 5.2(below) is a view of the image of six fans dividing the unit z-disk. To draw the images of fans with the same accuracy, we make use of the invariance of the function $x(z)$ under the monodromy groups.

5.1.3. *Other polyhedral cases.* For other polyhedral cases, situation is similar. The sphere Z is divided into $2N$ triangles. In Figure 5.3, for the tetrahedral and the octahedral cases, the images under Φ of N triangles are shown, and for the icosahedral case, $2N = 120$ triangles dividing the z-plane, the images of central ten triangles, and that of $N = 60$ triangles are shown.

5.2. **A Fuchsian monodromy group.** We study only the case $(k_0, k_1, k_\infty) = (\infty, \infty, \infty)$.

5.2.1. *Singular locus.* We find the singular locus of the image when $\mu_0 = \mu_1 = \mu_\infty = 0$. We have

$$Q = 1 - x + x^2, \quad R = (-1 + 2\,x)\left(x^2 - x + 2\right).$$

The singularities lie on the image of the curve

$$C : f := 16|x(1-x)|^4 - |Q|^2 = 0.$$

Note that this curve is symmetric with respect to the line $\Re x = 1/2$.

Recall that the condition (4.6) is stated as

$$h := \Im(Q^3\overline{R}^2) = 0, \qquad \Re(Q^3\overline{R}^2) > 0.$$

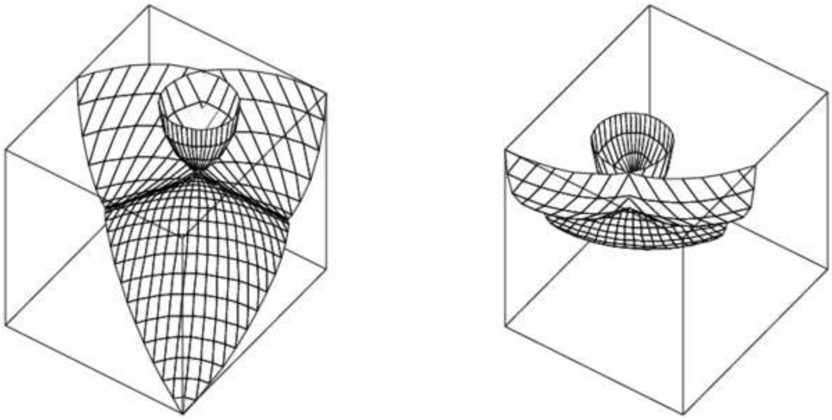


Image of a fan under Φ

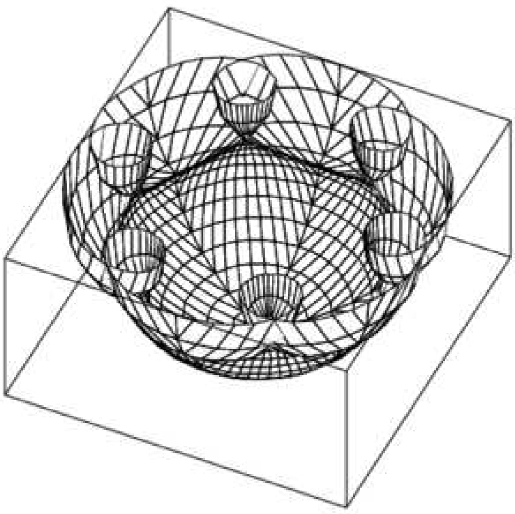


Image of six fans

FIGURE 5.2. Dihedral case

The curve $h = 0$ consists of the line $\Re x = 1/2$, the real axis, and a curve of degree 8. We can prove that, on the upper half x-plane, there is a unique point in the intersection points of the curves C and $h = 0$ satisfying the conditions (4.3) and (4.6) – this point is the intersection P of the curve C and the line $\Re(x) = 1/2$ – and that the image surface has a swallowtail singularity at P, and has cuspidal edges along $\mathscr{S}(C)$ outside $\mathscr{S}(P)$.

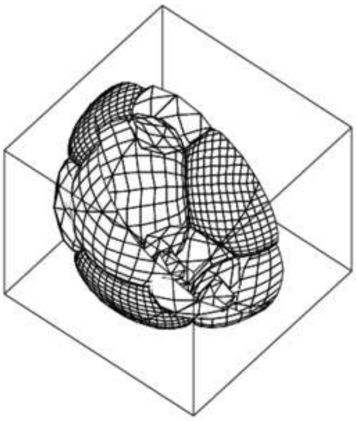
Actual computation proceeds as follows. The image curve has singularities at the image of the intersection of the curves C and $\{h = 0,\ \Re(Q^3\overline{R}^2) > 0\}$. We can show that there is only one such point: the intersection of C and the line $\Re x = 1/2$.

When $l = m = n = 0$, the coefficient q is expressed as

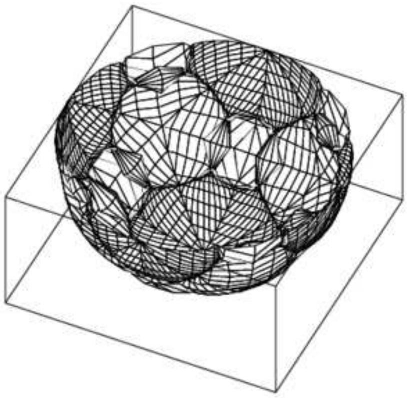
$$q = -\frac{1}{4}\frac{Q}{x^2(1-x)^2}, \qquad Q(x) = x^2 - x + 1.$$

If we put $x = s + it$, then $f := |Q|^2 - 4^2|x^2(1-x)^2|^2$ is a polynomial in s and t of order 8. If we put

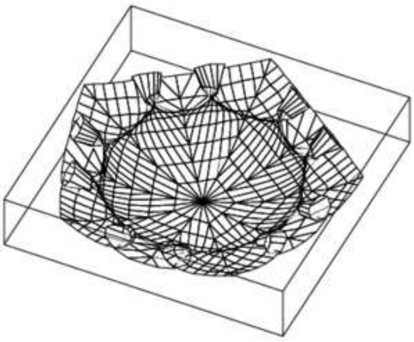
$$s = \frac{1}{2} + u, \quad u^2 = U, \quad t^2 = T,$$



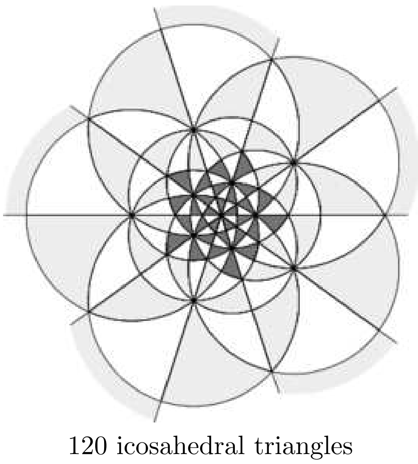
Tetrahedral
Image of twelve triangles



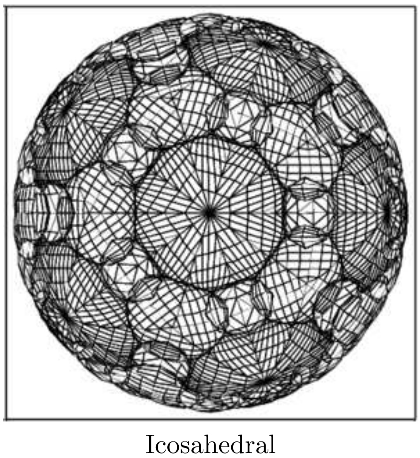
Octahedral
Image of 32 triangles



Icosahedral; Image of ten triangles



120 icosahedral triangles
and 60 central ones



Icosahedral
Image of 60 triangles

Figure 5.3. Other polyhedral cases

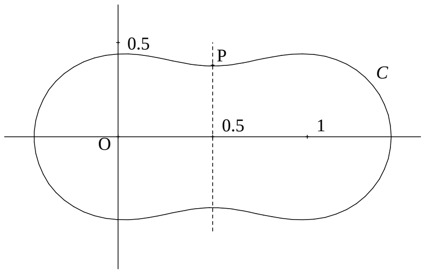


FIGURE 5.4. The curve $C : |Q| = 4|x(1-x)|^2$, when $(k_0, k_1, k_\infty) = (\infty.\infty, \infty)$

then f turns out to be a polynomial F in U and T of order 4:

$$F = \frac{1}{2} + \frac{5}{2}(U-T) - 5(U^2+T^2) + 6TU + 16(TU^2 - T^2U) + 16(U^3 - T^3) \\ - 16(U^4+T^4) - 64(T^3U - TU^3) - 96T^2U^2.$$

The polynomial R is expressed as

$$R = -4x^3(1-x)^3q'(x) = (2x-1)(x^2-x+2).$$

The imaginary part of $Q^3\overline{R}^2$ has the form $t(2s-1)G$, where G is a polynomial in U and T of order 4:

$$G = \frac{1323}{256} + \frac{189}{16}(U-T) + \frac{9}{8}(U^2+T^2) - \frac{99}{4}TU + 11(T^3-U^3) + 11(T^2U - TU^2) \\ - 5(U^4 - T^4) - 20(T^3U + TU^3) - 30T^2U^2.$$

Set

$$G_1 := 5F - 16G, \quad F_1 := 256F - 16G_1 \quad \text{and} \quad U - T =: S, \quad UT =: V.$$

Then we have

$$G_1 = -\frac{1283}{16} + 256S^3 - 43S^2 + 1024VS - \frac{353}{2}S + 340V,$$

which is *linear* in V. Solving V from the equality $G_1 = 0$, and substituting it into $F_1 = 0$, we get a rational function in S, whose numerator is a polynomial in S of degree 3. The roots of this polynomial can be computed. In this way, we can solve the system

$$|q| = 1, \quad \Im(Q^3\overline{R}^2) = 0,$$

and prove that a solution $x = \xi$ satisfies the condition (4.6) only if $\Re(\xi) = 1/2$. Substituting $x = \frac{1}{2} + it$ into the second equation (4.6), we have

$$2|R|^4 - x(1-x)(2R'Q - RQ')\overline{R}^2 = \frac{1}{64}t^2(7-4t^2)^2(21+440t^2-560t^4+256t^6). \tag{5.1}$$

Since $|q| \neq 1$ at $x = \frac{1}{2}(1 \pm \sqrt{7})$, we deduce that the real part of (5.1) does not vanish on the singular points. Hence there is a unique swallowtail in the image surface of the upper x-plane.

5.2.2. *Lambda function.* The inverse of the Schwarz map is a modular function known as the lambda function

$$\lambda : \boldsymbol{H}^2 = \{z \in \boldsymbol{C} \mid \Im z > 0\} \longrightarrow X.$$

The hyperbolic Schwarz map is expressed in terms of its derivatives. In this section we recall its definition and give a few properties. We begin with the theta functions:

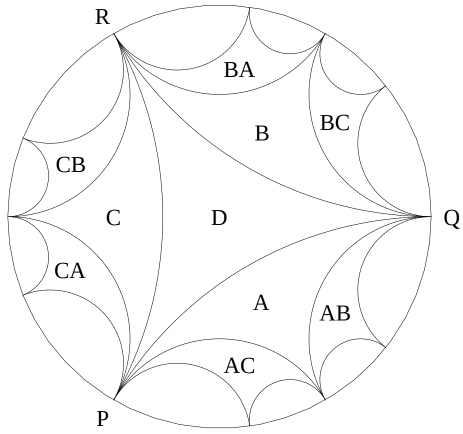


FIGURE 5.5. Schwarz triangles with three zero angles

for $z \in \boldsymbol{H}^2$, set $q = e^{\pi i z/2}$,

$$\theta_2 = \sum_{-\infty}^{\infty} q^{(2n-1)^2/2}, \quad \theta_3 = \sum_{-\infty}^{\infty} q^{2n^2}, \quad \theta_0 = \sum_{-\infty}^{\infty} (-1)^n q^{2n^2}.$$

Recall the well-known identity $\theta_3^4 - \theta_0^4 = \theta_2^4$. We define the lambda function as

$$\lambda(z) = \left(\frac{\theta_0}{\theta_3}\right)^4 = 1 - 16q^2 + 128q^4 - 704q^6 + 3072q^8 - 11488q^{10} + 38400q^{12} - \cdots;$$

note that $\lambda : \infty \mapsto 1, 0 \mapsto 0, 1 \mapsto \infty$, and that λ sends every triangle in Figure 5.5 onto the upper/lower half x-plane. In the figure, for symmetry reason, the Schwarz triangles tessellating the upper half plane $\boldsymbol{H}^2$ are shown in the Poincaré disk. The inverse of the Schwarz map is given by $x = \lambda(z)$. In the expression H of the hyperbolic Schwarz map given in §3, the derivatives λ' and λ''/λ' are used. They are computed as follows: Define the Eisenstein series E_2 by

$$E_2(z) = \frac{1}{24}\frac{\eta'(z)}{\eta(z)} = 1 - 24\sum_{n=1}^{\infty}\left(\sum_{d|n} d\right) e^{2\pi i n z} = 1 - 24(q^4 + 3q^8 + 4q^{12} + 7q^{16} + \cdots).$$

Then we have

$$\frac{\theta_0'}{\theta_0} - \frac{1}{6}E_2 = -\frac{1}{6}(\theta_2^4 + \theta_3^4), \quad \frac{\theta_2'}{\theta_2} - \frac{1}{6}E_2 = \frac{1}{6}(\theta_0^4 + \theta_3^4), \quad \frac{\theta_3'}{\theta_3} - \frac{1}{6}E_2 = -\frac{1}{6}(\theta_0^4 - \theta_2^4),$$

where

$$' = q\frac{d}{dq} = \frac{2}{\pi i}\frac{d}{dz},$$

and so we have

$$\lambda' = -2\theta_2^4 \lambda,$$

which leads to the q-series expansion

$$\frac{\lambda''}{\lambda'} = (\log \lambda')' = 4\frac{\theta_2'}{\theta_2} + \frac{\lambda'}{\lambda} = \frac{4}{6}E_2 + \frac{4}{6}(\theta_0^4 + \theta_3^4) - 2\theta_2^4.$$

This expression is useful for drawing the picture of the image of Φ, because q-series converge very fast.

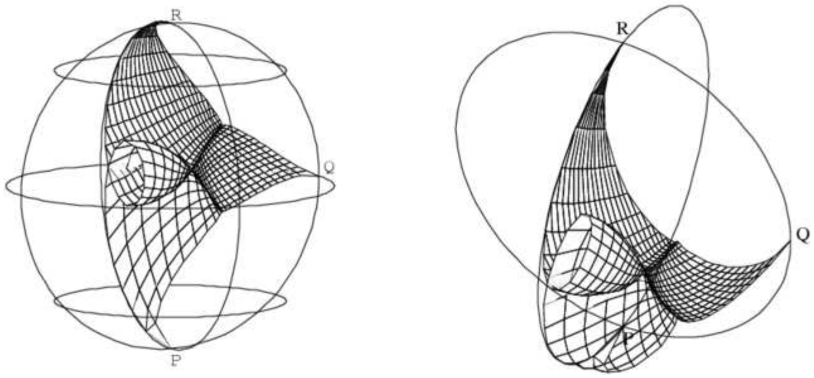


Image of one triangle D

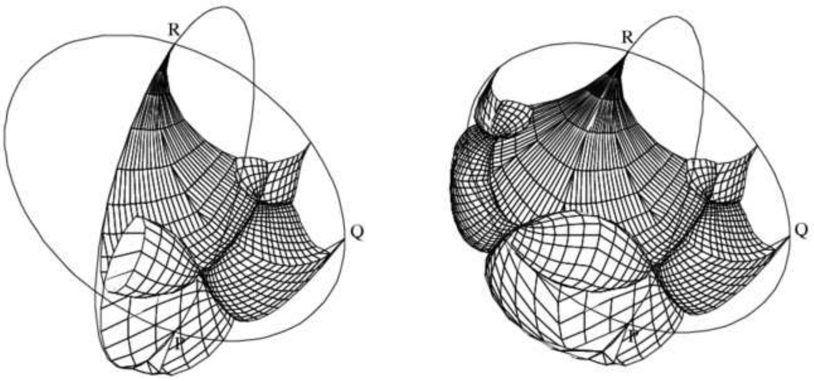


Image of $\{D, A\}$ Image of $\{D, A, B, BA\}$

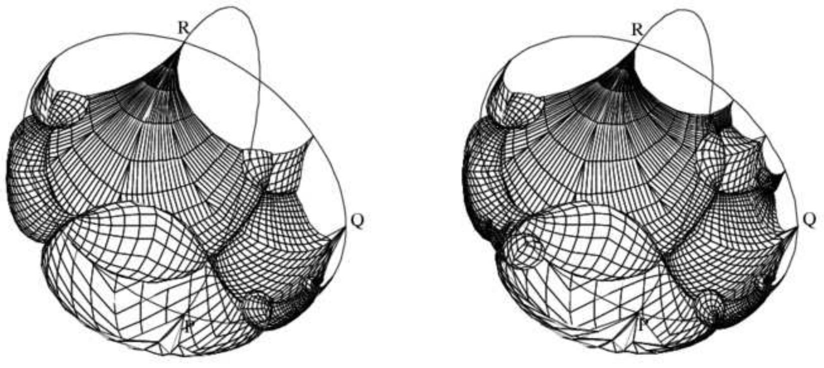


Image of $\{D, A, B, BA, C, CA\}$ Image of$\{D, A, B, C,$
$AB, BA, AC, CA, BC, CB\}$

FIGURE 5.6. Images of the hyperbolic Schwarz map when $k_0 = k_1 = k_\infty = \infty$

5.2.3. *Visualizing the image surface.* The image of the hyperbolic Schwarz map is shown in Figure 5.6. The first one is the image of the triangle $\{D\}$, the second one the two triangles $\{D, A\}$, the third one the four triangles $\{D, A, B, BA\}$, the fourth one the six triangles $\{D, A, B, BA, C, CA\}$, and the last one the ten triangles $\{D, A, B, C, AB, BA, AC, CA, BC, CB\}$.

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