If $\alpha \geq 8$, then this has six solutions where $\sin(3\theta) = 0$. The curve when $\alpha = 8$ is given by

$$(x(t), y(t)) = (-64 \sin t + 2 \sin(2t) + \sin(4t), 64 \cos t + 2 \cos(2t) - \cos(4t)).$$

See Figure 5; the inside curve is a circle drawn for the sake of comparison.

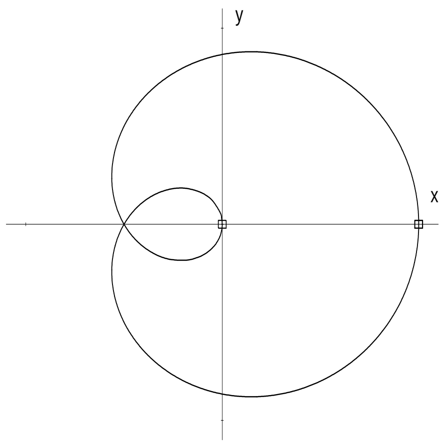

Figure 4

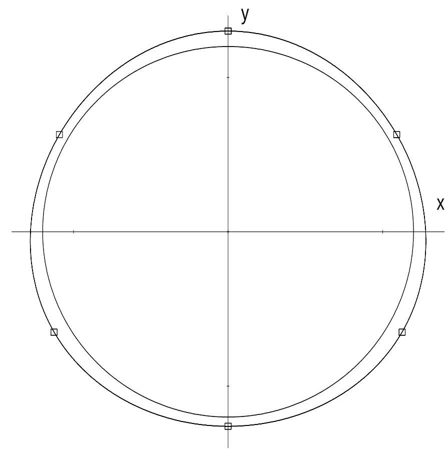

Figure 5

We remark that the study of an estimate of the least number of affine vertices of closed curves on general affine flat tori is open.

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which is the quotient of the whole plane relative to the group generated by g and h, where $g(x,y) = (x+y, y+1)$ and $h(x,y) = (x+\beta y+\alpha, y+\beta)$. One can see the curve $p(t) = (t(t-1)/2, t)$, $0 \le t \le 1$, projects down to a closed curve in the torus, whereas the affine curvature is everywhere zero. For the curve $p(t) = (\lambda^t, \mu^t)$ on the torus belonging to the case A1, we see that the affine curvature is negative, zero, or positive according as $\lambda > \mu^2$, $\lambda = \mu^2$, or $\mu^2 > \lambda > \mu$, respectively.

We next state a simple consequence on the number of affine vertices of a curve on a torus. Let p be a locally strictly convex closed curve on the euclidean torus $\mathbf{R}^2/\mathbf{Z}^2$. Then the affine curvature of a lifted curve into $\mathbf{R}^2$ is obviously periodic. Hence it has at least one maximum and one minimum. Hence

Proposition 6. Any locally strictly convex closed curve on the euclidean torus has at least two affine vertices.

That the number 2 is best possible is shown by the curve in Figure 3 with the affine vertices indicated by small boxes. An example homotopic to zero is the projection of the curve $(\cos(t/3)\cdot\cos t, \cos(t/3)\cdot\sin t)$, $0 \le t \le 3\pi$; see Figure 4. If we further assume that the curve is simple, and thus, homotopic to zero in view of Theorem 1, then the least number is 6 because of the theorem due to Herglotz and Radon. Refer to §19 of [B], where an example with six affine vertices is given. Here we give a new example that is explicitly defined in terms of the functions cos and sin.

Consider a curve $t \mapsto (x(t), y(t))$ with the condition $x'y'' - x''y' = 1$. Put $x' = \lambda\cos\theta$ and $y' = \lambda\sin\theta$, thus defining λ and θ as before. Then it is easy to see that the Blaschke normal ξ is given by

$$\xi = \frac{1}{\lambda}(-\sin\theta, \cos\theta) - \frac{1}{\theta'}\left(\frac{1}{\lambda}\right)'(\cos\theta, \sin\theta)$$

and the affine curvature is equal to

$$k_a = \frac{1}{\lambda^4} - \frac{\lambda''}{\lambda}.$$

Now, regard θ as a local parameter. Then, we see

$$\frac{dx}{d\theta} = \lambda^3\cos\theta, \qquad \frac{dy}{d\theta} = \lambda^3\sin\theta$$

and

$$k_a = \frac{1}{\lambda^4} + \frac{1}{\lambda^3}\frac{d^2}{d\theta^2}\left(\frac{1}{\lambda}\right). \tag{5.2}$$

The affine vertex is obtained by solving $dk/d\theta = 0$. For example, for the curve defined by $\lambda^3 = \alpha + \cos(3\theta)$, where $\alpha > 1$ is a constant, the affine vertices are points where

$$\sin(3\theta)(7\alpha\cos(3\theta) + 8 - \alpha^2) = 0.$$

identity (1.5) hold as required in Introduction, we need to replace ξ by

$$\tilde{\xi} = \lambda(U + \xi) \quad \text{where} \quad \lambda = (f'')^{-1/3} \quad \text{and} \quad U = \frac{1}{3} f'''(f'')^{-5/3} X.$$

Then, by definition in Introduction, we can see

$$k_a(0) = \frac{1}{3} a.$$

This gives a graphical meaning of the affine curvature. Refer to the equation (86) of [Bl], p. 14.

For a nondegenerate curve $p(t) = (x(t), y(t))$, we give another description of the affine curvature. By a change of parameter if necessary, we can assume $x'y'' - y'x'' = 1$ at least locally. Put $X = x'\partial/\partial x + y'\partial/\partial y$ and $\xi = x''\partial/\partial x + y''\partial/\partial y$. Then, for the flat affine connection D on the affine plane, we easily see that $D_X X = \xi$ (hence, $h(X, X) = 1$) and $D_X \xi = x'''\partial/\partial x + y'''\partial/\partial y$. Hence $D_X \xi$ is parallel to X by the assumption on the parameter and, if we put

$$x''' + kx' = y''' + ky' = 0,$$

then $k_a = k$ because of definition (1.5).

On the other hand, the ordinary differential equation

$$x''' + kx' = 0 \tag{5.1}$$

has a constant solution, say 1. Then three solutions 1, x, and y define an immersion of a curve into $\mathbf{R}^3(x^1, x^2, x^3)$ by $(x^1, x^2, x^3) = (1, x, y)$; the curve lies on the affine plane $\{x^1 = 1\}$. A linear transformation of $\mathbf{R}^3$ maps this curve to a new space curve and any projection into another affine plane defines a new plane curve, which is a projective transform of the original plane curve. Remark here that the condition $dk/dt = 0$ does not change under any projective transformation while the value itself dk/dt is not an invariant; refer to [W], Chapter III and [ST].

Definition. A point of the curve in the affine plane is called an *affine vertex* (a sextactic point, classically) if $dk/dt = 0$.

We have

Lemma 5. Assume $k \leq 0$. Then the equation (5.1) has no periodic solution other than constant functions.

Proof. Let the period be 1. Then

$$\int_0^1 k(x')^2 dt = -\int_0^1 x'x''' dt = [-x'x'']_0^1 + \int_0^1 (x'')^2 dt = \int_0^1 (x'')^2 dt,$$

which shows the lemma.

This lemma trivially implies Theorem 3 in Introduction. This theorem, however, does not hold for general affine flat tori. Consider a torus belonging to the case A3(i), Differentiation gives

$$\theta' = (x'y'' - y'x'')/\lambda^2.$$

Hence,

$$\theta(1) - \theta(0) = \int_0^1 (x'y'' - y'x'')/\lambda^2 dt.$$

The value of this integral is by definition an integer-multiple of 2π. We want to show that it is equal to zero. For this, let ℓ be a whole line passing through the two points $p(0)$ and $p(1)$. Its projection onto the torus defines a simple closed curve and the integral obviously vanishes. Therefore, it is enough to see that two simple curves, γ and the projection of ℓ, are isotopically homotopic on the torus, whereas this is known classically; see, say, Theorem 4.1 of [E]. Since a point of inflection is a point where the integrand vanishes, we see that the curve has at least two points of inflection and complete the proof of Theorem 2. Figure 2 shows the situation in the theorem, while the curve $p(t) = (\cos t + t/2, \sin t + 1)$ in Figure 3 defines a curve on the torus which has no points of inflection.

Corollary 4. Any simple closed curve on the standard Klein bottle has at least one point of inflection.

This number 1 is best possible; see also Figure 2.

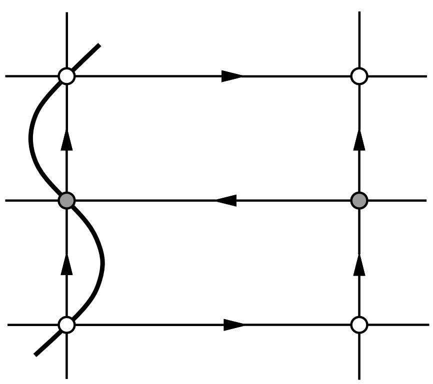

Figure 2

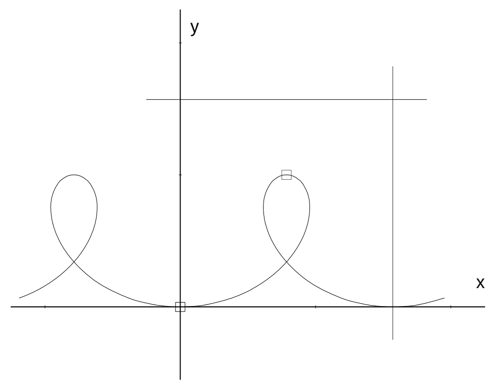

Figure 3

It seems interesting to find an analogue of Theorem 1 for affine flat Klein bottles.

§5. Affine curvature and affine vertices

Let $p(t) = (t, f(t))$ be a curve in the affine plane defined around the origin as the graph of a function f. Assume that the origin is not an inflection point. Then, by performing an affine transformation if necessary, we can assume f has the form

$$f(t) = \frac{1}{2}t^2 + \frac{a}{4!}t^4 + O(t^5)$$

around the origin. Let $X = d/dt = \partial/\partial x + f'\partial/\partial y$ be a tangent vector and $\xi = \partial/\partial y$ a transversal vector. Then, we see $h(X,X) = f''$ and $S = 0$. In order that $\tau = 0$ and the for the case $\epsilon = -1$. The projected curve is locally strictly convex and not homotopic to zero. Further, it is simple provided that the parameter a is sufficiently small.

We next treat the case C(i). Assume first $\lambda, \mu, a, d > 0$. Then, we can reduce the construction to the case where $k = 0$ or $\ell = 0$. Assume $k\ell \neq 0$ and let $m = (|k|, |\ell|)$ be the greatest common divisor; put $k = k'm$ and $\ell = \ell' m$ and choose p and q so that $pk' + q\ell' = 1$. Then $\tilde{g} = g_k^{\ell'} h_\ell^{-k'}$ and $\tilde{h} = g_k^p h_\ell^q$ generate the same holonomy group. The transformation $\tilde{g}$ is the lift of the transformation $g' = g^{\ell'} h^{-k'}$ and $\tilde{g}(r, \theta, \theta) = (R', \Theta', \Theta')$ holds where $g'(r, \theta) = (R', \Theta')$; namely, we have the case where $k = 0$. Now let $k = 0$ and put $(\lambda^t, \mu^t) = (R(t)\cos\Theta(t), R(t)\sin\Theta(t))$. Define a curve on the torus by

$$p(t) = [R(t), \Theta(t), \Theta(t)] \quad 0 \leq t < 1.$$

For the same reason as for A1, the curve is locally strictly convex. Assume, for some q, it holds that $h_\ell^q(p(t)) = p(s)$ for $0 \leq t, s < 1$. Since ℓ cannot be zero, we see $q = 0$ by comparing the third coordinates (the value of z). This means that the curve is simple and we conclude the present case. Next assume $\lambda, \mu < 0$ and $a, d > 0$. In this case, we construct a curve p as in the first case now for the group generated by g_k^2 and h_ℓ, which is a subgroup of index 2 of the modified holonomy group. Since the projection p is in the first quadrangle and that of $g_k p$ is in the opposite quadrangle, they cannot meet each other. Hence, p defines a closed simple curve also on the torus $\tilde{\mathbf{T}}/H^*$. The case where $\lambda, \mu, a, d < 0$ is reduced to the second case by replacing the generators g and h by g and gh.

We finally treat the case C(ii). Similarly to the case C(i), when $\lambda, \mu > 1$, we can assume $k = 0$ and make a lift of the curve $(\lambda^t + t\lambda^{t-1}, \lambda^t) \in \mathbf{R}^2$ as before to get a required simple closed curve. When $\lambda < 0$ and $\mu > 0$, construct a curve for the subgroup generated by g_k^2 and h_ℓ; since this curve lies over the upper half plane and its image by g_k is lying over the lower half plane, both cannot overlap each other and we end the construction. When $\lambda, \mu < 0$, it is enough to consider g and gh for generators.

§4. **Proof of Theorem 2**

Let $\gamma(t)$ be a closed curve in the euclidean torus $\mathbf{R}^2/\mathbf{Z}^2$ with period 1: $\gamma(t+1) = \gamma(t)$ and $p(t)$ a lift of γ on $\mathbf{R}^2$ which passes through the origin. We have two cases:

$$\text{(a)} \quad p(t+1) = p(t) \quad \text{and} \quad \text{(b)} \quad p(t+1) \neq p(t).$$

In the case (a), the curve p is closed in $\mathbf{R}^2$ and its projection γ is homotopic to zero. In the case (b), γ is not homotopic to zero and $p(t)$ is an immersed curve defined for $t \in \mathbf{R}$ with period 1.

We now consider the case (b). Fix the coordinates (x, y) of $\mathbf{R}^2$ and put

$$p(t) = (x(t), y(t)).$$

We assume that the curve is regular, namely, $\lambda = |p'(t)|$ is not zero. Then we can define an angle variable $\theta(t)$ (mod 2π) of tangent line by

$$x' = \lambda\cos\theta, \qquad y' = \lambda\sin\theta.$$

We next consider the inhomogeneous case and first treat the case B(i). The construction depends on (k, ℓ). When $k = \ell = 0$, the modified holonomy group is the same as H. In this case, a required curve is chosen to be the unit-circle in $\mathbf{R}^2 - \{0\}$

$$p(t) = (\cos 2\pi t, \sin 2\pi t) \quad \text{for} \quad 0 \leq t \leq 1$$

when $\epsilon = 1$, and the half unit-circle

$$p(t) = (\cos \pi t, \sin \pi t) \quad \text{for} \quad 0 \leq t \leq 1$$

when $\epsilon = -1$. In the case where $|k| + |\ell| \neq 0$, it is sufficient to take a certain cover of these curves; namely,

$$p(t) = [1, ((2\pi\ell t)), 2\pi\ell t] \quad \text{for} \quad 0 \leq t \leq 1$$

when $\epsilon = 1$ and

$$p(t) = (1, [\pi(2\ell + 1)t], \pi(2\ell + 1)t) \quad \text{for } 0 \leq t \leq 1$$

when $\epsilon = -1$. In the former case, $\ell = 0$ cannot occur. In any case, the simplicity of the projected curve in the torus can be checked easily. Figure 1 shows the case where $\epsilon = 1$ and $k = 2$; the upper surface denotes a fundamental domain in the right helicoid that is a two-fold cover of the lower annulus.

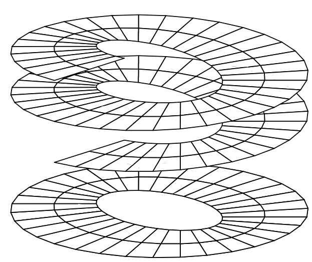

Figure 1

For the case B(ii), we consider an ellipse given by

$$\gamma(t) = (\cos(2\pi t), a \sin(2\pi t)) \quad 0 \leq t \leq 1,$$

where a is a constant to be chosen. Define $R(t)$ and $\Theta(t)$ by

$$(R(t) \cos \Theta(t), R(t) \sin \Theta(t)) = (\cos(2\pi t), a \sin(2\pi t))$$

and construct a curve p on the helicoid by putting

$$p(t) = [R(t), ((2\pi\ell t + \Theta(t))), 2\pi\ell t + \Theta(t)] \quad 0 \leq t \leq 1$$

for the case $\epsilon = 1$ and by

$$p(t) = [R(t), ((2\pi\ell t + \Theta(t) + \pi t)), 2\pi\ell t + \Theta(t) + \pi t] \quad 0 \leq t \leq 1$$

and $\delta = 0$ or 1 according as $\epsilon = 1$ or -1. Then, for $k, \ell \in \mathbf{Z}$, the transformations

$$\begin{aligned} g_k &: [r,\theta,z] \longrightarrow [R,\Theta,z+\Theta-\theta+2\pi k], \\ h_\ell &: [r,\theta,z] \longrightarrow [r,((\theta+\delta\pi)),z+\delta\pi+2\pi\ell] \end{aligned} \tag{2.5}$$

generate the modified holonomy group. The same result holds also for B(ii) by replacing R and Θ by those defined by the equation $(R\cos\Theta, R\sin\Theta) = (\lambda r\cos\theta + r\sin\theta, \lambda r\sin\theta)$. For the cases C(i) and C(ii), the elements

$$\begin{aligned} g_k &: [r,\theta,z] \longrightarrow [R,\Theta,z+\Theta-\theta+2\pi k], \\ h_\ell &: [r,\theta,z] \longrightarrow [R',\Theta',z+\Theta'-\theta+2\pi\ell] \end{aligned} \tag{2.6}$$

generate the modified holonomy group, where R, Θ, R', and Θ' are defined by

$$(R\cos\Theta, R\sin\Theta) = (\lambda r\cos\theta, \mu r\sin\theta),$$
$$(R'\cos\Theta', R'\sin\Theta') = (ar\cos\theta, dr\sin\theta)$$

for the case C(i) and by

$$(R\cos\Theta, R\sin\Theta) = (\lambda r\cos\theta + r\sin\theta, \lambda r\sin\theta),$$
$$(R'\cos\Theta', R'\sin\Theta') = (\mu r\cos\theta + \beta r\sin\theta, \mu r\sin\theta)$$

for the case C(ii). We remark here that the case $k = \ell = 0$ can occur for the cases B, but not for the cases C.

§3. Proof of Theorem 1

The ordinary Euclidean torus is A3(ii). We will prove Theorem 1 by constructing curves without inflection points for each case except A3(ii) in the previous classification.

Let us first consider the homogeneous case and begin with the case A1. The group $\tilde{G}$ is the multiplicative group $\mathbf{R}^+ \times \mathbf{R}^+$, which we identify with the space $\tilde{T}$ as the orbit of the point $(1,1)$. Then the group H corresponds to the lattice and the 1-parameter subgroup $\{\mathrm{diag}(\lambda^t, \mu^t)\}$ to the curve $p(t) = (\lambda^t, \mu^t)$. This curve joins two lattice points $(1,1)$ and (λ, μ) and is locally strictly convex, since $\lambda \neq \mu$ and we may assume $\lambda \neq 1$ and $\mu \neq 1$ because of the assumption; otherwise, replace generators. Its projection into the torus $\tilde{T}/H$ is not homotopic to zero. Assume $p(t)$ and $p(s)$, where $0 \le t, s < 1$, are equivalent under H. Then it holds that $(a^k\lambda^t, d^k\mu^t) = (\lambda^s, \mu^s)$ for some integer k and hence $\log a \log\mu - \log d \log\lambda = 0$, which is contrary to the assumption, i.e., to the proper discontinuity. Hence, we see that the projected curve is simple. This finishes the construction. Similarly, it is easy to see that the following curves have the required properties in each case. A case-by-case check will be omitted; look at the action of generators as in (2.3)-(2.4) for the cases B(iii) and C(iii).

A2(i) $\quad p(t) = (\lambda^t + t\lambda^{t-1}, \lambda^t)$,

A2(ii) $\quad p(t) = (1+\alpha t, \lambda^t)$,

A3(i) $\quad p(t) = (\frac{1}{2}t(t-1), t)$,

B(iii) $\quad p(t) = (\lambda^t\cos\alpha t, \lambda^t\sin\alpha t)$ when $k = \ell = 0$,

$\quad p(t) = [\lambda^t, (((\alpha+2\pi k)t)), (\alpha+2\pi k)t]$ otherwise,

C(iii) $\quad p(t) = [\lambda^t, (((\alpha+2\pi k)t)), (\alpha+2\pi k)t]$.

for the element $g' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} g$. Then the lifts of g to the space $\mathcal{H}$ are parametrized by an integer k: if the eingenvalues of g are positive, define

$$g_k[r,\theta,z] = [R,\Theta,\Theta + z - \theta + 2\pi k] \tag{2.1}$$

and, if the eigenvalues are negative, define

$$g_k[r,\theta,z] = [R,((\Theta' + \pi)),\Theta' + \pi + z - \theta + 2\pi k]. \tag{2.2}$$

Now the modified holonomy groups for the case B(iii) are parametrized by $(k,\ell) \in \mathbf{Z} \times \mathbf{Z}$ and each group is generated by

$$\begin{aligned} g_k &: [r,\theta,z] \longrightarrow [\lambda r, ((\theta + \alpha)), z + \alpha + 2\pi k], \\ h_\ell &: [r,\theta,z] \longrightarrow [r, ((\theta + 2\pi/a)), z + 2\pi/a + 2\pi\ell]; \end{aligned} \tag{2.3}$$

the corresponding torus is $\mathcal{H}/H$. Similarly, for the case C(iii), the modified holonomy groups are parametrized by $(k,\ell) \in \mathbf{Z} \times \mathbf{Z}$ and each group is generated by

$$\begin{aligned} g_k &: [r,\theta,z] \longrightarrow [\lambda r, ((\theta + \alpha)), z + \alpha + 2\pi k], \\ h_\ell &: [r,\theta,z] \longrightarrow [\mu r, ((\theta + \beta)), z + \beta + 2\pi\ell]. \end{aligned} \tag{2.4}$$

When the torus is not affinely homogeneous, the space $\mathbf{T}$ is known to be affinely isomorphic to $\mathbf{R}^2 - \{0\}$. If furthermore the holonomy group acts properly discontinuously, then it is isomorphic to $\mathbf{Z}$ or $\mathbf{Z} \times \mathbf{Z}_k$ for some k. More precisely, it is one of the groups with generators listed below.

B(i) $\quad \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}; \quad |a| > |d| > 1,\ ad > 0, \quad \epsilon = \pm 1;$

B(ii) $\quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}; \quad |\lambda| > 1, \quad \epsilon = \pm 1.$

If the holonomy group does not act properly discontinuously, then the possible H has two types and the generators of H are given as follows:

C(i) $\quad \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}; \quad |\lambda| > |\mu|, \quad \dfrac{\log|\lambda|}{\log|a|} = \dfrac{\log|\mu|}{\log|d|}$ is irrational;

C(ii) $\quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \mu & \beta \\ 0 & \mu \end{pmatrix}; \quad |\lambda| > 1,\ |\mu| > 1, \quad \dfrac{\lambda\beta}{\mu} = \dfrac{\log|\mu|}{\log|\lambda|}$ is irrational.

Refer to [FA] for this list. The modified holonomy group H^* is obtained from H by lifting the action of H to the helicoid as was explained above. For the case B(i), let us define R and Θ by

$$(R\cos\Theta, R\sin\Theta) = (ar\cos\theta, dr\sin\theta)$$

covering group of G. Then there exists a lattice subgroup $\tilde{H}$ of $\tilde{G}$ such that the torus is affinely equivalent to the quotient space $\tilde{G}/\tilde{H}$ with the affine structure induced from any open G-orbit in $\mathbf{R}^2$. See Theorem 3.3 of [NY]. These tori are affinely homogeneous.

According to [FA], we list the groups H of such tori. The numbering corresponds to that used in [FA] and the list consists of the generators of H.

A1 $\begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}, \quad \begin{pmatrix} a & \\ & d \end{pmatrix}; \quad \lambda > \mu > 0,\ d > a > 0,\ \log a \log \mu - \log d \log \lambda \neq 0;$

A2(i) $\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}, \quad \begin{pmatrix} \mu & \beta \\ & \mu \end{pmatrix}; \quad \lambda, \mu > 1,\ \beta \neq 0,\ \lambda\beta \log \mu \neq \mu \log \mu;$

A2(ii) $\begin{pmatrix} 1 & & \alpha \\ & \lambda & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & 1 \\ & \mu & \\ & & 1 \end{pmatrix}; \quad \log \lambda \neq \alpha \log \mu;$

A3(i) $\begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \beta & \alpha \\ & 1 & \beta \\ & & 1 \end{pmatrix}; \quad \alpha \neq \frac{1}{2}\beta(\beta - 1);$

A3(ii) $\begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix};$

B(iii) $\lambda \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}, \quad \begin{pmatrix} \cos(2\pi/a) & \sin(2\pi/a) \\ -\sin(2\pi/a) & \cos(1\pi/a) \end{pmatrix}, \quad \lambda > 1, a \in \mathbf{Z}^+;$

C(iii) $\lambda \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}, \quad \mu \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix},$

$\lambda, \mu > 1, \quad \lambda^k \neq \mu^\ell$ or $k\alpha \neq \ell\beta \bmod 2\pi$ for any $k, \ell \in \mathbf{Z}$ with $|k| + |\ell| \neq 0$.

The space $\tilde{T}$ is a quadrant for A1, a half-plane for A2, the whole plane for A3, and the punctured plane for B and C. The modified holonomy group for the cases A is equal to the holonomy group, while, for the cases B and C, we need to construct it as follows: We realize the universal cover of $\mathbf{R}^2 - \{0\}$ as the right helicoid $\mathcal{H}$ in $\mathbf{R}^3$, the set $\{[r, \theta, z],\ z - \theta \in 2\pi\mathbf{Z},\ 0 \leq \theta < 2\pi\}$ where [ ] denotes the cylinder coordinates of $\mathbf{R}^3$; $[r, \theta, z]$ corresponds to $(r\cos\theta, r\sin\theta, z)$ in the euclidean coordinates. The second argument θ is considered mod 2π, which we may denote by $((\theta))$ for clarity. Let g be an element in the holonomy group and let

$$g[r, \theta] = [R, \Theta] \qquad 0 \leq \theta, \Theta < 2\pi$$

be the representation of g relative to the polar coordinates [ ] in the plane. In the case where the eigenvalues of g are negative, let

$$g'[r, \theta] = [R, \Theta'] \qquad 0 \leq \theta, \Theta' < 2\pi$$

Such a ξ is determined uniquely up to sign and called the Blaschke normal; refer to [NS]. Now the scalar function

$$k_a = \frac{h(SX, X)}{|h(X, X)|} \tag{1.6}$$

is well-defined and is called the *affine curvature* of the curve.

We prove

Theorem 3. Any closed locally strictly convex curve in a euclidean torus has a point where $k_a > 0$.

A point where the affine curvature is stationary is classically called a *sextactic point*; we shall call it an *affine vertex* as an analogue to the notion of vertex in euclidean geometry. In §5, we give some discussions on affine vertices of an affine curve.

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§2. Description of affine flat tori

Affine flat tori, that is, tori with flat affine connections, were classified first by N.H. Kuiper for the convex case and then by [NY] and [FA] for all cases. We begin by reviewing the classification as we need it for our use in the following sections.

Let $A(2)$ denote the group of affine transformations of $\mathbf{R}^2$ and $\widetilde{A(2)}$ its universal covering group. Given an affine 2-torus T^2, we immerse the universal covering $\tilde{T}$ of T^2 into $\mathbf{R}^2$ by the developing map φ. The map φ induces a group homomorphism, denoted by φ_*, from $\pi_1(T^2)$ into $A(2)$. Then the quotient space $\tilde{T}/\ker\varphi_*$ is called the holonomy covering of T^2 and the image H of φ_* is called the affine holonomy group. The image $\mathbf{T} = \varphi(\tilde{T})$ is known to be one of the following: the whole plane $\mathbf{R}^2$, a half-plane, a quadrant, and the punctured plane $\mathbf{R}^2 - \{0\}$, and the group H is an abelian subgroup of $A(2)$ isomorphic to one of the groups $\mathbf{Z}^2$, $\mathbf{Z} \times \mathbf{Z}_k$ (k is a positive integer), and $\mathbf{Z}$. The affine connection is complete if and only if $\mathbf{T}$ is the whole plane. Notice that the action of H on $\mathbf{T}$ is not necessarily properly discontinuous in the case where $\mathbf{T}$ is the punctured plane. To take care of this case, we let $\tilde{\mathbf{T}}$ be the universal covering of $\mathbf{T}$ and H^* the lift of H, which is a subgroup of $\widetilde{A(2)}$. The group H^* is called the modified affine holonomy group. With this terminology, a structure theorem says that the lift of the developing map φ is a covering map and that the torus T^2 is affinely isomorphic to $\tilde{\mathbf{T}}/H^*$. Furthermore, the classification theorem due to [NY] is as follows.

Theorem B. Two affine structures on a torus T^2 are isomorphic if and only if the modified affine holonomy groups are conjugate in $\widetilde{A(2)}$.

A classification of maximal abelian subgroups of $A(2)$ is given in [NY] and [FA]. By virtue of their results, the case where some maximal abelian connected subgroup G that includes H is transitively acting on $\mathbf{T}$ is easy to describe. Specifically, let $\tilde{G}$ be the universal equivalence class of the connection D. Let D' be a torsion-free affine connection that is projectively equivalent to D. By definition, there is a 1-form ρ on M^2 such that

$$D'_X Y = D_X Y + \rho(X)Y + \rho(Y)X, \tag{1.2}$$

where X and Y are tangent vector fields of M. Then, with the same vector ξ, the equation (1.1) shows that the scalar $h(X, X)$ is the same for both D and D'.

We can now cite the following theorem on a closed curve on the projective plane with the standard projective structure.

Theorem A. (1) Any closed curve in the projective plane which is not homotopic to zero has at least one point of inflection. (2) Any simple closed curve in the projective plane that is not homotopic to zero has at least three points of inflection.

The second part of the theorem was known to Möbius; refer to the paper [Kn]. A proof of the theorem can be found in [SS].

Now let M^2 be a torus and D a torsion-free flat affine connection. We call such a pair (M^2, D) or simply M^2 an affine flat torus. Among such tori, a euclidean torus is by definition a torus which is affinely equivalent to the quotient of $\mathbf{R}^2$ by the integer lattice $\mathbf{Z}^2$. Now we can state the following theorems that are analogues of Theorem A.

Theorem 1. On any affine flat torus other than euclidean tori there exists a simple closed curve that is not homotopic to zero and has no points of inflection.

Theorem 2. In a euclidean torus any simple smooth closed curve that is not homotopic to zero has at least two points of inflection.

Contrary to the first part of Theorem A, one can see that Theorem 2 does not hold without the assumption of simpleness on the curve and that the number 2 is best; see examples in §4.

Let us next recall the definition of the affine curvature. When the curve is nondegenerate, the invariant h can be considered as a quadratic form which defines a metric on the curve. On the other hand, the change of the transversal vector field ξ along the curve is expressed by the equation

$$D_X \xi = -f_*(SX) + \tau(X)\xi, \tag{1.3}$$

where SX denotes the tangent component. The endomorphism S is called the affine shape operator. For each transversal vector field ξ, we associate a 1-dimensional volume form ω by

$$\omega(X) = \det\, |X, \xi|, \tag{1.4}$$

where det is the determinant function on the ambient space $\mathbf{R}^2$. Then, it is known that, if the curve is nondegenerate, there exists a transversal vector field ξ such that $\tau = 0$ and

$$|\omega(X)| = |h(X, X)|^{1/2}. \tag{1.5}$$

Inflection points and affine vertices of closed curves on 2-dimensional affine flat tori

Takeshi Sasaki

Dedicated to Professor Katsumi Nomizu on his 70th birthday

ABSTRACT: This paper deals with a geometric problem on inflection points and affine vertices for closed curves in an affine flat torus. We show that the least number of inflection points lying on a closed curve that is not homotopic to zero is 2 if the torus is affinely equivalent to a euclidean torus and 0 otherwise. We consider also the number of affine vertices on a strictly convex closed curve on a flat torus. An explicit example of a closed curve with six affine vertices is given.

Keywords: inflection point, affine vertex, affine flat torus, closed curve

Math. Subject Classification: 53A15, 52A10

§1 Introduction

In this paper we consider closed curves on a 2-dimensional torus with flat affine connection and show some global properties regarding the number of inflection points and affine vertices on such curves.

To begin with let us recall the definition of inflection points. Let M^2 be a 2-manifold with a torsion-free affine connection D and I an interval. An immersion of I into M^2 is called an affine curve in this paper. Let f be an affine curve: $f : I \to M^2$ and ξ a transversal vector field defined along the mapping f. Then, for any tangent vector field X of I, we have the decomposition

$$D_X X = f_*(\nabla_X X) + h(X,X)\xi, \tag{1.1}$$

where $\nabla_X X$ denotes the tangent component. For another choice of ξ, say, $\xi' = \lambda\xi + f_* Z$, where Z is tangent to I, the coefficient $h(X,X)$ changes to $h(X,X)/\lambda$; thus, the property that $h(X,X)(a) = 0$ at a point a is independent of the choice of ξ. In this case, we say that the point a is a point of inflection or, simply, an *inflection point*. Otherwise, the curve is said to be *nondegenerate*; geometrically, this is equivalent to local strict convexity of the affine curve. The notion of inflection is seen to be dependent only on the projective