Homogeneous two-manifolds with an invariant two-form

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Abstract: We study a 2-dimensional manifold that is homogeneous acted on by a 3-dimensional Lie group G, and that has a 2-form invariant under G. We show that such a manifold can be realized as a surface in the affine 3-space and list such realizations.

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Introduction

Let M be a 2-dimensional manifold that is homogeneous acted on by a 3-dimensional Lie group G of diffeomorphisms. We assume that the manifold has a 2-form ρ invariant under G. Our aim is to show that such a manifold can be realized as a surface in the affine 3-space. More precisely, there exists an immersion $f : M \ni (p,q) \mapsto (\xi,\eta,\zeta) \in \mathbb{R}^3$ such that the action of G on M is the pull-back of the restriction on $S = f(M)$ of the natural affine action on $\mathbb{R}^3$ preserving the surface S, and that the form ρ is the pull-back of the natural area form on S. Thus, the manifold is endowed with an equiaffine homogeneous structure; refer to [3, 1, 2, 5]. We make a list of such surfaces S together with the action of G. The realization f is far from being unique. Variety of f reduces to that of area-preserving diffeomorphisms, which are discussed in the last section.

1 Homogeneous two-manifolds with an invariant two-form

1.1 Notations

We consider a 2-dimensional manifold M that is homogeneous acted on by a 3-dimensional Lie group G of diffeomorphisms. We assume that the manifold has a 2-form ρ invariant under G. Let (p, q) be a (local) coordinate system on M and set

$$\rho = \frac{dp \wedge dq}{\phi}.$$

We choose two 1-parameter subgroups of G so that the infinitesimal generators X and Y of respective 1-paramter group are independent in the sense that $\rho(X, Y) \neq 0$.

Let L_X and ι_X denote the Lie derivation and the inner derivative with respect to X, respectively. Since $d\rho = 0$, the identity $d\iota_X + \iota_X d = L_X$ leads to $d(\iota_X \rho) = 0$; hence, $\iota_X \rho = d\eta$ locally for a function η. Similarly, $\iota_Y \rho = d\xi$ for a function ξ. As $\rho(X, Z) = Z\eta$ for any vector field Z, we have $X\eta = 0$ and $Y\eta = \rho(X, Y)$. Similarly, we have $Y\xi = 0$ and $X\xi = \rho(X, Y)$. These imply

$$X = \rho(X, Y)\partial_\xi \quad \text{and} \quad Y = \rho(X, Y)\partial_\eta,$$

where $\partial_\xi = \partial/\partial\xi$ and $\partial_\eta = \partial/\partial\eta$, and $\rho(\partial_\xi, \partial_\eta)\rho(X, Y) = 1$. In particular, the pair (ξ, η) defines a coordinate system. We set

$$\psi = \varphi \frac{\partial(\xi, \eta)}{\partial(p, q)}.$$

Then we have

$$\rho(X, Y) = \phi\Delta, \quad X = \psi\partial_\xi, \quad Y = \psi\partial_\eta,$$

and

$$\rho = \frac{d\xi \wedge d\eta}{\psi}.$$

1.2 Lie algebras

From the expression of X and Y above, we compute the commutators:

$$H := [X, Y] = \psi\psi_\xi\partial_\eta - \psi\psi_\eta\partial_\xi,$$

$$[H, X] = \psi\{(\psi\psi_\eta)_\xi\partial_\xi - (\psi\psi_\xi)_\xi\partial_\eta\},$$

$$[H, Y] = \psi\{(\psi\psi_\eta)_\eta\partial_\xi - (\psi\psi_\xi)_\eta\partial_\eta\}.$$

We now assume that the algebra generated by X, Y and H, which we denote simply by $\mathbb{R}\langle X, Y, H\rangle$, is 3-dimensional. Then we must have the bracket relations such as

$$[H, X] = aX + bY + cH \quad \text{and} \quad [H, Y] = a'Y + b'X + c'H,$$

where a, b, *etc.* are constants. The constants cannot be arbitrary. The Jacobi identity $[X, [Y, H]] + [Y, [H, X]] + [H, [X, Y]] = 0$ implies $(ac' - b'c)X + (bc' - a'c)Y - (a + a')H = 0$. Hence,

$$a' = -a, \quad ac' = b'c \quad \text{and} \quad a'c = bc'.$$

The relations above imply that ψ satisfies the system of differential equations

$$\begin{aligned}(\psi\psi_\xi)_\xi &= -b - c\psi_\xi, \quad (\psi\psi_\eta)_\xi = a - c\psi_\eta,\\ (\psi\psi_\xi)_\eta &= a - c'\psi_\xi, \quad (\psi\psi_\eta)_\eta = b' - c'\psi_\eta.\end{aligned}$$

For the system to be integrable, it is necessary to have

$$c\psi_\eta = c'\psi_\xi. \tag{1.1}$$

Then, we see that ψ is determined by

$$\psi\psi_\xi = a\eta - b\xi - c\psi + c_1 \quad \text{and} \quad \psi\psi_\eta = b'\eta + a\xi - c'\psi + c_2, \tag{1.2}$$

where c_1 and c_2 are integration constants. We need to have $c'c_1 = cc_2$ from (1.1).

We next consider the case where $\mathbb{R}\langle X, Y, H\rangle$ is not 3-dimensional for any choice of X and Y. We use the following lemma whose proof is given in the appendix.

Lemma 1.1 *Let $\mathfrak{g}$ be a 3-dimensional Lie algebra. Assume that it has the property $[A, B] \in \mathbb{R}\{A, B\}$ for any A and B. Then $\mathfrak{g}$ is abelian or isomorphic to the algebra $\mathbb{R}\langle A, B, C\rangle$ with the bracket relations $[A, B] = B$, $[A, C] = C$ and $[B, C] = 0$.*

According to the lemma, we may assume that there exist two vector fields X and Y such that $[X, Y] = 0$, namely $H = 0$. Then ψ is constant; we may assume $\psi = 1$, that is, $X = \partial_\xi$, $Y = \partial_\eta$, and $\rho = d\xi \wedge d\eta$. If the given 3-dimensional Lie group is abelian, then the action is not effective; hence, we may assume the algebra is not abelian. By the lemma above, there is an element Z such that $[Z, X] = X$ and $[Z, Y] = Y$. It is easy to see that $Z = -\xi\partial_\xi - \eta\partial_\eta$ as a vector field. However, Z generates a 1-parameter group of multiplications $(\xi, \eta) \to (\lambda\xi, \lambda\eta)$ and it cannot preserve the form ρ. Hence, in the following consideration, we assume that the Lie algebra $\mathfrak{g}$ of the Lie group G is spanned by X, Y and H.

1.3 Integral curves

We introduce the third variable ζ, and work in the (ξ,η,ζ)-space. When we regard ψ as a funtion of (ξ,η), $\zeta=\psi$ defines a surface S. We set $\pi:(\xi,\eta,\zeta)\mapsto(\xi,\eta)$. Then any integral curve of the vector field $(\pi|_S)^*X$ is given by the system

$$\frac{d\xi}{dt}=\zeta,\quad \frac{d\eta}{dt}=0,\quad \frac{d\zeta}{dt}=a\eta-b\xi-c\zeta+c_1,$$

in view of (1.2). Namely,

$$\frac{d}{dt}\begin{pmatrix}\xi\\ \eta\\ \zeta\\ 1\end{pmatrix}=\begin{pmatrix}0&0&1&0\\ 0&0&0&0\\ -b&a&-c&c_1\\ 0&0&0&0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\\ \zeta\\ 1\end{pmatrix}=:A\begin{pmatrix}\xi\\ \eta\\ \zeta\\ 1\end{pmatrix},$$

where the matrix A is defined. Similarly, for the vector field Y, we have

$$\frac{d}{dt}\begin{pmatrix}\xi\\ \eta\\ \zeta\\ 1\end{pmatrix}=\begin{pmatrix}0&0&0&0\\ 0&0&1&0\\ a&b'&-c'&c_2\\ 0&0&0&0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\\ \zeta\\ 1\end{pmatrix}=:B\begin{pmatrix}\xi\\ \eta\\ \zeta\\ 1\end{pmatrix}.$$

The matrices A and B generate a Lie subalgra in $\mathfrak{gl}(4,\mathbb{R})$, which is isomorphic to $\mathfrak{g}$. Note

$$C:=[A,B]=\begin{pmatrix}a&b'&-c'&c_2\\ b&-a&c&-c_1\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}.$$

1.4 Surfaces on which the integral curves lie

We want to determine the function ψ to see an explicit form of the surface S. We have three cases: (1) $cc'\neq0$, (2) either $c=0$ and $c'\neq0$, or $c\neq0$ and $c'=0$, and (3) $c=c'=0$.

In the case (1), ψ is determined by the system

$$\psi\psi_\xi=a\eta-b\xi-c\psi+c_1,\quad c\psi_\eta=c'\psi_\xi.$$

If we set $x=c\xi+c'\eta$ and $y=c\eta-c'\xi$, then the equations are transformed to

$$cc'\psi\psi_x=ax-cc'\psi+c'c_1,\quad \psi_y=0.$$

Hence, ψ is a function of the variable x. In the case (2) where $c\neq0$ and $c'=0$, we see that $a=b'=c_2=0$ and the function ψ is determined by the system

$$\psi\psi_\xi=-b\xi-c\psi+c_1,\quad \psi_\eta=0.$$

Hence, ψ is a function of ξ. In both cases above, we can write ψ as

$$\psi = F(c\xi + c'\eta), \tag{1.3}$$

where $F(x)$ is a solution of an ordinary differential equation of the form

$$F\frac{dF}{dx} = \alpha x + \beta F + \gamma,$$

and where α, β and γ are constants. We remark that the Lie algebra $\mathfrak{g}$ is solvable in the cases (1) and (2); refer to Section 2.

In the case (3) where $c = c' = 0$, as we have

$$\psi\psi_\xi = a\eta - b\xi + c_1, \quad \psi\psi_\eta = b'\eta + a\xi + c_2,$$

we get

$$\psi^2 = 2a\xi\eta + b'\eta^2 - b\xi^2 + 2c_1\xi + 2c_2\eta + c_3, \tag{1.4}$$

where c_3 is an integration constant. We remark that the algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2 = \mathfrak{sl}(2,\mathbb{R})$ or to $\mathfrak{so}_3 = \mathfrak{so}(3,\mathbb{R})$ unless $a^2 + bb' = 0$. It is solvable when $a^2 + bb' = 0$. We refer to Section 2.

Example 1.2 Let $\mathcal{H} = \{z = x + iy; y > 0\}$ be the upper half-plane with the Poincaré metric. The associated area-form $\rho = dx \wedge dy/y^2$ is invariant under the action of $G = \mathrm{PSL}(2,\mathbb{R})$. The algebra $\mathfrak{g}$ is $\mathfrak{sl}_2$. Let X and Y be the vector fields associated to the 1-parameter subgroups generated by the elements $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. It is easy to see that

$$X = \partial_x \quad \text{and} \quad Y = (y^2 - x^2)\partial_x - 2xy\partial_y.$$

From the identity $\rho(X,Y) = -2x/y$, we see that

$$\partial_\eta = -\frac{y}{2x}\partial_x \quad \text{and} \quad \partial_\xi = -\frac{y(y^2 - x^2)}{2x}\partial_x + y^2\partial_y.$$

Hence, we have $\xi = -1/y$ and $\eta = -x^2/y - y$ up to additive integration constants. Thus, we have a realization

$$\mathcal{H} \ni (x,y) \mapsto (\xi,\eta,\zeta) = \left(-\frac{1}{y}, -y - \frac{x^2}{y}, -\frac{2x}{y}\right) \in \mathbb{R}^3,$$

which is globally defined on $\mathcal{H}$. The surface S is written as $\zeta^2 = 4(\xi\eta - 1)$, which is the case where $a = 2$, $c = c' = b = b' = c_1 = c_2 = 0$ and $c_3 = -4$ in (1.4).

Example 1.3 In the paper [4], it is shown that the surface

$$V : p^2 + q^2 + r^2 - pqr - 4 = 0$$

in the space $\mathbb{R}^3$ with coordinates p, q and r has an action of $\mathrm{SL}(2,\mathbb{R})$. The Lie algebra is generated by vectors X and Y defined by

$$X = \frac{1}{2} f(q)\phi\partial_p \quad \text{and} \quad Y = \frac{1}{2} f(p)\phi\partial_q,$$

where

$$\phi = 2r - pq, \quad f(p) = \frac{2}{\sqrt{p^2-4}} \log \frac{p + \sqrt{p^2-4}}{2}.$$

Set $H = [X,Y]$; then we have $[H,X] = -2X$ and $[H,Y] = 2Y$. The action has the invariant form $\rho = dp \wedge dq/\phi$. Then, we see that $\partial_p = (1/2)f(p)\partial_\xi$ and $\partial_q = (1/2)f(q)\partial_\eta$ and

$$\xi = \frac{1}{2}\left(\log \frac{p + \sqrt{p^2-4}}{2}\right)^2 \quad \text{and} \quad \eta = \frac{1}{2}\left(\log \frac{q + \sqrt{q^2-4}}{2}\right)^2.$$

From the identity $\phi^2 = (p^2-4)(q^2-4)$ holding on the surface V, we get the surface S defined as

$$\zeta^2 = 4\xi\eta,$$

which is a cone in $\mathbb{R}^3$ with coordinates ξ, η, ζ. This is the case where $a = 2$, $c = c' = b = b' = c_1 = c_2 = c_3 = 0$.

Example 1.4 Let G be the upper-triangular group. Every element $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ acts on the plane $\mathbb{R}^2$ with coordinates x and y by $(x,y) \mapsto (x + ay + c, y + b)$. The Lie algebra is nilpotent and the vector fields corresponding to $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ are $X = y\partial_x$ and $Y = \partial_y$, respectively. The 2-form $\rho = dx \wedge dy$ is invariant. Taking $\rho(X,Y) = y$ into account, we can see $(\xi,\eta,\zeta) = (y^2/2, x, y)$. The surface S is the cylinder over the curve $\zeta^2 = 2\xi$.

2 Classification of the actions and the associated surfaces

We list the possible Lie alebras $\mathfrak{g}$ and the associated surfaces S. We allow linear changes of the vector fields X and Y.

2.1 Classification of the actions

We first assume $c \neq 0$ and replace Y by $Y' = Y - (c'/c)X$. We then have $[X, Y'] = H$ and $[H, Y'] = 0$. Namely, the case where $cc' \neq 0$ reduces to the case $c \neq 0$ and $c' = 0$. Let us assume so. Replacing X by $(1/c)X$ and H by $(1/c)H$, we may assume $c = 1$; we then have

$$[X, Y] = H, \quad [H, X] = bY + H, \quad [H, Y] = 0.$$

The algebra $\mathfrak{g}$ is solvable.

We next consider the case where $c = c' = 0$. We assume $a^2 + bb' = 0$. When $b' \neq 0$, set $X' = X - (a/b')Y$. Then $b = -a^2/b'$ implies

$$[X', Y] = H, \quad [H, X'] = 0, \quad [H, Y] = b'X'.$$

By multiplying constans to the vectors, we may assume $b' = \pm 1$. When $b' = 0$, we get

$$[X, Y] = H, \quad [H, X] = bY, \quad [H, Y] = 0,$$

in which case we may assume $b = 0$ or $b = \pm 1$. Namely, up to exchange of X and Y, we have the following cases:

$$[X, Y] = H, \quad [H, X] = \pm Y, \quad [H, Y] = 0.$$

$$[X, Y] = H, \quad [H, X] = 0, \quad [H, Y] = 0.$$

We third consider the case where $c = c' = 0$ and $a^2 + bb' > 0$. We replace X by $X' = X + \rho Y$, Y with $Y' = Y + \sigma X$ and H with $H' = (1 - \rho\sigma)H$. Here we set $\rho = (-a + \sqrt{a^2 + bb'})/b'$ and $\sigma = (a - \sqrt{a^2 + bb'})/b$ when $bb' \neq 0$, $\rho = b/2a$ and $\sigma = 0$ when $b \neq 0$ and $b' = 0$, $\rho = 0$ and $\sigma = -b'/2a$ when $b = 0$ and $b' \neq 0$, and $\rho = \sigma = 0$ when $b = b' = 0$. Then, we see

$$[X', Y'] = H', \quad [H', X'] = \tau X', \quad [H', Y'] = -\tau Y'$$

for an appropriate constant τ. Namely, the algebra is isomorphic to $\mathfrak{sl}_2$. In the following, we may assume $\tau = 2$.

We finally consider the case where $c = c' = 0$ and $a^2 + bb' < 0$. We replace X by $X' = X - (a/b')Y$. Then, we get

$$[X', Y] = H, \quad [H, X'] = (a^2 + bb')/b'Y, \quad [H, Y] = b'X'.$$

By myltiplying constants, we have two cases

$$[X, Y] = H, \quad [H, X] = -Y, \quad [H, Y] = X$$

and

$$[X,Y]=H, \quad [H,X]=Y, \quad [H,Y]=-X,$$

according as $b'>0$ and $b'<0$, respectively. The algebra is isomorphic to $\mathfrak{so}(2,1)$ or to $\mathfrak{so}(3)$, respectively.

Summarizing the above consideration, we have proved the following lemma.

Lemma 2.1 *The Lie algebra* $\mathfrak{g}$ *is isomorphic to one of the following, described by the bracket relations of bases* X, Y *and* H*:*

1. $[H,X]=2X$, $[H,Y]=-2Y$, *where* $\mathfrak{g}\cong\mathfrak{sl}_2$.
2. $[H,X]=\epsilon Y$, $[H,Y]=-\epsilon X$, *where* $\epsilon=\pm1$ *and* $\mathfrak{g}\cong\mathfrak{so}_3$ *or* $\mathfrak{g}\cong\mathfrak{so}(2,1)$.
3. $[H,X]=\pm Y$, $[H,Y]=0$, *where* $\mathfrak{g}$ *is solvable.*
4. $[H,X]=0$, $[H,Y]=0$, *where* $\mathfrak{g}$ *is nilpotent.*
5. $[H,X]=bY$, $[H,Y]=0$, *where* $\mathfrak{g}$ *is solvable.*

2.2 List of surfaces S

We give an expression of surfaces for each case above relative to appropriately chosen affine coordinates and give an explicit form of the action. The invariant 2-form, which is $(d\xi\wedge d\eta)/\zeta$, is also expressed in terms of the new coordinates.

Case 1: $c=c'=b=b'=0$, $a=2$ and $a'=-2$:
The surface is $\zeta^2=4\xi\eta+2c_1\xi+2c_2\eta+c_3$. We may assume $c_1=c_2=0$ by translation of the coordinates ξ and η. The surface is a one-sheeted hyperboloid, a cone or one sheet of two-sheeted hyperboloid, according as $c_3>0$, $c_3=0$ or $c_3<0$. The group $G=\mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ acts on the space of symmetric matrices $\{h=\begin{pmatrix}2\xi & \zeta\\ \zeta & 2\eta\end{pmatrix}\}$, which is identified with the space $\mathbb{R}^3$ with coordinates ξ, η, ζ, by the action $h\mapsto {}^t ghg$ for $g\in G$. This action leaves invariant any surface $\det(h)=c_3$. The invariant two form is

$$\rho=\frac{d\xi\wedge d\eta}{\sqrt{4\xi\eta+c_3}},$$

which is the volume form of the invariant metric

$$\frac{\eta^2 d\xi d\xi+(2\xi\eta+c_3)d\xi d\eta+\xi^2 d\eta d\eta}{4\xi\eta+c_3}.$$

We remark that the metric is the affine metric of the surface; we refer to [3]. It is degenerate when $c_3=0$.

Case 2: $c = c' = a = a' = 0$, $b = \epsilon$ and $b' = -\epsilon$:
The surface is $\zeta^2 = -\epsilon(\xi^2 + \eta^2) + 2c_1\xi + 2c_2\eta + c_3$. We may assume $c_1 = c_2 = 0$ by translation. When $\epsilon = 1$, it is a sphere and the associated group is SO(3). The form ρ is the area form of the Euclidean sphere. When $\epsilon = -1$, the surface is the same as in the first case by a coordinate change, say $\zeta = \xi' + \eta'$, $\xi = \xi' - \eta'$ and $\eta = \zeta'$. This is due to the isomorphism $\mathfrak{sl}_2 \cong \mathfrak{so}(2,1)$.

Case 3: $c = c' = a = a' = b' = 0$ and $b = \epsilon = \pm 1$:
The surface is

$$\zeta^2 = -\epsilon\xi^2 + 2c_1\xi + 2c_2\eta + c_3.$$

We may assume $c_1 = 0$ by translation and recall the notation $C = [A, B]$. Then we have

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & c_2 \\ \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The elements B and C generate an abelian normal subgroup. We denote the action of the 1-parameter subgroup by $(\xi, \eta, \zeta) \mapsto (\xi_t, \eta_t, \zeta_t)$. Then, $\exp(tA)$ is written as the rotation

$$(\xi_t, \eta_t, \zeta_t) = (\xi\cos t + \zeta\sin t, \eta, -\xi\sin t + \zeta\cos t)$$

when $\epsilon = 1$ and the hyperbolic rotation

$$(\xi_t, \eta_t, \zeta_t) = (\xi\cosh t + \zeta\sinh t, \eta, \xi\sinh t + \zeta\cosh t)$$

when $\epsilon = -1$. The 1-parameter subgroup $\exp(tB)$ is written as

$$(\xi_t, \eta_t, \zeta_t) = (\xi, \eta + t\zeta + \frac{1}{2}t^2c_2, \zeta + tc_2)$$

and $\exp(tC)$ is written as

$$(\xi_t, \eta_t, \zeta_t) = (\xi + c_2t, \eta + \epsilon t\xi + \frac{1}{2}\epsilon t^2c_2, \zeta).$$

The form ρ is invariant under these transformations.
Subcase 3.1: When $c_2 = 0$, the surface is the cylinder over the curve $\zeta^2 + \epsilon\xi^2 = c_3$.

Subcase 3.2: Assume $c_2 \neq 0$. Then we may set $c_3 = 0$: the surface is $\zeta^2 + \epsilon\xi^2 = 2c_2\eta$, which is an elliptic or a hyperbolic paraboloid. The form $\rho = d\xi \wedge d\eta/\zeta$ is equal to $d\xi \wedge d\zeta/c_2$. Hence, this form is the volume form of the flat metric $d\xi^2 + \epsilon d\zeta^2$ up to a constant, which is the affine metric of the surface.

Case 4: $c = c' = a = a' = b = b' = 0$:

The surface is $\zeta^2 = 2c_1\xi + 2c_2\eta + c_3$. If $c_1 = c_2 = 0$, then $[A, B] = 0$, which is not included in our consideration; in fact, in this case the surface is a plane on which $\mathbb{R}^2$ acts as the group of translations. Otherwise, by an affine change of coordinates, the surface is the cylinder over the curve $\zeta^2 = 2\xi$. This is the case in Example 1.4.

Case 5: $a = c' = a' = b' = 0$ and $c = 1$:
The surface is the cylinder over the curve $\zeta = f(\xi)$, where f is a solution of the equation

$$ff_\xi + f = -b\xi.$$

The action of the groups $\exp(tB)$ and $\exp(tC)$ is $(\xi, \eta, \zeta) \mapsto (\xi, \eta + t\zeta, \zeta)$ and $(\xi, \eta, \zeta) \mapsto (\xi, \eta + bt\xi + t\zeta, \zeta)$, respectively. To see the action of $\exp(tA)$, we integrate the equation above. If we set $f = u(\xi)\xi$, then the equation becomes

$$\frac{d\xi}{\xi} = -\frac{udu}{u^2 + u + b}.$$

Subcase 5.1: We first consider the case $4b - 1 > 0$. We set $\alpha = \sqrt{4b - 1}$. As $\int du/(u^2 + u + b) = 2/\alpha \arctan((2u + 1)/\alpha)$, we have the curve

$$2\zeta + \xi = \alpha\xi \tan\left(\frac{\alpha}{2} \log k^2(\zeta^2 + \zeta\xi + b\xi^2)\right),$$

where k is an integration constant. If we set $\xi' = \alpha k\xi/2$ and $\zeta' = k(2\zeta + \xi)/2$, then the curve is written as

$$\zeta' = \xi' \tan\left(\frac{\alpha}{2} \log(\xi'^2 + \zeta'^2)\right).$$

Relative to the polar coordinates (r, θ) where $\xi' = r\cos\theta$ and $\zeta' = r\sin\theta$, it is written as

$$r = \exp(\theta/\alpha);$$

namely, we obtain a logarithmic spiral. The action of $\exp(tA)$ is $r_t = r\exp(-t/2)$ and $\theta_t = \theta - \alpha t/2$. The invariant 2-form is written as $2d\xi \wedge d\log r$.
Subcase 5.2: In the case where $4b - 1 = 0$, the curve is seen to be

$$\xi' = \zeta' \log(k\zeta'),$$

where k is an integration constant, $\xi' = -\xi$ and $\zeta' = 2\zeta + \xi$. The action of $\exp(tA)$ is $\xi'_t = \exp(-t/2)(\xi' + t\zeta'/2)$ and $\zeta'_t = \exp(-t/2)\zeta'$. The invariant 2-form is $-2d\xi' \wedge d\eta/(\xi' + \zeta')$.
Subcase 5.3: For the last case where $4b - 1 < 0$, we set $\beta = \sqrt{1 - 4b}$. Then, relative to the coordinates $\xi' = 2\zeta + (1 + \beta)\xi$ and $\zeta' = 2\zeta + (1 - \beta)\xi$, the curve is written as

$$(1 - \beta)\log\zeta' = (1 + \beta)\log\xi' + k,$$

where k is an integration constant. The action of $\exp(tA)$ is $\xi'_t = \exp(-(1-\beta)t/2)\xi'$ and $\zeta'_t = \exp(-(1+\beta)t/2)\zeta'$. The invariant 2-form is $d\log\xi' \wedge d\eta$ up to a constant.

Summarizing the argument, we have proved the following theorem.

Theorem 2.2 *Let M be a two-manifold with an action of a 3-dimenisonal non-abelian Lie group and assume that it has an invariant 2-form ρ. Then, there exists locally an immersion $f : M \ni (p,q) \mapsto (\xi,\eta,\zeta) \in \mathbb{R}^3$ such that the action of the group on M is the pull-back of the restriction on $S = f(M)$ of the natural affine action on $\mathbb{R}^3$ preserving the surface S, and that the form ρ is the pull-back of the area form $d\xi \wedge d\eta/\zeta$ on S. This area form is the metrical volume form when S is not cylindrical. The following list exhausts the surfaces up to affine transformation. Here, $\epsilon = \pm 1$, and c_i, k, α, β are constants.*

1 : $\zeta^2 = 4\xi\eta + c_3$.

2 : $\zeta^2 = -\epsilon(\xi^2 + \eta^2) + c_3$.

3.1 : *Cylinder over* $\zeta^2 + \epsilon\eta^2 = c_3$.

3.2 : $\zeta^2 + \epsilon\eta^2 = 2c_1\xi$.

4 : *Cylinder over* $\zeta^2 = 2\xi$.

5.1 : *Cylinder over* $\zeta = \eta\tan\left(\alpha\log(\eta^2+\zeta^2)\right), \quad \alpha > 0$.

5.2 : *Cylinder over* $\eta = \zeta\log(k\zeta)$.

5.3 : *Cylinder over* $(1-\beta)\log\zeta = (1+\beta)\log\eta + k, \quad 0 < \beta < 1$.

3 Nonrigidity of area-preserving analytic diffeomorphisms

Conversely, for M, ϕ and ρ given as above, and for one of the surfaces listed above, say, $S : \zeta^2 = 4\xi\eta + c_3$, we consider a solution (ξ,η) of the differential equations

$$\phi(p,q)\partial(\xi,\eta)/\partial(p,q) = \sqrt{4\xi\eta + c_3}, \tag{3.1}$$

which is the condition that the map $(p,q) \mapsto (\xi,\eta)$ is volume-preserving. The target surface is $\zeta^2 = 4\xi\eta + c_3$ with the volume form $d\xi \wedge d\eta/\sqrt{4\xi\eta + c_3}$. We define Δ by $\phi\Delta = \phi(p,q)\partial(\xi,\eta)/\partial(p,q)$. Then obviously $\rho = d\xi \wedge d\eta/(\phi\Delta)$. We next define two vector fields X and Y by

$$X = \phi\Delta\partial_\xi \quad \text{and} \quad Y = \phi\Delta\partial_\eta.$$

Then, it is easy to see that $H := [X, Y]$ satisfies $[H, X] = 2X$ and $[H, Y] = -2Y$. Hence, the Lie group generated by X and Y, which is locally isomorphic to SL_2, acts on M.

Here arises a question: Is the solution (ξ, η) of the equation (3.1) unique under a suitable condition?

As we see below, it is far from beging unique. Let us rewrite the equation in a more accessible coordinate system. By identifying the surface with the upper half-plane (see Example 1.2), the equation is written as

$$\phi(p, q)\partial(x, y)/\partial(p, q) = 1/y^2$$

with (x, y) valued in the upper half-plane. By changing the coordinates further from (x, y) to $(u = 1/y, v = x)$, we get the equation

$$\phi(p, q)\partial(u, v)/\partial(p, q) = 1.$$

Hence, the question is reduced to finding a suitable condition for the uniqueness assertion of the equation

$$\partial(u, v)/\partial(U, V) = 1$$

relative to the mapping (U, V) to (u, v). There would be no such.

Example 3.1 Let us work on the upper half-plane $\mathcal{H}$ with coordinates $z = x + iy$. Since there are uncontrollably many volume-preserving diffeomorphisms in general, we restrict our consideration to real analytic volume-preserving automorphisms which commute with the extended modular group Γ generated by $\mathrm{SL}(2, \mathbf{Z})$ and a reflection $r : z = x + iy \mapsto z = -x + iy$.

Claim: There are volume-preserving harmonic flows T which commute with the action of Γ.

Let us write the vector field generating the flow T by

$$V = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}.$$

The vector field V preserves the volume form $dx \wedge dy/y^2$ if and only if

$$\alpha = y^2\xi_y, \quad \beta = -y^2\xi_x \quad \text{for some} \quad \xi(x, y).$$

The flow commutes with the action of Γ if and only if

$$\begin{aligned} \xi(g(x, y)) &= \xi(x, y), && g \in SL(2, \mathbf{Z}), \\ \xi(g(x, y)) &= -\xi(x, y), && g \in \Gamma - SL(2, \mathbf{Z}). \end{aligned}$$

Such a function ξ exists; there are in fact many. If J denotes the automorphic function, the so-called j-invariant, the odd powers of its imaginary part $\chi(x,y) = \mathrm{Im}\ J(z)$ and their linear combinations (over $\mathbb{R}$) can serve as ξ. Indeed, since the inverse function of J is the Schwarz map of the hypergeometric equation of type $(2,3,\infty)$, we have $J(rz) = \overline{J(z)}$, and so $\chi(rz) = -\chi(z)$. In particular this implies that ξ vanishes along the walls of the Weyl chamber of the Coxeter group Γ; in other words, the flow generated by V preserves the walls. Furthermore, the flow when $\xi = \chi^3$, for example, preserves the wall *pointwise*.

This example shows that a seemingly-strong boundary condition still does not assure the uniqueness.

Appendix. Proof of Lemma 1.1.

Let $\{A, B, C\}$ be a basis of $\mathfrak{g}$. By the assumption

$$[A,B] = aA + bB, \quad [A,C] = cA + dC, \quad [B,C] = eB + fC$$

for some constants a, b, etc. Since $[A, B + C]$ is also a linear combination of A and $B + C$, it holds $d = b$. Similarly, $e = c$ and $f = -a$. Hence,

$$[A,B] = aA + bB, \quad [A,C] = cA + bC, \quad [B,C] = cB - aC.$$

Assume $a \neq 0$. Then $\{A' = aA + bB, B, C\}$ is a basis and

$$[A',B] = aA', \quad [A',C] = cA', \quad [B,C] = cB - aC.$$

Namely, we have the case where $b = 0$. Hence, we may assume $ab = 0$ and, taking into acoount the symmetry, it is enough to treat the case $a = 0$:

$$[A,B] = bB, \quad [A,C] = cA + bC, \quad [B,C] = cB.$$

Assume next $b \neq 0$. Then, $\{A, B, C' = cA + bC\}$ is a new basis and

$$[A,B] = bB, \quad [A,C'] = bC', \quad [B,C'] = 0.$$

Namely, we have the case where $c = 0$. Hence, it is enough to consider the case $bc = 0$. By symmetry we may assume $c = 0$. If $b = 0$ further, then $\mathfrak{g}$ is abelian. If $b \neq 0$, then replace A by bA to get

$$[A,B] = B, \quad [A,C] = C, \quad [B,C] = 0.$$

Thus, we have Lemma 1.1.

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