

# On the generalized Gauss hypergeometric system of three variables

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## §1. Introduction

A partition  $\lambda$  is any sequence  $(\lambda_1, \dots, \lambda_n)$  of nonnegative integers such that  $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ . The length of partition  $\lambda$  is the number of  $\lambda_i \neq 0$  and is denoted by  $l(\lambda)$ . The weight is defined to be  $|\lambda| = \sum \lambda_i$ . For a given fixed parameter  $k$  and for a partition  $\lambda = (\lambda_1, \dots, \lambda_n)$  of length  $l(\lambda) \leq n$ , we define

$$a_\lambda = \prod_{i=1}^n (a - k(i-1), \lambda_i), \quad a \in \mathbf{C},$$

where

$$(a, m) = a(a+1) \cdots (a+m-1), \quad a \in \mathbf{C}, \quad m \in \mathbf{N}.$$

Let  $J_\lambda(x; k)$  be the Jack polynomials of  $n$  variables  $x = (x^1, x^2, \dots, x^n)$ . Then the *generalized Gauss hypergeometric function*  ${}_2F_1(a, b; c; k; x)$  of  $n$  variables associated with Jack polynomial is defined as

$${}_2F_1(a, b; c; k; x) = \sum_{\lambda} \frac{a_\lambda b_\lambda J_\lambda(x; k)}{c_\lambda k^{|\lambda|} \langle J_\lambda, J_\lambda \rangle},$$

where the summation runs over all partitions  $\lambda$  of length  $\leq n$ . Here,  $a$ ,  $b$ , and  $c$  are complex numbers such that  $(c)_\lambda \neq 0$  for all  $\lambda$  and  $\langle \cdot, \cdot \rangle$  denotes certain inner product of the space of symmetric polynomials that depends on the parameter  $k$ ; refer to [7]. The series above was defined by Korányi[4], Macdonald, and Kaneko[2] independently. It is characterized by the following fact.

**Theorem** (Z. Yan[8]) *The hypergeometric function  ${}_2F_1(a, b; c; k; x)$  is the unique (up to constant) solution of the following system of differential equations which is symmetric in the  $n$  variables  $x^1, \dots, x^n$ , and analytic at  $x^1 = \dots = x^n = 0$ :*

$$(1.1) \quad \begin{aligned} & x^i(1-x^i)z_{x^i x^i} + \{c - (n-1)k - (a+b+1-nk)x^i\}z_{x^i} - k \sum_{j=1}^n x^j z_{x^j} \\ & + k(1-x^i) \sum_{j \neq i} \frac{x^i z_{x^i} - x^j z_{x^j}}{x^i - x^j} - abz = 0; \quad i = 1, \dots, n. \end{aligned}$$

We call this system the *generalized Gauss hypergeometric system associated with Jack polynomials* and denote it by  $HJ(a, b, c; k; (x^1, \dots, x^n))$  if one needs to specify variables or  $HJ(a, b, c; k)$  if not and by  $HJ$  for short. In [5], Mimachi considered integrals of Selberg type and computed the Gauss-Manin system satisfied by these integrals. He has shown that the system has a subsystem that is equivalent to the  $HJ$  above. The aim of this paper is to give the Pfaffian form explicitly for  $HJ$  when  $n = 3$  and consider local behaviors of solutions around some singular points. Note that the associated connection form is a logarithmic 1-form with constant coefficients. See Theorem 2.1.

We give a few remarks:

**Remarks.** (1) When  $n = 1$ , the function  ${}_2F_1(a, b; c; k; x)$  is independent of the parameter  $k$  and is the classical Gauss hypergeometric function  ${}_2F_1(a, b; c; x)$ . When  $n = 2$ , the system is equivalent to Appell's hypergeometric system  $F_4$ ; refer to [1] and [3]. Its explicit expression is cited in the appendix.

(2) When  $k = 0$ , the system is the  $n$ -times tensorial product of Gauss hypergeometric equations; namely, we have

$${}_2F_1(a, b; c; 0; (x^1, \dots, x^n)) = \prod_{i=1}^n {}_2F_1(a, b; c; x^i),$$

where  ${}_2F_1(a, b; c; x^i)$  is the Gauss hypergeometric function. This identity means that the system  $HJ$  can be regarded as a deformation of the tensorial product of the Gauss hypergeometric equations.

(3) The following identity is an analogue of classical Kummer relations([8]).

$$\begin{aligned} {}_2F_1(a, b, c; k; x) &= \prod(1 - x^i)^{-a} {}_2F_1\left(a, c - b, c; k; -\frac{x^1}{1 - x^1}, \dots, -\frac{x^n}{1 - x^n}\right) \\ &= \prod(1 - x^i)^{c-a-b} {}_2F_1(c - a, c - b, c; k; x). \end{aligned}$$

Further, we can see by computation that the system  $HJ$  has the following symmetry.

**Lemma 1.1.** *The function*

$$\prod_{i=1}^n (x^i)^\rho {}_2F_1(a + \rho, b + \rho, c + 2\rho; k; (x^1, \dots, x^n))$$

is a solution of the system  $J(a, b, c; k; (x^1, \dots, x^n))$ , where  $\rho = (n - 1)k + 1 - c$ .

**Lemma 1.2.** *By the coordinate transformation  $\tau: x = (x^1, \dots, x^n) \rightarrow \xi = (\xi^1, \dots, \xi^n)$ , where  $\xi^i = 1/x^i$ , the system  $HJ(a, b, c; k; x)$  relative to the unknown  $z$  is transformed to the system  $HJ(a, a - c + 1 - (n - 1)k, a - b + 1 + (n - 1)k; k; \xi)$  relative to the new unknown  $f := z \prod_{i=1}^n (x^i)^a$ .*

**Lemma 1.3.** *By the coordinate transformation  $\sigma: x = (x^1, \dots, x^n) \rightarrow \xi = (\xi^1, \dots, \xi^n)$ , where  $\xi^i = 1 - x^i$ , the system  $HJ(a, b, c; k; x)$  is transformed to the system  $HJ(a, b, a + b + 1 - c + (n - 1)k; k; \xi)$  relative to the same unknown.*

## §2. Pfaffian form of the generalized Gauss hypergeometric system of three variables

The system  $HJ$  when  $n = 3$  is written as follows; here, we write  $(x, y, z)$  for  $(x^1, x^2, x^3)$  and  $w$  for the unknown.

$$\begin{aligned} &x(1 - x)w_{xx} + (c - 2k - (a + b + 1 - 3k)x)w_x - k(xw_x + yw_y + zw_z) \\ &+ k(1 - x) \left( \frac{xw_x - yw_y}{x - y} + \frac{xw_x - zw_z}{x - z} \right) - abw = 0, \end{aligned}$$

$$\begin{aligned}
& y(1-y)w_{yy} + (c-2k - (a+b+1-3k)y)w_y - k(xw_x + yw_y + zw_z) \\
& \quad + k(1-y) \left( \frac{yw_y - zw_z}{y-z} + \frac{yw_y - xw_x}{y-x} \right) - abw = 0, \\
& z(1-z)w_{zz} + (c-2k - (a+b+1-3k)z)w_z - k(xw_x + yw_y + zw_z) \\
& \quad + k(1-z) \left( \frac{zw_z - xw_x}{z-x} + \frac{zw_z - yw_y}{z-y} \right) - abw = 0.
\end{aligned}$$

We define a frame  $e$  by

$$e = {}^t(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8),$$

$$\begin{aligned}
e_1 &= w, \quad e_2 = xw_x, \quad e_3 = yw_y, \quad e_4 = zw_z, \quad e_5 = xy \left( w_{xy} - k \frac{w_x - w_y}{x-y} \right), \\
e_6 &= yz \left( w_{yz} - k \frac{w_y - w_z}{y-z} \right), \quad e_7 = zx \left( w_{zx} - k \frac{w_z - w_x}{z-x} \right), \\
e_8 &= xyz \left\{ w_{xyz} + k \left( \frac{1}{x-z} + \frac{1}{y-z} \right) w_{xy} + k \left( \frac{1}{y-x} + \frac{1}{z-x} \right) w_{yz} \right. \\
& \quad \left. + k \left( \frac{1}{z-y} + \frac{1}{x-y} \right) w_{zx} - 4k^2 \frac{1}{(x-y)(x-z)} w_x \right. \\
& \quad \left. - 4k^2 \frac{1}{(y-z)(y-x)} w_y - 4k^2 \frac{1}{(z-x)(z-y)} w_z \right\}.
\end{aligned}$$

Then a calculation shows the following formula.

**Theorem 2.1.** *The system  $HJ(a, b; c; k)$  is equivalent to the Pfaffian system*

$$de = \Omega e,$$

where

$$\begin{aligned}
\Omega &= P_1 \frac{dx}{x} + P_2 \frac{dy}{y} + P_3 \frac{dz}{z} + P_4 \frac{dx}{x-1} + P_5 \frac{dy}{y-1} + P_6 \frac{dz}{z-1} \\
& \quad + P_7 \frac{d(x-y)}{x-y} + P_8 \frac{d(y-z)}{y-z} + P_9 \frac{d(z-x)}{z-x}.
\end{aligned}$$

The  $8 \times 8$  matrices  $P_i$  are given as follows.

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-c+2k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -k & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1-c+2k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & 0 & -k & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-c+2k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-c+2k \end{pmatrix},$$

$$\begin{aligned}
P_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-c+2k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-c+2k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-c+2k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & -k & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-c+2k \end{pmatrix}, \\
P_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -k & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1-c+2k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k & -k & 1 \\ 0 & 0 & 0 & 0 & 0 & 1-c+2k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-c+2k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-c+2k \end{pmatrix}, \\
P_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -ab & c' & -k & -k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(k-a)(k-b) & 0 & c' & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(k-a)(k-b) & 0 & -k & c' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(2k-a)(2k-b) & 0 & c' \end{pmatrix}, \\
P_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -ab & -k & c' & -k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(k-a)(k-b) & 0 & 0 & c' & 0 & -k & 0 \\ 0 & 0 & 0 & -(k-a)(k-b) & 0 & c' & -k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(2k-a)(2k-b) & c' \end{pmatrix}, \\
P_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -ab & -k & -k & c' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(k-a)(k-b) & 0 & -k & c' & 0 & 0 \\ 0 & -(k-a)(k-b) & 0 & 0 & -k & 0 & c' & 0 \\ 0 & 0 & 0 & 0 & 0 & -(2k-a)(2k-b) & 0 & c' \end{pmatrix},
\end{aligned}$$

$$P_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k & k & 0 & 0 & 0 & 0 & 0 \\ 0 & k & -k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k & k & 0 \\ 0 & 0 & 0 & 0 & 0 & k & -k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k & k & 0 & 0 & 0 & 0 \\ 0 & 0 & k & -k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_9 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k & 0 & k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & -k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & k & 0 & 0 \\ 0 & 0 & 0 & 0 & k & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where  $c' = c - a - b - 1$ .

### §3. Singular points of the system $HJ$

We regard the system  $HJ$  as lying on the space  $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ . The set of singular divisors consists of 12 planes:

$$\{x = 0\}, \quad \{y = 0\}, \quad \{z = 0\}, \quad \{x = 1\}, \quad \{y = 1\}, \quad \{z = 1\},$$

$$\{x = \infty\}, \quad \{y = \infty\}, \quad \{z = \infty\}, \quad \{x = y\}, \quad \{y = z\}, \quad \{z = x\}.$$

Singular points, where pass through three or more divisors, are  $(p, q, r)$ , where  $p, q,$  and  $r$  are one of 0, 1, and  $\infty$ . Up to permutation of coordinates, we classify them into three classes:

- (1)  $(0, 0, 0), (1, 1, 1), (\infty, \infty, \infty)$
- (2)  $(0, 1, 1), (1, 0, 0), (1, \infty, \infty), (\infty, 0, 0), (\infty, 1, 1), (0, \infty, \infty)$
- (3)  $(\infty, 0, 1)$

Through each point of (1) pass six divisors and through each point of (2) four divisors. At the point of (3), the crossing is normal. Now recall Lemma 1.2 and Lemma 1.3. These lemmas show that each singular point can be transformed to one of the following by the coordinate transformations  $\tau$  and  $\sigma$  for  $n = 3$ :

$$(0, 0, 0), \quad (1, 0, 0), \quad (\infty, 0, 1).$$

We are going to find solutions around the points  $(0, 0, 0)$  and  $(1, 0, 0)$  in the following sections.

### §4. Solutions around the point $(0, 0, 0)$

Let us consider the system around the point  $(0,0,0)$ . To make the system look nice, we blow up the space  $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$  at the origin by introducing coordinates  $(u, \eta, \zeta)$  defined by

$$x = u, \quad y = u\eta, \quad z = u\zeta.$$

The Pfaffian system relative to the new coordinates  $(u, \eta, \zeta)$  is

$$de = \Omega e,$$

where

$$\begin{aligned} \Omega = & Q \frac{du}{u} + P_2 \frac{d\eta}{\eta} + P_3 \frac{d\zeta}{\zeta} + P_4 \frac{du}{u-1} + P_7 \frac{d\eta}{\eta-1} + P_9 \frac{d\zeta}{\zeta-1} \\ & + P_8 \frac{d(\eta-\zeta)}{\eta-\zeta} + P_5 \frac{d(u\eta)}{u\eta-1} + P_6 \frac{d(u\zeta)}{u\zeta-1}, \end{aligned}$$

$$Q = P_1 + P_2 + P_3 + P_7 + P_8 + P_9$$

$$= \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -c+1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -c+1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c+1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2(k-c+1) & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2(k-c+1) & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2(k-c+1) & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3(2k-c+1) \end{pmatrix}.$$

Note that, on the plane  $\{u = 0\}$ , we have five divisors

$$\eta = 0, \quad \eta = 1, \quad \zeta = 0, \quad \zeta = 1, \quad \eta = \zeta.$$

The configuration of these divisors is the same as that of Appell's system  $F_1$ .

The eigenvalues of  $Q$  are  $0, 1-c, 2(1-c+k)$ , and  $3(1-c+2k)$ ; the multiplicities are  $1, 3, 3, 1$ , respectively. Hence, assuming that any of  $1-c, 2(1-c+k)$ , and  $3(1-c+2k)$  is not integer, we can find eight independent solutions, one is holomorphic, three are of the form  $u^{1-c} \cdot \phi(u, v, w)$ , the other three of the form  $u^{2(1-c+k)} \cdot \phi(u, v, w)$ , and the last of the form  $u^{3(1-c+2k)} \cdot \phi(u, v, w)$ , where  $\phi(u, v, w)$  is holomorphic relative to  $u$ . This follows from the next lemma.

**Lemma 4.1.** (Kato[3]) *Let  $B(x^1, \dots, x^r)$  and  $\Omega_i(x^1, \dots, x^r)$  ( $i = 2, \dots, r$ ) be  $n \times n$  matrices holomorphic at the point  $(0, a^2, \dots, a^r)$  and  $A$  a constant matrix. Put  $\Omega_1 = A/x^1 + B$ ,  $\Omega = \Omega_1 dx^1 + \dots + \Omega_r dx^r$ , which satisfies  $d\Omega = \Omega \wedge \Omega$ . Suppose moreover that  $\rho$  is an eigenvalue of  $A$  but any positive integer is not an eigenvalue of  $A - \rho I_n$ . Let  $m$  be the corank of  $A - \rho I_n$  and  $\omega_i(x^2, \dots, x^r) = \Omega_i(0, x^2, \dots, x^r)$  ( $i = 2, \dots, r$ ),  $\omega = \omega_2 dx^2 + \dots + \omega_r dx^r$ . Then*

$$(4.1) \quad de = \Omega e$$

has  $m$  linearly independent solutions of the form  $e = (x^1)^\rho f$ , where  $f$  is holomorphic at the point  $(0, a^2, \dots, a^r)$ . Put  $f_0(x^2, \dots, x^r) = f(0, x^2, \dots, x^r)$  for any solution  $e = x^\rho f$  of the equation; then,  $f_0$  satisfies

$$(4.2) \quad (A - \rho I_n)f_0 = 0 \quad \text{and} \quad df_0 = \omega f_0.$$

Conversely, for each solution  $f_0(x^2, \dots, x^r)$  of (4.2), there exists a unique solution  $e = x^\rho f$  of (4.1) satisfying  $f(0, x^2, \dots, x^r) = f_0(x^2, \dots, x^r)$ .

Let us consider the eigenvalue  $\rho = 1 - c$ . To get a solution of the form

$$e = u^\rho f, \quad f = f_0(\eta, \zeta) + u f_1(u, \eta, \zeta),$$

solve the condition  $(Q - \rho I)f_0 = 0$  in Lemma 4.1. We see that its general solution has the form

$$f_0 = {}^t(e_1, \rho e_1 - e_3 - e_4, e_3, e_4, 0, 0, 0, 0).$$

By definition, it is a solution of the equation

$$df_0 = \omega f_0,$$

$$\omega = P_2 \frac{d\eta}{\eta} + P_3 \frac{d\zeta}{\zeta} + P_7 \frac{d\eta}{\eta - 1} + P_9 \frac{d\zeta}{\zeta - 1} + P_8 \frac{d(\eta - \zeta)}{\eta - \zeta}.$$

Then, the vector  $\tilde{f} = {}^t(e_1, e_3, e_4)$ , which is the essential part of  $f_0$ , satisfies the induced system

$$(4.3) \quad d\tilde{f} = \omega_0 \tilde{f},$$

$$\omega_0 = R_1 \frac{d\eta}{\eta} + R_2 \frac{d\zeta}{\zeta} + R_3 \frac{d\eta}{\eta - 1} + R_4 \frac{d\zeta}{\zeta - 1} + R_5 \frac{d(\eta - \zeta)}{\eta - \zeta},$$

where

$$R_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \rho + 2k & 0 \\ 0 & -k & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -k \\ 0 & 0 & \rho + 2k \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 0 & 0 & 0 \\ k\rho & -2k & -k \\ 0 & 0 & 0 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k\rho & -k & -2k \end{pmatrix}, \quad R_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -k & k \\ 0 & k & -k \end{pmatrix}.$$

In fact,  $R_1$  (resp.  $R_2$ ) is the submatrix of  $P_2$  (resp.  $P_3$ ) consisting of the  $(i, j)$ -th components where  $i, j = 1, 3, 4$ . So is for  $R_2$ . As for  $R_3$ , let us consider  $de_3$ : A look at the matrix  $P_7$  shows that the coefficient of the form  $d\eta/(\eta - 1)$  is  $ke_2 - e_3$  that is equal to  $k(\rho e_1 - e_3 - e_4) - e_3$ . Since  $de_1$  and  $de_4$  have no component including the form  $d\eta/(\eta - 1)$ , we get the matrix  $R_3$ . The remaining matrices are obtained similarly.

Now compare the system (4.3) with the Appell's system (A1.1) for  $F_1$  in the appendix. Then we can see that the Pfaffian system (4.3) is the same as Appell's system  $F_1(c - 1, k, k; c - k; \eta, \zeta)$ . As for the eigenvalue  $2(1 - c + k)$ , we can proceed similarly. We get the following proposition.

**Proposition 4.2.** Assume  $2k - c + 1 \notin \mathbf{Z}$ . Then, (1) any solution of the system corresponding to the eigenvalue  $1 - c$  can be written as

$$f = f_0(\eta, \zeta) + u f_1(u, \eta, \zeta),$$

where  $f_0$  is a solution of Appell's system  $F_1(c-1, k, k; c-k; \eta, \zeta)$ . (2) Solutions corresponding to the eigenvalue  $2(1 - c + k)$  have a similar expression, the first term corresponding to  $F_1(4k+1-c, k, k; 3k+2-c; \eta, \zeta)$ . The number of linear independent solutions is 3 in each case.

Thus, we have six solutions, which, together with the generalized Gauss hypergeometric function  ${}_2F_1(a, b, c; k; x, y, z)$  and its associate  $(xyz)^{2k+1-c} {}_2F_1(a-c+1+2k, b-c+1+2k, 4k+2-c; k; x, y, z)$  (refer to Lemma 1.1), form a fundamental set of solutions.

## §5. Solutions around the point $(1, 0, 0)$

We introduce the coordinates  $(u, y, z) = (x - 1, y, z)$ . The Pfaffian form is

$$\begin{aligned} \Omega = & P_1 \frac{du}{1+u} + P_2 \frac{dy}{y} + P_3 \frac{dz}{z} + P_4 \frac{du}{u} + P_5 \frac{dy}{y-1} + P_6 \frac{dz}{z-1} \\ & + P_7 \frac{d(u-y)}{1+u-y} + P_8 \frac{d(y-z)}{y-z} + P_9 \frac{d(z-u)}{z-u-1}, \end{aligned}$$

Now we are interested in the point  $(u, y, z) = (0, 0, 0)$ . The crucial matrix is  $P_4$  and its eigenvalues are 0 and  $c' = c - a - b - 1$  with multiplicity 4 for each. We assume  $c' \notin \mathbf{Z}$  and consider the eigenvalue 0. The matrix equation  $P_4^t(e_1, e_2, \dots, e_8) = 0$  consists of

$$\begin{aligned} -abe_1 + c'e_2 - ke_3 - ke_4 &= 0, \\ -(k-a)k - b)e_3 + c'e_5 - ke_6 &= 0, \\ -(k-a)k - b)e_4 + c'e_7 - ke_6 &= 0, \\ -(2k-a)(2k-b)e_6 + c'e_8 &= 0. \end{aligned}$$

Its general solutions are determined by  $f = (e_1, e_3, e_4, e_6)$ . Since the Pfaff system at  $u = 0$  is reduced to

$$de = \Omega_0 e,$$

where

$$\Omega_0 = P_2 \frac{dy}{y} + P_3 \frac{dz}{z} + (P_5 + P_7) \frac{dy}{y-1} + (P_6 + P_9) \frac{dz}{z-1} + P_8 \frac{d(y-z)}{y-z},$$

the associated system written relative to  $f$  is computed to be

$$df = \left( A_1 \frac{dy}{y} + A_2 \frac{dz}{z} + A_3 \frac{dy}{y-1} + A_4 \frac{dz}{z-1} + A_5 \frac{d(y-z)}{y-z} \right) f,$$

where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1-c+2k & 0 & 0 \\ 0 & -k & 0 & 1 \\ 0 & 0 & 0 & 1-c+2k \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -k & 1 \\ 0 & 0 & 1-c+2k & 0 \\ 0 & 0 & 0 & 1-c+2k \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -ab & c' - k & -k & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(k-a)(k-b) & c' - k \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -ab & -k & c' - k & 0 \\ 0 & -(k-a)(k-b) & 0 & c' - k \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -k & k & 0 \\ 0 & k & -k & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now refer to the Pfaffian form of the  $HJ$  for two variables (A2.1) in the appendix. Then, we can see that the obtained system is  $HJ(a, b, c - k; k; y, z)$ . For the eigenvalue  $c' (= c - a - b - 1)$ , the situation is similar and we get

**Proposition 5.1.** *Assume  $c' \notin \mathbf{Z}$ . Then, (1) any solution of the system corresponding to the eigenvalue 0 can be written as*

$$f = f_0(y, z) + u f_1(u, y, z),$$

where  $f_0$  is a solution of the system  $HJ(a, b, c - k; k; y, z)$ . (2) Solutions corresponding to the eigenvalue  $c'$  have a similar expression

$$u^{c-a-b} (f_0(y, z) + u f_1(u, y, z)),$$

where  $f_0(y, z) = ((1 - y)(1 - z))^{c-a-b} g(y, z)$  and  $g(y, z)$  is a solution of the system  $HJ(c - a, c - b, c - k; k; y, z)$ . The number of linear independent solutions is 4 in each case.

**Remark.** We can get a higher order development of solutions relative to the variable  $u$  as the following procedure; we treat the case where the eigenvalue is 0. The system  $J(a, b, c; k)$  consists of the following three equations:

$$\begin{aligned} & -(1 + u)w_{uu} + \{c - a - b - 1 - (a + b + 1)u\}w_u - k(yw_y + zw_z) \\ & -ku \left( \frac{y}{1 + u - y} + \frac{z}{1 + u - z} \right) w_u + ku \left( \frac{y}{1 + u - y} w_y + \frac{z}{1 + u - z} w_z \right) \\ & -abw = 0, \\ & y(1 - y)w_{yy} + \{c - 2k - (a + b + 1 - 3k)y\}w_y - k(yw_y + zw_z) \\ & + k(1 - y) \frac{yw_y - zw_z}{y - z} + k(1 - y) \frac{yw_y}{y - u - 1} - abw + k \frac{u(1 + u)}{y - u - 1} w_u = 0, \\ & z(1 - z)w_{zz} + \{c - 2k - (a + b + 1 - 3k)z\}w_z - k(zw_z + yw_y) \\ & + k(1 - z) \frac{zw_z - yw_y}{z - y} + k(1 - z) \frac{zw_z}{z - u - 1} - abw + k \frac{u(1 + u)}{z - u - 1} w_u = 0. \end{aligned}$$

Assume that a solution has the development

$$w = f_0(y, z) + f_1(y, z)u + f_2(y, z)u^2 + \dots$$

Then the second and the third equations induce the system for  $f = f_0$ :

$$\begin{aligned} y(1-y)f_{yy} + \{c - 2k - (a+b+1)y\}f_y + k(1-z)\frac{yf_y - zf_z}{y-z} - abf &= 0, \\ z(1-z)f_{zz} + \{c - 2k - (a+b+1)z\}f_z + k(1-y)\frac{yf_y - zf_z}{y-z} - abf &= 0. \end{aligned}$$

This means that the  $f_0(y, z)$  is a solution of the  $HJ(a, b, c - k; k; y, z)$ , as we have seen earlier. For higher order terms, we get from the first equation that

$$\begin{aligned} \alpha f_1 &= abf_0 + k(yf_{0y} + zf_{0z}), \\ 2(\alpha - 1)f_2 &= (\beta + ab)f_1 + k(yf_{1,y} + zf_{1,z}) \\ &\quad + k\left(\frac{y}{1-y} + \frac{z}{1-z}\right)f_1 - k\left(\frac{y}{1-y}f_{0,y} + \frac{z}{1-z}f_{0,z}\right), \\ &\quad \dots\dots\dots \\ (m+1)(\alpha - m)f_{m+1} &= (m(m-1) + m\beta + ab)f_m + k(yf_{m,y} + zf_{m,z}) \\ &\quad + \frac{ky}{1-y}(mf_m + A_1) + \frac{kz}{1-z}(mf_m + A_2) \\ &\quad - \frac{ky}{1-y}(f_{m-1,y} + A_3) - \frac{kz}{1-z}(f_{m-1,z} + A_4), \quad m \geq 2, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \frac{m-1}{y-1}f_{m-1} + \frac{m-2}{(y-1)^2}f_{m-2} + \dots + \frac{1}{(y-1)^{m-1}}f_1, \\ A_2 &= \frac{m-1}{z-1}f_{m-1} + \frac{m-2}{(z-1)^2}f_{m-2} + \dots + \frac{1}{(z-1)^{m-1}}f_1, \\ A_3 &= \frac{1}{y-1}f_{m-2,y} + \dots + \frac{1}{(y-1)^{m-1}}f_{0,y}, \\ A_4 &= \frac{1}{z-1}f_{m-2,z} + \dots + \frac{1}{(z-1)^{m-1}}f_{0,z}; \end{aligned}$$

Here

$$\alpha = c - a - b - 1 \quad \text{and} \quad \beta = a + b + 1.$$

From this relation, we can claim that, if  $\alpha \notin \mathbf{Z}$ , then  $f_m(y, z)$  are recursively determined. Here arises a problem whether, with these  $f_m$ , is the series  $w$  convergent.

## Appendix

### A1. Appell's hypergeometric system $F_1(\alpha, \beta_1, \beta_2, \gamma)$

Appell's hypergeometric function  $F_1$  is the function defined by the power series

$$F_1(\alpha, \beta_1, \beta_2, \gamma; x, y) = \sum_{m, n \geq 0} \frac{(\alpha, m+n)(\beta_1, m)(\beta_2, n)}{(\gamma, m+n)m!n!} x^m y^n.$$

It satisfies the system  $F_1(\alpha, \beta_1, \beta_2, \gamma)$ :

$$\begin{aligned} x(1-x)z_{xx} + (1-x)yz_{xy} + \{\gamma - (\alpha + \beta_1 + 1)x\}z_x - \beta_1yz_y - \alpha\beta_1z &= 0, \\ y(1-y)z_{yy} + (1-y)xz_{xy} + \{\gamma - (\alpha + \beta_2 + 1)y\}z_y - \beta_2xz_x - \alpha\beta_2z &= 0, \end{aligned}$$

which is equivalent to the Pfaffian system

$$(A1.1) \quad de = Ae, \quad e = {}^t(w, xw_x, yw_y),$$

where

$$\begin{aligned} A &= A_1 \frac{dx}{x} + A_2 \frac{dy}{y} + A_3 \frac{dx}{x-1} + A_4 \frac{dy}{y-1} + A_5 \frac{d(x-y)}{x-y}, \\ A_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 - \gamma + \beta_2 & 0 \\ 0 & -\beta_2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\beta_1 \\ 0 & 0 & 1 - \gamma + \beta_1 \end{pmatrix} \\ A_3 &= \begin{pmatrix} 0 & 0 & 0 \\ -\alpha\beta_1 & \gamma - \alpha - \beta_1 - 1 & -\beta_1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha\beta_2 & -\beta_2 & \gamma - \alpha - \beta_2 - 1 \end{pmatrix}, \\ A_5 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta_2 & \beta_1 \\ 0 & \beta_2 & -\beta_1 \end{pmatrix}. \end{aligned}$$

## A2. Appell's hypergeometric system $F_4(\alpha, \beta; \gamma_1, \gamma_2)$ and $HJ(a, b, c; k)$

Appell's hypergeometric function  $F_4$  is the function defined by the power series

$$F_4(\alpha, \beta; \gamma_1, \gamma_2; x, y) = \sum_{m, n \geq 0} \frac{(\alpha, m+n)(\beta, m+n)}{(\gamma_1, m)(\gamma_2, n)m!n!} x^m y^n.$$

It satisfies the system  $F_4(\alpha, \beta, \gamma_1, \gamma_2)$ :

$$\begin{aligned} x(1-x)w_{xx} - 2xyw_{xy} - y^2w_{yy} + (\gamma_1 - (\alpha + \beta + 1)x)w_x - (\alpha + \beta + 1)yw_y - \alpha\beta w &= 0, \\ y(1-y)w_{yy} - 2xyw_{xy} - x^2w_{xx} + (\gamma_2 - (\alpha + \beta + 1)y)w_y - (\alpha + \beta + 1)xw_x - \alpha\beta w &= 0. \end{aligned}$$

By the change of coordinates

$$x = uv, \quad y = (1-u)(1-v)$$

the system is transformed to

$$\begin{aligned} u(1-u)w_{uu} + (\gamma_1 - (\alpha + \beta + 1 - 2\epsilon)u)w_u - \epsilon(uw_u + vw_v) - \alpha\beta w \\ + \epsilon(1-u) \frac{uw_u - vw_v}{u-v} &= 0, \\ v(1-v)w_{vv} + (\gamma_1 - (\alpha + \beta + 1 - 2\epsilon)v)w_v - \epsilon(uw_u + vw_v) - \alpha\beta w \\ + \epsilon(1-v) \frac{vw_v - uw_u}{v-u} &= 0, \end{aligned}$$

which is nothing but  $HJ(\alpha, \beta, \alpha + \beta + 1 - \gamma_2; \alpha + \beta + 1 - \gamma_1 - \gamma_2; u, v)$  when  $n = 2$ . Here,  $\epsilon = \alpha + \beta + 1 - \gamma_1 - \gamma_2$ ; Refer to [3].

The Pfaffian form of  $HJ(a, b; c; k; u, v)$  for two variables are given as follows: Define a frame  $e$  by

$$e = {}^t \left( w, uw_u, vw_v, uv \left( w_{uv} - k \frac{w_u - w_v}{u - v} \right) \right).$$

Then, the Pfaff system

$$(A2.1) \quad de = Be, \quad B = B_1 \frac{du}{u} + B_2 \frac{dv}{v} + B_3 \frac{du}{u-1} + B_4 \frac{dv}{v-1} + B_5 \frac{d(u-v)}{u-v},$$

is given by the matrices

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & k-c+1 & 0 & 0 \\ 0 & -k & 0 & 1 \\ 0 & 0 & 0 & k-c+1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -k & 1 \\ 0 & 0 & k-c+1 & 0 \\ 0 & 0 & 0 & k-c+1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -ab & c-a-b-1 & -k & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(k-a)(k-b) & c-a-b-1 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -ab & -k & c-a-b-1 & 0 \\ 0 & -(k-a)(k-b) & 0 & c-a-b-1 \end{pmatrix}, \quad B_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -k & k & 0 \\ 0 & k & -k & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

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