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and $\rho(\gamma')$ is equal to $-(\rho \cdot \rho')(\gamma \cdot \gamma')$, where $(\rho \cdot \rho')$ and $(\gamma \cdot \gamma')$ are intersection numbers of curves ρ and ρ' and γ and γ' , respectively. One has only to sum up.

For a general $S(q, \ell)$ we define the cycles $c'_1(q, \ell), \dots, c'_7(q, \ell)$ and $c_1(q, \ell), \dots, c_7(q, \ell)$ by tracing c'_1, \dots, c'_7 and c_1, \dots, c_7 continuously along a path C connecting the original configuration and (q, ℓ) in X . Notice that these cycles depend only on the homotopy class of the path C and that the intersection matrix $(c'_i \cdot c'_j)_{1 \leq i, j \leq 7}$ and the relation $c_i \cdot c'_j = \delta_{ij}$ are preserved.

§A4. Periods of the surface $S(q, \ell)$

The integral of the 2-form $\eta(q, \ell)$ over a 2-cycle is called a period of the surface $S(q, \ell)$. Since the integral over an algebraic cycle vanishes, we have only to consider the following seven periods

$$\omega_i(q, \ell) = \int_{c_i(q, \ell)} \eta(q, \ell) \quad (1 \leq i \leq 7).$$

They satisfy the Riemann relation and the Riemann inequality:

$$\sum_{1 \leq i, j \leq 7} A_{ij} \omega_i(q, \ell) \omega_j(q, \ell) = 0,$$

$$\sum_{1 \leq i, j \leq 7} A_{ij} \omega_i(q, \ell) \overline{\omega_j(q, \ell)} > 0.$$

The correspondence ϕ

$$X \ni (q, \ell) \mapsto \omega(q, \ell) = (\omega_1(q, \ell), \dots, \omega_7(q, \ell)) \in \mathbb{P}^6$$

is called the period map of the family $\mathcal{S} = \cup_{(q, \ell) \in X} S(q, \ell)$. By virtue of the Riemann relation and inequality, the image of ϕ is contained in one of the connected components of

$$\{x = (x_1, \dots, x_7) \in \mathbb{P}^6 \mid xAx^T = 0, xA\bar{x}^T = 0\},$$

which is called the 5-dimensional bounded symmetric domain D_A of type IV. Since the cycles $c_1(q, \ell), \dots, c_7(q, \ell)$ depend on the path C connecting the original configuration and (q, ℓ) in X , this map is not single-valued. In order to describe the multi-valuedness, let us introduce the monodromy group. By the continuation along a loop C in X starting and ending at the original configuration, there is a matrix $M_C \in GL(7, \mathbb{C})$ such that ω changes into $M_C \omega$; the matrix M_C is called the circuit matrix of ω along the loop C . This gives the homomorphism (called the monodromy representation) of the fundamental group of X into $GL(7, \mathbb{C})$, whose image is called monodromy group. Since the multi-valuedness is caused by the change of the cycles c_1, \dots, c_7 , and the intersection matrix $A = (c'_i \cdot c'_j)_{1 \leq i, j \leq 7}$ is preserved, the monodromy group is a subgroup of $\{g \in GL(7, \mathbb{Z}) \mid gAg^T = A\}$.

We think that the monodromy group is commensurable with the reflection group treated in [MY].

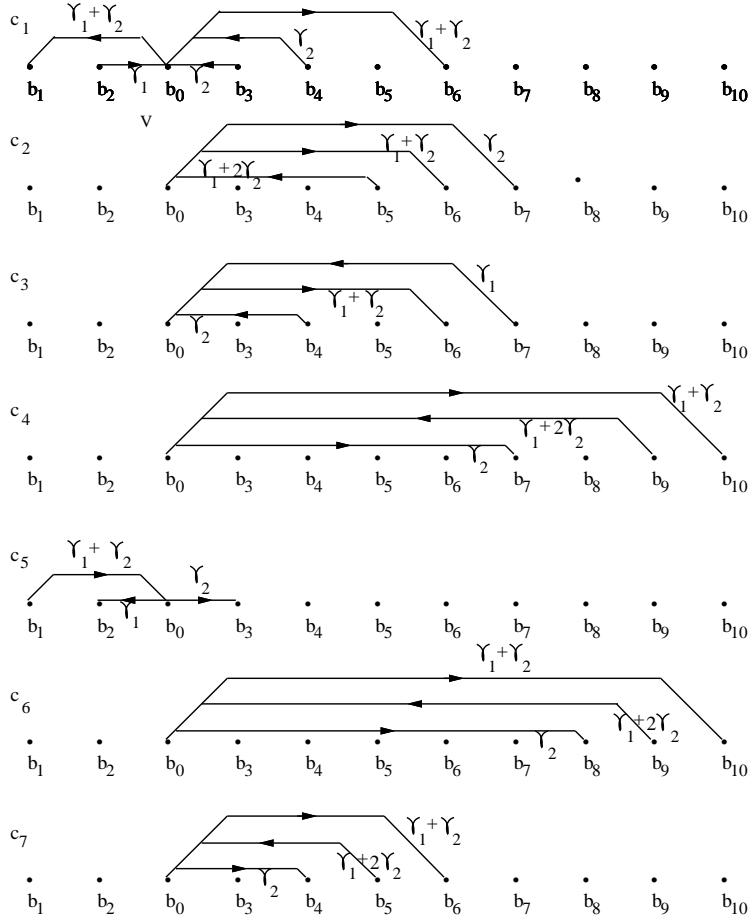


Figure 5

In fact, one can easily check that at each terminal point of the curves the generating cycles in question vanish; so they are closed surfaces.

Proposition A3.2. *The cycles c_i ($1 \leq i \leq 7$) are dual to c'_j ($1 \leq j \leq 7$):*

$$c_i \cdot c'_j = \delta_{ij} \quad (1 \leq i, j \leq 7).$$

The intersection matrix $(c'_i \cdot c'_j)_{i,j}$ is

$$2A = 2 \begin{pmatrix} U & & \\ & U & \\ & & -I_3 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Sketch of a proof. To compute the intersection number of two such 2-cycles, we study how they intersect on each fiber of the intersection points of the two base arcs. Since we do not concern about the intersection numbers among c_1, \dots, c_7 , the intersection points in question are on regular fibers. For two curves ρ and ρ' in B intersecting at a point b and for $\gamma, \gamma' \in H_1(\pi^{-1}(b), \mathbb{Z})$, the intersection number $\rho(\gamma) \cdot \rho'(\gamma')$ of two surfaces $\rho(\gamma)$

is denote by $\rho(\gamma)$; the curve ρ is called the base arc and γ the generating cycle. We get closed real surfaces

$$c'_1 := (\rho_7^{-1}\rho_6^{-1}\rho_5^{-1}\rho_4^{-1})(\gamma_2), \quad c'_2 := (\rho_7\rho_8\rho_9\rho_{10})(\gamma_1),$$

$$c'_3 := (\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6)(\gamma_2), \quad c'_4 := (\rho_4\rho_5\rho_6\rho_7)(\gamma_1);$$

in fact we have

$$N_4N_5N_6N_7 = N_7N_8N_9N_{10} = N_1N_2N_3N_4N_5N_6 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We need three other closed surfaces. Consider two open surfaces

$$(\rho_1\rho_2)(\gamma_2) \quad \text{and} \quad \rho_3(\gamma_1)$$

which have common boundary in $\pi^{-1}(b_0)$; glue them together to get a closed surface c'_5 , which is denoted by

$$(\rho_1\rho_2)(\gamma_2) + \rho_3(\gamma_1).$$

In the same way we get two closed surfaces

$$c'_6 := \rho_8(\gamma_1) + (\rho_9\rho_{10})(\gamma_2), \quad c'_7 := \rho_4(\gamma_1) + (\rho_5\rho_6)(\gamma_2).$$

Proposition A3.1. *The cycles c'_i ($1 \leq i \leq 7$) are orthogonal to AC .*

Proof. It is clear that a regular fiber $\pi^{-1}(b)$, $b \in B - \cup_{i=1}^{10}\rho_i$ of the elliptic surface (S, π, B) does not intersect c'_1, \dots, c'_7 . Since the exceptional curves, which arise from the intersection points of the conic and the four lines except E_{1q} , are on singular fibers of (S, π, B) , they do not intersect c'_1, \dots, c'_7 . Since we have

$$2\tilde{\ell}_2 \equiv F + E_{1q} - E_{2q_1} - E_{2q_2} - \sum_{j \neq 2} E_{2j},$$

we see how c'_1, \dots, c'_7 intersect $\tilde{\ell}_2$ instead of E_{1q} . By the construction of c'_1, \dots, c'_7 , they do not touch the ramification locus of $p' \circ p$, which includes $\tilde{\ell}_2$.

For b_i ($1 \leq i \leq 10$), let μ_i denote an oriented curve in $B^+ = \{t \in \mathbb{C} \mid \text{Im}(t) > 0\}$ starting from b_0 and ending at b_i . The following is another set of closed surfaces in S :

$$c_1 := \mu_1(\gamma_1 + \gamma_2) + \mu_2(-2\gamma_1) + \mu_3(-\gamma_2) + \mu_4(-\gamma_2) + \mu_6(\gamma_1 + \gamma_2),$$

$$c_2 := \mu_5(-\gamma_1 - 2\gamma_2) + \mu_6(\gamma_1 + \gamma_2) + \mu_7(\gamma_2),$$

$$c_3 := \mu_4(-\gamma_2) + \mu_6(\gamma_1 + \gamma_2) + \mu_7(-\gamma_1),$$

$$c_4 := \mu_7(\gamma_2) + \mu_9(-\gamma_1 - 2\gamma_2) + \mu_{10}(\gamma_1 + \gamma_2),$$

$$c_5 := \mu_1(-\gamma_1 - \gamma_2) + \mu_2(\gamma_1) + \mu_3(\gamma_2),$$

$$c_6 := \mu_8(\gamma_2) + \mu_9(-\gamma_1 - 2\gamma_2) + \mu_{10}(\gamma_1 + \gamma_2),$$

$$c_7 := \mu_4(\gamma_2) + \mu_5(-\gamma_1 - 2\gamma_2) + \mu_6(\gamma_1 + \gamma_2).$$

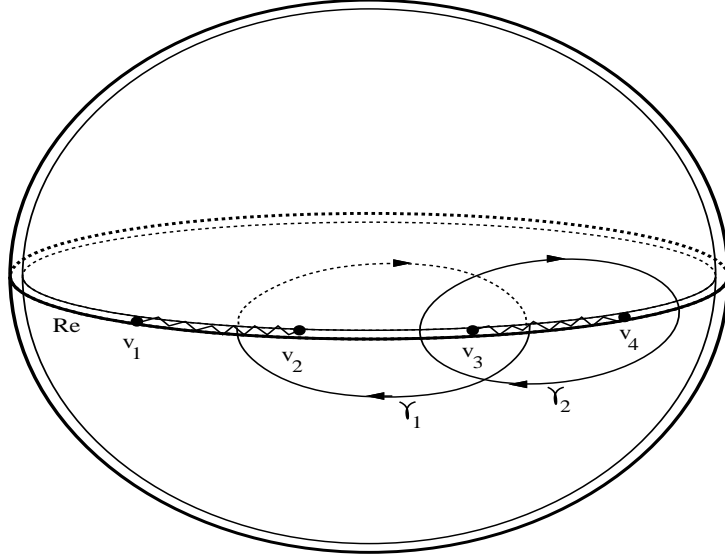


Figure 3

Let us choose paths ρ_i ($1 \leq i \leq 10$) with base point b_0 in $B - \{b_1, \dots, b_{10}\}$ as indicated in the Figure 4, which represent generators of the fundamental group of $B - \{b_1, \dots, b_{10}\}$

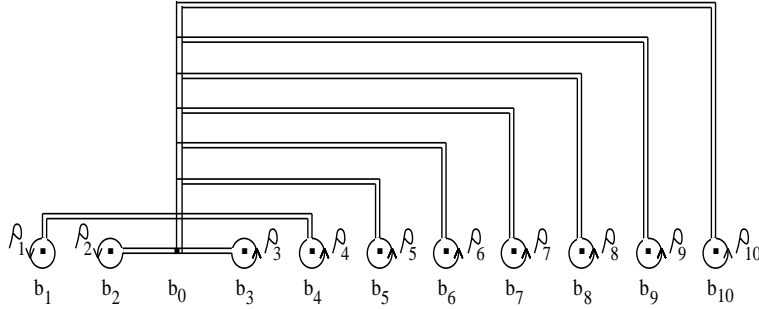


Figure 4

Let N_i ($1 \leq i \leq 10$) be the circuit matrices of $(\gamma_1, \gamma_2)^T$ along ρ_i . By tracing deformations of the γ_j 's along ρ_i , we have

$$\begin{aligned}
 N_1 &= \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}, & N_2 &= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, & N_3 &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \\
 N_4 &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, & N_5 &= \begin{pmatrix} 5 & 8 \\ -2 & 3 \end{pmatrix}, & N_6 &= \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}, & N_7 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \\
 N_8 &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, & N_9 &= \begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}, & N_{10} &= \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}.
 \end{aligned}$$

Now we construct 2-cycles on S . For a curve ρ starting from b_0 and a 1-cycle γ on $\pi^{-1}(b_0)$, there is, by the local flatness of the fibering $S \rightarrow B$, a continuous family of 1-cycles γ_b of $\pi^{-1}(b)$ ($b \in \rho$) such that $\gamma_{b_0} = \gamma$; the union $\cup_{b \in \rho} \gamma_b$, which is a real 2-dimensional surface,

§A3. Cycles c'_1, \dots, c'_7 and c_1, \dots, c_7 in $H_2(S(q, \ell), \mathbb{Z})$

Let us fix a configuration (q, ℓ) defined for real w_{ij} and z_{ij} as in Figure 2, and construct 2-cycles on the K3 surface $S = S(q, \ell)$ by using the structure of the elliptic surface.

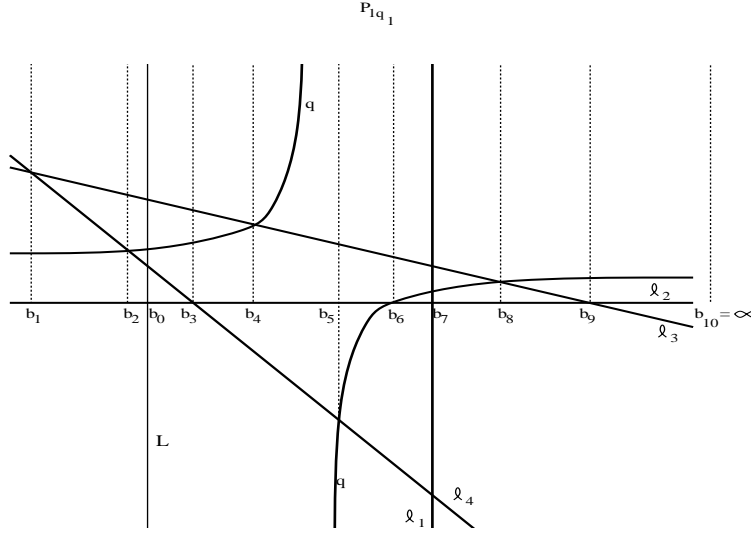


Figure 2

For the elliptic surface (S, π, B) , each singular fiber F_i ($1 \leq i \leq 10$) lies over $b_i \in B = \ell_2$, where

$$b_1 := r(P_{34}), \quad b_2 := r(P_{4q_1}), \quad b_3 := r(P_{24}), \quad b_4 := r(P_{3q_1}), \quad b_5 := r(P_{4q_2}),$$

$$b_6 := r(P_{2q_2}), \quad b_7 := r(P_{\ell_1}), \quad b_8 := r(P_{3q_2}), \quad b_9 := r(P_{23}), \quad b_{10} := r(P_{2q_1});$$

The singular fiber F_7 is of type I_0^* and the others are of type I_2 .

Let us study the global monodromy of the elliptic surface (S, π, B) . In order to do so, let us choose a base point b_0 in B as indicated in Figure 2, express the fiber $\pi^{-1}(b_0)$ as the double cover of the line L defined by P_{1q} and b_0 branching at four intersection points $v_1 < \dots < v_4$ of L and (q, ℓ) except P_{1q} . Let us choose a basis γ_1 and γ_2 of $H_1(\pi^{-1}(b_0), \mathbb{Z})$ as in Figure 3.

where $a_i \in \mathbb{Z}$ ($1 \leq i \leq 22$) and

$$(c'_8, \dots, c'_{22}) = (F, E_{1q}, E'_{1q}, \dots, E_{4q_1}, E_{4q_2}, E_{12}, \dots, E_{34}).$$

Then we have $a_i = c_0 \cdot c'_i = 0$ for $1 \leq i \leq 7$ by definition and the fact that $\text{rank}(AC) = 15$ implies vanishing of the remaining coefficients. Hence they are independent.

Next consider

$$c'_0 := c' - \sum_{i=1}^7 (c' \cdot c_i) c'_i$$

for any cycle $c' \in AC^\perp$. Then we easily see that $c'_0 \cdot c'_i = 0$ for all i because $c'_i \cdot c_j = \delta_{ij}$ for $1 \leq i, j \leq 7$ and c', c'_i ($1 \leq i \leq 7$) $\in AC^\perp$. Since $H_2(S, \mathbb{Z})$ is a unimodular lattice of rank 22, we conclude that $c'_0 = 0$, which implies that any $c' \in AC^\perp$ can be expressed as a linear combination of c'_i ($1 \leq i \leq 7$) over \mathbb{Z} .

(2) Since $H_2(S, \mathbb{Z})$ is unimodular, it suffices to show that the lattice $\langle c_1, \dots, c_7 \rangle \oplus \overline{AC}$ is also unimodular. Since the discriminant of $AC^\perp \oplus AC$ is -2^{22} , we only have to show that $[\langle c_1, \dots, c_7 \rangle \oplus \overline{AC} : AC^\perp \oplus AC]$ equals to 2^{11} in view of Lemma A2.1; it is equivalent to

$$[\langle c_1, \dots, c_7 \rangle \oplus AC : AC^\perp \oplus AC] = 2^7$$

in view of Proposition A2.2. We express c_i in terms of c'_j ($1 \leq j \leq 22$) as follows:

$$c_i := \sum_{j=1}^{22} a'_{i,j} c'_j.$$

Then, inverting the relation $(c_i \cdot c'_k) = \sum_{j=1}^{22} a'_{i,j} (c'_j \cdot c'_k)$, we have

$$(a'_{i,1}, \dots, a'_{i,22}) = (c_i \cdot c'_1, \dots, c_i \cdot c'_{22}) ((c'_j \cdot c'_k)_{1 \leq j, k \leq 22})^{-1}.$$

Since $c_i \cdot c'_j = \delta_{ij}$ and $(c_i \cdot c'_k) \in \mathbb{Z}$, there exist $E_i \in AC$ ($1 \leq i \leq 7$) such that

$$(c_1, \dots, c_7)^T = \frac{1}{2} A (c'_1, \dots, c'_7)^T + \frac{1}{2} (E_1, \dots, E_7)^T$$

by referring to the intersection matrix of $AC \oplus \overline{AC}$. Hence we have the assertion.

(3) By definition, we have

$$\overline{AC} \subset NS(S)$$

and, by (2) above, any element $E \in NS(S)$ has the expression

$$E = a_1 c_1 + \dots + a_7 c_7 + E', \quad a_i \in \mathbb{Z}, \quad E' \in \overline{AC}.$$

If the rank of $NS(S)$ is equal to the rank of AC , then every c'_j belongs to $NS(S)^\perp$. Since $c_i \cdot c'_j = \delta_{ij}$, and $E \cdot c'_j = E' \cdot c'_j = 0$, we have $a_j = 0$, i.e., $E = E' \in \overline{AC}$.

which implies

$$\text{rank}(AC) = 15 \quad \text{and} \quad \text{disc}(AC) = 2^{15},$$

where $\text{rank}(L)$ and $\text{disc}(L)$ denote the rank of and the discriminant of a lattice L , respectively. The following lemma is well known.

Lemma A2.1. *Let L be a nondegenerate lattice and M a submodule of L . Then we have*

- (1) $\text{rank}(M) + \text{rank}(M^\perp) = \text{rank}(L)$,
- (2) if $\text{rank}(M) = \text{rank}(L)$ then $[L : M]^2 = d(M)d(L)^{-1}$.

Let $\tilde{\ell}_i$ be the reduced divisor of the strict transform of ℓ_i under the morphism $p' \circ p : S \rightarrow \mathbb{P}^2$. Let \overline{AC} be the \mathbb{Z} -submodule of $NS(S)$ generated by AC and $\tilde{\ell}_i$ ($1 \leq i \leq 4$).

Proposition A2.2. $[\overline{AC} : AC] = 2^4$.

Proof. Consider on S the rational function

$$(p' \circ p)^* \frac{f}{\ell_i(z_i, t)},$$

where f is a linear form defining the line $p' \circ p(F)$ in \mathbb{P}^2 ; since its divisor is linearly equivalent to

$$F + E_{1q} - \{2\tilde{\ell}_i + E_{iq} + E'_{iq} + \sum_{j \neq i} E_{ij}\},$$

we have

$$\tilde{\ell}_i \equiv \frac{1}{2} \{F + E_{1q} - E_{iq} - E'_{iq} - \sum_{j \neq i} E_{ij}\},$$

where \equiv denotes linear equivalence. Notice that, in the above expressions for $1 \leq i \leq 4$, the cycle E'_{iq} appears only in the expression of $\tilde{\ell}_i$, which proves the proposition.

In the next section we explicitly construct 2-cycles $c'_1, \dots, c'_7, c_1, \dots, c_7 \in H_2(S, \mathbb{Z})$ such that $c'_i \in AC^\perp$ ($1 \leq i \leq 7$), $c'_i \cdot c_j = \delta_{ij}$ and that the intersection matrix $(c'_i \cdot c'_j)$ ($1 \leq i, j, \leq 7$) is given by $2A$ where

$$A = \begin{pmatrix} U & & \\ & U & \\ & & -I_3 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Let us assume for the moment the existence of such cycles.

Proposition A2.3. (1) *The cycles c'_1, \dots, c'_7 form a \mathbb{Z} -basis of AC^\perp .*

(2) $H_2(S, \mathbb{Z}) = \langle c_1, \dots, c_7 \rangle \oplus \overline{AC}$ (as a \mathbb{Z} -module).

(3) *If the rank of the Néron-Severi group $NS(S)$ of $S = S(q, \ell)$ is equal to that of AC , then we have $NS(S) = \overline{AC}$.*

Proof. (1) We first see that 22 cycles $c_1, \dots, c_7, F, E_{ij}$ ($1 \leq i < j \leq 4$) and E_{kq}, E'_{kq} ($1 \leq k \leq 4$) are linearly independent. In fact, assume that it holds a linear relation

$$c_0 := \sum_{i=1}^7 a_i c_i + \sum_{i=8}^{22} a_i c'_i = 0$$

Appendix
Structure of the cycles of double covers of the projective
plane branching along a conic and four lines

§A1. Double cover of \mathbb{P}^2 branching along a conic and four lines

Let

$$q = \{t = (t^1, t^2, t^3) \in \mathbb{P}^2 \mid q(w, t) = \sum_{1 \leq i, j \leq 3} w_{ij} t^i t^j = t w t^T = 0\}, \quad w_{ij} = w_{ji}$$

$$\ell_k = \{t \in \mathbb{P}^2 \mid \ell_k(z_k, t) = \sum_{i=1}^3 z_{ik} t^i = 0\}, \quad 1 \leq k \leq 4.$$

be a non-degenerate conic and four lines in the complex projective plane \mathbb{P}^2 lying in general position, namely, with the property that no intersection point $\ell_i \cap \ell_j$ of two lines is not on another line ℓ_k nor on the conic q , and no line ℓ_k is tangent to the conic q . Let

$$X' = \left\{ (w, z) \left| \begin{array}{ll} w = w^T \in M(3, 3), & z = (z_1, \dots, z_4) \in M(3, 4) \\ \det(w) \neq 0, & \det(z_i, z_j, z_k) \neq 0 \ (i < j < k) \\ z_i^T w^{-1} z_i \neq 0, & (z_i \wedge z_j)^T w (z_i \wedge z_j) \neq 0 \ (i < j) \end{array} \right. \right\}$$

denote the space parametrizing q and the ℓ_k 's; the groups $GL(3, \mathbb{C})$ and $(\mathbb{C}^*)^5$ act on this space as in §1. The quotient space

$$X = GL(3, \mathbb{C}) \backslash X' / (\mathbb{C}^*)^5$$

is the configuration space of a conic and four lines in general position in \mathbb{P}^2 . Let (q, ℓ) represent an element of X , where q stands for a conic and ℓ a system of four lines ℓ_k ($1 \leq k \leq 4$). Let $S'(q, \ell)$ be the double cover of \mathbb{P}^2 branching along the union $q \cup (\cup_{k=1}^4 \ell_k)$ and let $p' : S'(q, \ell) \rightarrow \mathbb{P}^2$ be the projection; there are 14 singular points of $S'(q, \ell)$, which arise from 14 intersection points $\{P_{kq}, P'_{kq}\} = \ell_k \cap q$ and $P_{ij} = \ell_i \cap \ell_j$ ($i \neq j$). Denote by $S(q, \ell)$ the minimal smooth model of $S'(q, \ell)$ that is obtained from $S'(q, \ell)$ by replacing each singular points, coming from P_{kq}, P'_{kq} and P_{ij} , by rational curve E_{kq}, E'_{kq} and E_{ij} of self-intersection number -2 ; let $p : S(q, \ell) \rightarrow S'(q, \ell)$ be the birational morphism. On $S(q, \ell)$ there is a unique nowhere-vanishing holomorphic 2-form

$$\eta(q, \ell) = (p' \circ p)^* q(w, t)^{-1/2} \prod_{k=1}^4 \ell_k(z_k, t)^{-1/2} (t^3 dt^1 \wedge dt^2 + t^1 dt^2 \wedge dt^3 + t^2 dt^3 \wedge dt^1)$$

up to multiplicative constant. Let

$$r : \mathbb{P}^2 - \{P_{1q}\} \rightarrow \ell_2$$

be the projection with center P_{1q} onto the line ℓ_2 . Then there is an unique morphism $\pi : S(q, \ell) \rightarrow \ell_2$ such that the diagram

$$\begin{array}{ccccc} S' & \xleftarrow{p'} & S(q, \ell) - E_{1q} & \xrightarrow{\subset} & S(q, \ell) \\ p' \downarrow & & & & \downarrow \pi \\ \mathbb{P}^2 & & & & \\ r \searrow & & \ell_2 & \xrightarrow{=} & \ell_2 \end{array}$$

We second treat the integral (7.2). The group of symmetry is now $(GL_3 \times (\mathbb{C}^*)^4) / \mathbb{C}^*$ of dimension 12 and the number of coefficients is $6 \times 2 + 3 \times 2 = 18$; the number of variables, that is, the dimension of the quotient space is 6. The requirement on the parameter is

$$(7.4) \quad 2(a_1 + a_2) + a_3 + a_4 = 3.$$

We use the notation

$$q_k = \sum_{i,j} w_{ij}^k t^i t^j = t w^k t^T \quad \text{for } k = 1, 2; \quad t = (t^1, t^2, t^3)^T, \quad w = (w_{ij}), \quad w_{ij}^k = w_{ji}^k$$

$$l_i = \sum z_{ji} t^j = \langle t, z_i \rangle, \quad i = 1, 2$$

$$\varphi_2(a; z, w; t) = q_1^{a_1-1} q_2^{a_2-1} \ell_1^{a_3-1} \ell_2^{a_4-1}$$

and

$$u(a; z, w) = u(a_i; z_{ij}, w_{ij}^k) = \int \varphi_2(a; z, w; t) dt.$$

Then the calculation similar to the above shows that the system consists of the following differential equations.

$$(I_2) \quad \sum_j z_{jk} \frac{\partial u}{\partial z_{jk}} = (a_{k+2} - 1)u \quad \text{for } k = 1, 2; \quad \sum_{i \geq j} w_{ij}^k \frac{\partial u}{\partial w_{ij}^k} = (a_k - 1)u \quad \text{for } k = 1, 2,$$

$$(II_2) \quad \sum_{k=1}^2 z_{\ell k} \frac{\partial u}{\partial z_{mk}} + \sum_{k=1}^2 \left(w_{\ell m}^k \frac{\partial u}{\partial w_{\ell \ell}^k} + \sum_{j=1}^2 w_{\ell j}^2 \frac{\partial u}{\partial w_{mj}^k} \right) = -\delta_{\ell m} u,$$

$$(III_2) \quad \begin{cases} \frac{\partial^2 u}{\partial z_{j_2} \partial z_{i_1}} = \frac{\partial^2 u}{\partial z_{i_1} \partial z_{j_2}}, \\ (2 - \delta_{j\ell})(2 - \delta_{im}) \frac{\partial^2 u}{\partial w_{\ell m}^p \partial w_{ij}^r} = (2 - \delta_{\ell m})(2 - \delta_{ij}) \frac{\partial^2 u}{\partial w_{j\ell}^p \partial w_{im}^r}. \end{cases}$$

Remark. We believe that both systems are holonomic and the rank is 9 or 8, respectively, which however we do not proved yet.

By letting $\lambda_k = 1$, we get

$$(I_1) \quad \sum_{i \geq j} w_{ij}^k \frac{\partial u}{\partial w_{ij}^k} = (a_k - 1)u.$$

The differentiation of 2° gives

$$\begin{aligned} \frac{\partial u}{\partial g_p^i} &= \sum_{k; p \leq \ell} w_{iq}^k g_\ell^q \frac{\partial u}{\partial w_{p\ell}^k} + \sum_{k; \ell \leq p} g_\ell^q w_{qi}^k \frac{\partial u}{\partial w_{\ell p}^k} \\ &= \frac{-1}{(\det g)^{-2}} (ip\text{-cofactor}) \cdot u. \end{aligned}$$

Then, letting $g = 1$, we have

$$(II_1) \quad \sum_k \left(w_{\ell m}^k \frac{\partial}{\partial w_{\ell\ell}^k} + \sum_j w_{\ell j}^k \frac{\partial}{\partial w_{mj}^k} \right) = -\delta_{\ell m} u.$$

In addition to these equations of equivariance, the following sets of equations are satisfied: By differentiation, we have

$$\begin{aligned} \frac{\partial u}{\partial w_{ij}^k} &= \int (2 - \delta_{ij})(a_k - 1) t^i t^j \varphi_1 / q_k dt, \\ \frac{\partial^2 u}{\partial w_{ij}^k \partial w_{pq}^k} &= \int (2 - \delta_{ij})(2 - \delta_{pq})(a_k - 1)(a_k - 2) t^i t^j t^p t^q \varphi_1 / q_k^2 dt, \\ \frac{\partial^2 u}{\partial w_{ij}^k \partial w_{pq}^\ell} &= \int (2 - \delta_{ij})(2 - \delta_{pq})(a_k - 1)(a_\ell - 1) t^i t^j t^p t^q \varphi_1 / q_k q_\ell dt, \end{aligned}$$

and, by the symmetry relative to i, j, p, q , we get

$$(III_1) \quad (2 - \delta_{iq})(2 - \delta_{jp}) \frac{\partial^2 u}{\partial w_{ij}^k \partial w_{pq}^\ell} = (2 - \delta_{ij})(2 - \delta_{pq}) \frac{\partial^2 u}{\partial w_{iq}^k \partial w_{pj}^\ell}.$$

The combination of (k, ℓ) is 6:

$$(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)$$

and the combination of $(ij; pq)$ are 6:

$$(11; 22), (11; 33), (22; 33), (11; 23), (22; 13), (33; 12);$$

hence we have 36 equations. The system of equations associated with the integral (7.1) is eventually the set of (I_1) , (II_1) and (III_1) . The number of variables is $18 - 11 = 7$.

§7. Systems of differential equations analogous to $Q(3, 4)$

In this section, we give a sketch how to derive the systems of differential equations associated with the integrals

$$(7.1) \quad \int q_1^{a_1-1} q_2^{a_2-1} q_3^{a_3-1} \omega_2,$$

$$(7.2) \quad \int q_1^{a_1-1} q_2^{a_2-1} \ell_1^{a_3-1} \ell_2^{a_4-1} \omega_2,$$

which are further generalizations of the integral (1.1) in the special case where $k = 3$ and $n = 4$. Here q, q_i are quadratic forms on \mathbb{P}^2 . These integrals are also related to certain families of $K3$ -surfaces when the complex parameters are equal to $1/2$.

First, we treat the integral (7.1). We level the quadratic forms as

$$q_k = \sum_{i,j} w_{ij}^k t^i t^j = t w^k t^T \quad \text{for } 1 \leq k \leq 3;$$

$$t = (t^1, t^2, t^3)^T, \quad w = (w_{ij}), \quad w_{ij}^k = w_{ji}^k,$$

where t^T means the transposed vector of t . Put

$$\varphi_1(a; w; t) = q_1^{a_1-1} q_2^{a_2-1} q_3^{a_3-1}$$

and consider the integral

$$u(a; w) = u(a_i; w_{ij}^k) = \int \varphi_1(a; w; t) dt.$$

In order that the integral is well-defined, we require the condition

$$(7.3) \quad 2(a_1 + a_2 + a_3) = 3.$$

The number of coefficients of q_k is 18. The group $(GL_3 \times (\mathbb{C}^*)^3)/\mathbb{C}^*$ of dimension 11 acts on the integrals as shown in §2:

The action of the diagonal element $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ induces the formula

$$1^\circ \quad u(a; \lambda_k w_{ij}^k) = \prod_k (\lambda_k)^{a_k-1} u(a; w_{ij}^k)$$

and the action of the element $g \in GL_3$ yields

$$2^\circ \quad u(a; g w^k g^T) = (\det g)^{-1} u(a; w^k).$$

Differentiating the first identity by λ_k , we have

$$\frac{\partial u}{\partial \lambda_k} = \sum_{i \geq j} w_{ij}^k \frac{\partial u}{\partial w_{ij}^k} = (a_k - 1) (\lambda_k)^{a_k-2} u;$$

we have

$$(6.9) \quad I(z^1, 0, 0, 0, z^5) \\ = (-1)^{1-a_2+a_5} 2^{a_5-1} (z^1)^{1-a_5-a_2} (z^5)^{1-a_5-a_3} J(0, z^5, 0, z^5, z^1, z^1-1).$$

and, when $z^1 = z^2 = z^3 = 0$, by the transformation

$$h_1 = 1, \quad h_2 = z^4, \quad h_3 = z^5, \quad h_4 = 1; \quad g = \text{diag} \left(1, \frac{1}{z^4}, \frac{1}{z^5} \right),$$

we have

$$(6.10) \quad I(0, 0, 0, z^4, z^5) \\ = 2^{a_5-1} (z^4)^{1-a_5-a_2} (z^5)^{1-a_5-a_3} J(1, 0, 0, z^5, -z^4, -1).$$

Then, combining the formulas (6.4)-(6.10), we get

$$(6.11) \quad \int_{D_2} (yt_2 - xt_3 + yt_2^2 - (x+1)t_2t_3)^{a_5-1} t_2^{a_2-1} t_3^{a_3-1} (1+t_2+t_3)^{a_4-1} dt \\ = (-1)^{\gamma+\gamma'} 2^{1-a_5} C^{-1} x^{\gamma-1} y^{\gamma'-1} F_2(\alpha, \beta, \beta', \gamma, \gamma'; -x, y),$$

where

$$a_1 = 1 - \beta, \quad a_2 = \alpha - \gamma' + 1, \quad a_3 = \alpha - \gamma + 1, \\ a_4 = 1 - \beta', \quad a_5 = \gamma + \gamma' - \alpha - 1, \\ D_2 = \{(1, t_2, t_3); \quad t_2 \leq 0, \quad t_3 \leq 0, \quad x(1+t_2) + t_2 \geq 0, \\ yt_2 - xt_3 + yt_2^2 - (x+1)t_2t_3 \geq 0\};$$

As for F_4 , we have

$$(6.12) \quad \int_{D_4} (-yt_2 + xt_3 + t_2t_3)^{a_5-1} t_2^{a_2-1} t_3^{a_3-1} (1+t_2+t_3)^{a_4-1} dt \\ = (-1)^{1+\alpha+\gamma'} 2^{1+\alpha-\gamma-\gamma'} C^{-1} x^\gamma y^{\gamma'} F_4(\alpha, \beta, \gamma, \gamma'; x, -y),$$

where

$$a_1 = \beta - \gamma - \gamma', \quad a_2 = \alpha - \gamma' + 1, \quad a_3 = \alpha - \gamma + 1, \quad a_4 = 1 - \beta, \quad a_5 = \gamma + \gamma' - \alpha, \\ D_4 = \{(1, t_2, t_3); t_2 \leq 0, \quad t_3 \leq 0, \quad x + t_2 \geq 0, \quad -yt_2 + xt_3 + t_2t_3 \geq 0\}.$$

The variables (x, y) are assumed to be real positive in the calculation of both integrals; the formulas themselves hold by analytic continuation around the origin.

for both and the coefficient is

$$C = (-1)^{\gamma-\alpha} \frac{\sin \pi(\alpha - \gamma)}{\pi} \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\gamma + \gamma' - \alpha - 1)}.$$

Refer to [Y1, 2]. We will convert these integrals so that they look like the integrals treated in §5. For this purpose and for later use, consider generally the integral

$$(6.6) \quad I(z^1, z^2, z^3, z^4, z^5) \\ = \int (t_1 t_2 - t_1 t_3 - t_2 t_3)^{a_5-1} (t_1 + z^1 t_2)^{a_1-1} (t_2 + z^2 t_3)^{a_2-1} (t_3 + z^3 t_1)^{a_3-1} \\ \cdot (t_1 + z^4 t_2 + z^5 t_3)^{a_4-1} dt.$$

Define

$$\begin{aligned} h_1 &= 1 + z^2 z^3 z^4 - z^3 z^5, \\ h_2 &= -z^1 + z^4 + z^1 z^3 z^5, \\ h_3 &= z^1 z^2 - z^2 z^4 + z^5, \\ h_4 &= 1 + z^1 z^2 z^3, \\ g &= \begin{pmatrix} h_1 & 0 & h_3 z^3 \\ h_1 z^1 & h_2 & 0 \\ 0 & h_2 z^2 & h_3 \end{pmatrix}^{-1}. \end{aligned}$$

Then the change of variables from z to y by

$$(y_{ij}) = g \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} g^T$$

induces the transformation

$$(6.7) \quad I(z) = h_1^{a_1-1} h_2^{a_2-1} h_3^{a_3-1} h_4^{a_4-1} (\det g)^{-1} J(y)$$

where

$$(6.8) \quad J(y) = J(y_{11}, y_{22}, y_{33}, y_{12}, y_{13}, y_{23}) \\ = \int (\sum y_{ij} t_i t_j)^{a_5-1} t_1^{a_1-1} t_2^{a_2-1} t_3^{a_3-1} (t_1 + t_2 + t_3)^{a_4-1} dt.$$

Now the calculation shows the following formulas (6.11) and (6.12) realizing Appell's functions F_2 and F_4 in terms of the function J .

When $z^2 = z^3 = z^4 = 0$, by the transformation

$$h_1 = 1, \quad h_2 = -z^1, \quad h_3 = z^5, \quad h_4 = 1; \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -\frac{1}{z^1} & 0 \\ 0 & 0 & -\frac{1}{z^5} \end{pmatrix},$$

as the cycle of integration. Define

$$(6.1) \quad F(a; y) = (-1)^{a_2+a_3} \frac{\Gamma(a_2 + a_3 + a_4)}{\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)} \int_{\Delta} q^{a_5-1} t_1^{a_1-1} t_2^{a_2-1} t_3^{a_3-1} (t_1 + t_2 + t_3)^{a_4-1} dt.$$

Then we have

Proposition 6.1. *The function F has the following expansion around the origin.*

$$(6.2) \quad \begin{aligned} & F(a_1, a_2, a_3, a_4, a_5; y^1, y^2, y^3, y^4, y^5) \\ &= \sum_{i,j,k,\ell,m=1}^{\infty} \frac{(1 - a_5, i + j + k + \ell + m)(a_2, 2i + k + m)(a_3, 2j + \ell + m)}{(1, i)(1, j)(1, k)(1, \ell)(1, m)(a_2 + a_3 + a_4, 2i + 2j + k + \ell + 2m)} \\ & \quad \times (y^1)^i (y^2)^j (y^3)^k (y^4)^{\ell} (y^5)^m. \end{aligned}$$

Proof is given by a straightforward calculation: expand q^{a_5-1} relative to $(t_1, t_2, t_3) = (1, -t, -s)$ and get

$$\sum \frac{(1 - a_5, i + j + k + \ell + m)}{(1, i)(1, j)(1, k)(1, \ell)(1, m)} (y^1)^i (y^2)^j (y^3)^k (y^4)^{\ell} (y^5)^m \cdot t^{2i+k+m} s^{2j+\ell+m};$$

then use the formula

$$\int_{\{t,s,1-t-s \geq 0\}} t^{p-1} s^{q-1} (1-t-s)^{r-1} dt ds = \frac{\Gamma(p)\Gamma(q)\Gamma(r)}{\Gamma(p+q+r)}.$$

This expansion shows in particular

$$(6.3) \quad F(a_1, a_2, a_3, a_4, a_5; 0, 0, y^3, y^4, 0) = F_1(1 - a_5, a_2, a_3, a_2 + a_3 + a_4; y^3, y^4),$$

where F_1 is Appell's first hypergeometric function.

Appell's second and fourth hypergeometric functions have the relation with the integral (1.3) in the following way. Recall the integral representations of them:

$$(6.4) \quad \begin{aligned} & F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) \\ &= C \int_D (u + v - uv)^{\gamma+\gamma'-\alpha-2} u^{\alpha-\gamma'} v^{\alpha-\gamma} (1-xu)^{-\beta} (1-yv)^{-\beta'} du dv, \end{aligned}$$

$$(6.5) \quad \begin{aligned} & F_4(\alpha, \beta, \gamma, \gamma'; x, y) \\ &= C \int_D (u + v - uv)^{\gamma+\gamma'-\alpha-2} u^{\alpha-\gamma'} v^{\alpha-\gamma} (1-xu-yv)^{-\beta} du dv, \end{aligned}$$

where the domain of integration is

$$D = \{(u, v) \in \mathbb{R}^2; 0 \leq u \leq 1, v \leq 0, u + v - uv \leq 0\}$$

Then the computation of the symbols of [1]-[24] can be easily done. Since the coefficients c_{ij} are rational functions of y^i , a necessary check can be done by restricting the variables to special values. In this way we can get the following.

Theorem 5.3. *The rank of the system $Q(3,4)$ is 7.*

The contiguity relations in Proposition 4.1 can be rephrased as follows.

Proposition 5.4. *The function $w(a; x)$ satisfies the contiguity relations:*

$$\begin{aligned}
(1) \quad & S_{ij}w(a) = (a_j - 1)w(a + 1_i - 1_j), \\
& T_iw(a) = (a_4 - 1)w(a + 1_i - 1_4), \\
& \sum_i S_{ij}w(a) = (a_j - 1)w(a - 1_j + 1_4), \\
(2) \quad & \frac{1}{2 - \delta_{ij}} \frac{\partial}{\partial x_{ij}} w(a) = (a_5 - 1)w(a + 1_i + 1_j - 1_5), \\
& \sum_j \frac{1}{2 - \delta_{ij}} \frac{\partial}{\partial x_{ij}} w(a) = (a_5 - 1)w(a + 1_i + 1_4 - 1_5), \\
& \sum_{i \geq j} \frac{\partial}{\partial x_{ij}} w(a) = (a_5 - 1)w(a + 2_4 - 1_5), \\
(3) \quad & \sum_{i,j} x_{ij} \tilde{L}_{jk,i\ell} w(a) = (a_k - 1)(a_\ell - 1 - \delta_{k\ell})w(a - 1_k - 1_\ell + 1_5), \\
& \sum_{i,j} x_{ij} \tilde{M}_{i,j\ell} w(a) = (a_4 - 1)(a_\ell - 1)w(a - 1_\ell - 1_4 + 1_5), \\
& \sum_{ij} x_{ij} (T_i T_j - \delta_{ij} T_j) w(a) = (a_4 - 1)(a_4 - 2)w(a - 2_4 + 1_5).
\end{aligned}$$

§6. Power-series solutions and related hypergeometric functions

In this section we give an explicit solution of the system $Q(3,4)$ and show how Appell's hypergeometric functions of two variables can be written by use of the integral (1.1). Refer to [Y2] for Appell's hypergeometric functions.

The space $\{x = (x_{ij})\}$ admits a \mathbb{C}^* -action as was mentioned in the previous section. To kill this action, we introduce the new coordinates $y = (y^1, y^2, y^3, y^4, y^5)$ related with x by

$$x_{11} = 1, \quad x_{22} = -y^1, \quad x_{33} = -y^2, \quad x_{12} = y^3/2, \quad x_{13} = y^4/2, \quad x_{23} = -y^5/2.$$

Then the quadratic form q is

$$q = t_1^2 - y^1 t_2^2 - y^2 t_3^2 + y^3 t_1 t_2 + y^4 t_1 t_3 - y^5 t_2 t_3.$$

We here lower indices to t for simplicity. We choose

$$\Delta = \{t_1 = 1, t_2 \leq 0, t_3 \leq 0, 1 + t_2 + t_3 \geq 0\}$$

In terms of T_i and S_{ij} , these can be rewritten as follows:

$$\begin{array}{ll}
[7] & T_1 S_{23} = T_2 S_{13} \\
[9] & T_2 S_{31} = T_3 S_{21} \\
[11] & (T_1 + 1) S_{31} = T_3 S_{11} \\
[13] & (T_2 + 1) S_{32} = T_3 S_{22} \\
[15] & (T_3 + 1) S_{23} = T_2 S_{33} \\
[16] & S_{22} S_{11} - S_{12} S_{21} = S_{11} - S_{21} \\
[18] & S_{33} S_{22} - S_{23} S_{32} = S_{22} - S_{32} \\
[20] & S_{23} S_{11} - S_{13} S_{21} = -S_{21} \\
[22] & S_{22} S_{31} - S_{32} S_{21} = S_{31} \\
[24] & S_{33} S_{21} - S_{23} S_{31} = S_{21}. \\
[8] & T_1 S_{32} = T_3 S_{12} \\
[10] & (T_1 + 1) S_{21} = T_2 S_{11} \\
[12] & (T_2 + 1) S_{12} = T_1 S_{22} \\
[14] & (T_3 + 1) S_{13} = T_1 S_{33} \\
[17] & S_{33} S_{11} - S_{13} S_{31} = S_{11} - S_{31} \\
[19] & S_{32} S_{11} - S_{12} S_{31} = -S_{31} \\
[21] & S_{13} S_{22} - S_{23} S_{12} = -S_{12} \\
[23] & S_{33} S_{12} - S_{13} S_{32} = S_{12}
\end{array}$$

Thus we have reduced the system:

Proposition 5.2. *The system $Q(3, 4)$ reduced on the space $G'' \setminus X''(3, 4)$ with respect to the indeterminate $w(a; x)$ consists of (5.14) and the equations [1]-[24].*

In order to reduce $Q(3, 4)$ on the space $G(3, 4) \setminus X''(3, 4)$, we introduce the set of new variables $y = (y^1, y^2, y^3, y^4, y^5)$ by

$$y^1 = \frac{x_{12}}{x_{11}}, \quad y^2 = \frac{x_{13}}{x_{11}}, \quad y^3 = \frac{x_{22}}{x_{11}}, \quad y^4 = \frac{x_{23}}{x_{11}}, \quad y^5 = \frac{x_{33}}{x_{11}}.$$

Then w has the expression

$$w(a; x) = (x_{11})^{a_5-1} f(a; y)$$

for a new function $f(a; y)$ of y . The system $Q(3, 4)$ relative to y can be expressed by a system in the following form:

$$\frac{\partial^2 f}{\partial y^i \partial y^j} = c_{ij} \frac{\partial^2 f}{\partial y^a \partial y^b} + \sum_i c_i \frac{\partial f}{\partial y^i} + c f$$

for all $\{i, j\} \neq \{a, b\}$, where a and b are indices fixed appropriately, say, $a = 4$ and $b = 5$. If the matrix (c_{ij}) is known to be nondegenerate, then the rank of the system is equal to 7 by a general theory of such systems, refer to [SY]. The actual computation of the coefficients c_{ij} can be done successfully by looking at the symbols of the equations [1]-[24], though we omit to give them explicitly. In this computation we must take into account the condition (5.14) that is written as follows:

$$X_{11} w|_{x_{11}=1} = - \sum_{i=1}^5 y^i \frac{\partial f}{\partial y^i}.$$

where

$$(5.18) \quad \tilde{L}_{jk,i\ell} = S_{i\ell}S_{jk} + (\delta_{ij} - \delta_{ik})S_{jk} - \delta_{j\ell}S_{ik},$$

$$(5.19) \quad \tilde{M}_{i,j\ell} = T_iS_{j\ell} - (\delta_{ij} - \delta_{i\ell})S_{j\ell}.$$

Now we see that Lemma 3.1 and its consequence in §4 imply the following lemma.

Lemma 5.1. *The differential operators T_i and S_{jk} form a Lie algebra by the following rules:*

$$\begin{aligned} [S_{i\ell}, S_{jk}] &= \delta_{\ell j}S_{ik} - \delta_{ik}S_{j\ell} + (\delta_{ij} - \delta_{j\ell})S_{i\ell} + (\delta_{ik} - \delta_{ij})S_{jk}, \\ [T_i, S_{jk}] &= (\delta_{ij} - \delta_{ik})S_{jk} + \delta_{ij}T_i - \delta_{ik}T_j, \\ [T_i, T_j] &= 0. \end{aligned}$$

To see the structure of this Lie algebra, put

$$U_{jk} = T_j + S_{jk}.$$

Then the bracket relations turn out to be

$$[U_{i\ell}, U_{jk}] = \delta_{j\ell}U_{ik} - \delta_{ik}U_{j\ell}, \quad [T_i, U_{jk}] = \delta_{ij}U_{ik} - \delta_{ik}U_{ji}, \quad [T_i, T_j] = 0.$$

Since $T_j = 1 - a_j + U_{jj}$, the algebra is $\mathbb{C}\{1, U_{jk}\}$ unless a_j are all equal to 1. Thus in this case, the algebra is isomorphic to $\mathbb{C} \oplus \mathfrak{gl}_3$.

The set of equations (5.17) gives the following equations:

$$\begin{aligned} [1] \quad & 4 \frac{\partial^2 w}{\partial x_{11} \partial x_{22}} = \frac{\partial^2 w}{\partial x_{12} \partial x_{12}} \\ [2] \quad & 4 \frac{\partial^2 w}{\partial x_{22} \partial x_{33}} = \frac{\partial^2 w}{\partial x_{23} \partial x_{23}} \\ [3] \quad & 4 \frac{\partial^2 w}{\partial x_{11} \partial x_{33}} = \frac{\partial^2 w}{\partial x_{13} \partial x_{13}} \\ [4] \quad & 2 \frac{\partial^2 w}{\partial x_{11} \partial x_{23}} = \frac{\partial^2 w}{\partial x_{12} \partial x_{13}} \\ [5] \quad & 2 \frac{\partial^2 w}{\partial x_{22} \partial x_{13}} = \frac{\partial^2 w}{\partial x_{12} \partial x_{23}} \\ [6] \quad & 2 \frac{\partial^2 w}{\partial x_{33} \partial x_{12}} = \frac{\partial^2 w}{\partial x_{13} \partial x_{23}}. \end{aligned}$$

We give numbers to the equations in (5.15) and (5.16) as follows:

$$\begin{array}{lll} [7] \quad M_{1,23} = M_{2,13} & [8] \quad M_{3,12} = M_{1,32} & [9] \quad M_{2,31} = M_{3,21} \\ [10] \quad M_{2,11} = M_{1,21} & [11] \quad M_{3,11} = M_{1,31} & [12] \quad M_{1,22} = M_{2,12} \\ [13] \quad M_{3,22} = M_{2,32} & [14] \quad M_{1,33} = M_{3,13} & [15] \quad M_{2,33} = M_{3,23} \\ [16] \quad L_{11,22} = L_{21,12} & [17] \quad L_{11,33} = L_{31,13} & [18] \quad L_{22,33} = L_{32,23} \\ [19] \quad L_{11,32} = L_{31,12} & [20] \quad L_{11,23} = L_{21,13} & [21] \quad L_{22,13} = L_{12,23} \\ [22] \quad L_{31,22} = L_{21,32} & [23] \quad L_{12,33} = L_{32,13} & [24] \quad L_{21,33} = L_{31,23}. \end{array}$$

Since

$$\tilde{Y}_{ij} = X_{ij}, \quad \tilde{Z}_i = Z_i - a_i,$$

we can see by using (4.8) and (4.12) that

$$\tilde{Q}_{jk} = -\delta_{jk} + \frac{h_j}{h_k}(H_j + R_{jk}).$$

Hence

$$(5.10) \quad S_{jk} := \tilde{Q}_{jk}|_{h=1} = -\delta_{jk} + R_{jk},$$

where $h = 1$ denotes the condition $h_1 = h_2 = h_3 = 1$. We can also see that

$$(5.11) \quad \begin{aligned} \tilde{Q}_{i\ell}\tilde{Q}_{jk}w \Big|_{h=1} &= (\delta_{i\ell} - R_{i\ell})(\delta_{jk} - R_{jk})w + (\delta_{ij} - \delta_{ik})R_{jk}w \\ &= S_{i\ell}S_{jk}w + (\delta_{ij} - \delta_{ik})S_{jk}w. \end{aligned}$$

Similarly, from the identity

$$\frac{\tilde{\partial}}{\partial \zeta_i} = -h_i(a_i + X_i + H_i),$$

we have

$$(5.12) \quad \begin{aligned} \frac{\tilde{\partial}}{\partial \zeta_i}\tilde{Q}_{j\ell}w \Big|_{h=1} &= (a_i + X_i)(\delta_{j\ell} - R_{j\ell})w - (\delta_{ij} - \delta_{i\ell})R_{j\ell}w \\ &= T_i S_{j\ell}w - (\delta_{ij} - \delta_{i\ell})S_{j\ell}w, \end{aligned}$$

where

$$(5.13) \quad T_i = -a_i - X_i.$$

The equivariance condition (4.9) is now $R_{jj}w = -a_jw$, which is automatically satisfied in view of the definition of R_{jk} . The condition (4.10) is $-\sum(a_j + X_j)w = (a_4 - 1)w$, which is the same as the following consequence of (4.11):

$$(5.14) \quad \sum_{i \geq j} X_{ij}w = (a_5 - 1)w.$$

The system (4.17) now consists of

$$(5.15) \quad \tilde{L}_{jk, i\ell}v = \tilde{L}_{ik, j\ell}v,$$

$$(5.16) \quad \tilde{M}_{i, j\ell}v = \tilde{M}_{j, i\ell}v,$$

$$(5.17) \quad (2 - \delta_{ik})(2 - \delta_{j\ell})\frac{\partial^2 v}{\partial x_{ij}\partial x_{k\ell}} = (2 - \delta_{ij})(2 - \delta_{k\ell})\frac{\partial^2 v}{\partial x_{ik}\partial x_{j\ell}},$$

Relative to these coordinates, the function v can be written as

$$(5.5) \quad v(a; \zeta, \eta) = \phi(\zeta, \eta)w(a; x),$$

where

$$(5.6) \quad \begin{aligned} \phi(\zeta, \eta) &= h_1^{a_1} h_2^{a_2} h_3^{a_3} \\ &= \zeta_1^{-a_1} \zeta_2^{-a_2} \zeta_3^{-a_3}. \end{aligned}$$

The aim of the following computation is to rewrite the system (III) and its consequence (4.15) for the indeterminate w ; thus, killing the action by $(\mathbb{C}^*)^3$. We use the abbreviation

$$(5.7) \quad X_{ij} = x_{ij} \frac{\partial}{\partial x_{ij}}, \quad Y_{ij} = \eta_{ij} \frac{\partial}{\partial \eta_{ij}}, \quad Z_i = \zeta_i \frac{\partial}{\partial \zeta_i}, \quad H_i = h_i \frac{\partial}{\partial h_i}.$$

The action of the operators Z_i and Y_{ij} on functions of (h, x) is seen to be

$$\begin{aligned} Z_i &= \zeta_i \sum_{k \geq \ell} \frac{\partial}{\partial \zeta_i} \left(\frac{\eta_{k\ell}}{\zeta_k \zeta_\ell} \right) \frac{\partial}{\partial x_{k\ell}} + \zeta_i \sum_k \frac{\partial}{\partial \zeta_i} \left(\frac{1}{\zeta_k} \right) \frac{\partial}{\partial h_k} \\ &= -\zeta_i \sum \left(\frac{\delta_{ik}}{\zeta_k} + \frac{\delta_{i\ell}}{\zeta_\ell} \right) \frac{\eta_{k\ell}}{\zeta_k \zeta_\ell} \frac{\partial}{\partial x_{k\ell}} + \zeta_i \left(\frac{-\delta_{ik}}{\zeta_k^2} \right) \frac{\partial}{\partial h_k} \\ &= -\sum_{i \geq \ell} \frac{\eta_{i\ell}}{\zeta_i \zeta_\ell} \frac{\partial}{\partial x_{i\ell}} - \sum_{k \geq i} \frac{\eta_{ki}}{\zeta_k \zeta_i} \frac{\partial}{\partial x_{ki}} - \frac{1}{\zeta_i} \frac{\partial}{\partial h_i} \\ &= -X_i - H_i, \\ Y_{ij} &= Y_{ij} \left(\frac{\eta_{ij}}{\zeta_i \zeta_j} \right) \frac{\partial}{\partial x_{ij}} \\ &= X_{ij}, \end{aligned}$$

where

$$(5.8) \quad X_i := X_{ii} + \sum_{j=1}^3 X_{ij}.$$

Define

$$(5.9) \quad \begin{aligned} R_{jk} &= a_j + X_j - \frac{x_{kj}}{x_{jj}} X_{jj} - \sum_{\ell} \frac{x_{k\ell}}{x_{j\ell}} X_{j\ell} \\ &= a_j + (x_{jj} - x_{jk}) \frac{\partial}{\partial x_{jj}} + \sum_{\ell=1}^3 (x_{j\ell} - x_{k\ell}) \frac{\partial}{\partial x_{j\ell}}. \end{aligned}$$

Given a differential operator P acting on functions of (η, ζ) , we define the operator \tilde{P} by the formula

$$\tilde{P} = \phi^{-1} \cdot P \cdot \phi.$$

those in (3) yield the difference equations. Moreover, two equations (4.10) and (4.11) give also difference equations in view of (1) and (2). These difference equations comprise the Gauss relations in the present case, which however we do not write down here.

§5. Further reduction of $Q(3,4)$

This section treats the system $Q(3,4)$. Because of the invariance (I), the system can be reduced onto the configuration space $G(3,4)\backslash X(3,4)$. Since the dimension of $X(3,4)$ is 18 and that of $G(3,4)$ is 13, the quotient space has dimension 5, while the rank of $Q(3,4)$ is 7 as we shall see later. This implies that the difference of the dimension of the domain of definition and the dimension of the space of solutions is two, which falls into the special case that was fully considered in the paper [SY]. A precise form of the induced system is indispensable for application.

In the previous section we have computed its intermediate reduction on the space $GL_3\backslash X'(3,4)$. Since $G(3,4)/GL_3 \cong (\mathbb{C}^*)^4$, it remains to kill the action $(\mathbb{C}^*)^4$. However, we have several possibilities to do this. In fact, the process we followed in the previous section was restricted to the subspace $X'(k,n)$ of the space $X(k,n)$. For the understanding of the configuration space $G(k,n)\backslash X(k,n)$, we need to know the action of $G(k,n)$ on $X(k,n)$. A part of this action will be discussed in Appendix. With this general remark in mind, we choose the open dense subspace $X''(3,4)$ of $X(3,4)$ defined by

$$X''(3,4) = \{(z, w) \in X(3,4); \text{ every } 3 \times 3\text{-minor of } z \neq 0\},$$

on which acts the group $G'' = GL_3 \times (\mathbb{C}^*)^3$ as we specify below. Then the quotient space $G''\backslash X''(3,4)$ is a \mathbb{C}^* -fiber space over the base space $G(3,4)\backslash X''(3,4)$; that is, we still have a freedom of \mathbb{C}^* -action, which will be taken care of later. We use the notations in the previous section freely. For simplicity, we write ζ_i for ζ_{i4} and $M_{i,j\ell}$ for $M_{i4,j\ell}$. Indices in this section range from 1 to 3.

For a given (ζ, η) let us find (h_1, h_2, h_3) , (g_1, g_2, g_3) , and x_{ij} so that

$$(5.1) \quad \begin{pmatrix} 1 & & \zeta_1 \\ & 1 & \zeta_2 \\ & & 1 & \zeta_3 \end{pmatrix} = \text{diag}(g_1, g_2, g_3) \begin{pmatrix} 1 & & 1 \\ & 1 & 1 \\ & & 1 & 1 \end{pmatrix} \text{diag}(h_1, h_2, h_3, 1),$$

$$(5.2) \quad \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \text{diag}(g_1, g_2, g_3) \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \text{diag}(g_1, g_2, g_3).$$

One solution is

$$(5.3) \quad g_1 = \zeta_1, \quad g_2 = \zeta_2, \quad g_3 = \zeta_3, \quad h_1 = \frac{1}{\zeta_1}, \quad h_2 = \frac{1}{\zeta_2}, \quad h_3 = \frac{1}{\zeta_3}, \quad x_{ij} = \frac{\eta_{ij}}{\zeta_i \zeta_j}.$$

This defines a change of coordinates from (ζ, η) to (h, x) , where $h = (h_1, h_2, h_3)$. The converse relation is

$$(5.4) \quad \zeta_1 = \frac{1}{h_1}, \quad \zeta_2 = \frac{1}{h_2}, \quad \zeta_3 = \frac{1}{h_3}, \quad \eta_{ij} = \frac{x_{ij}}{h_i h_j}.$$

The operators A_{pq} , B_{pq} , and C_{pq} defined in §2 have the corresponding operators on the space (ζ, η) , which are listed below by use of the same letters. The index range is as was mentioned above.

$$(4.18) \quad \begin{aligned} A_{j\ell} &= Q_{j\ell}, & A_{jq} &= \frac{\partial}{\partial \zeta_{jq}}, \\ A_{p\ell} &= \sum_i \zeta_{ip} Q_{i\ell}, & A_{pq} &= \sum_i \zeta_{ip} \frac{\partial}{\partial \zeta_{iq}}, \\ B_{j\ell} &= \frac{1}{2-\delta_{j\ell}} \frac{\partial}{\partial \eta_{j\ell}}, & B_{jq} &= B_{qj} = \sum_j \frac{\zeta_{jq}}{2-\delta_{ij}} \frac{\partial}{\partial \eta_{ij}}, \\ B_{pq} &= \sum_{i \geq j} \zeta_{ip} \zeta_{jq} \frac{\partial}{\partial \eta_{ij}}, & C_{j\ell} &= \sum_{i,m} \eta_{im} L_{mj, i\ell}, \\ C_{jq} &= C_{qj} = \sum_{i,m} \eta_{im} M_{mq, ij}, & C_{pq} &= \sum_{i,m} \eta_{im} \frac{\partial^2}{\partial \zeta_{ip} \partial \zeta_{mq}}. \end{aligned}$$

These operators satisfy the commutation relations of Lemma 3.1. In particular, $\{Q_{j\ell}, \partial/\partial \zeta_{jq}, \sum \zeta_{ip} Q_{i\ell}, \sum \zeta_{ip} \partial/\partial \zeta_{iq}\}$ generate the Lie algebra isomorphic to \mathfrak{gl}_n .

The contiguity relations (3.4)-(3.6) can be rewritten as follows.

Proposition 4.1. *The function $v(a; \zeta, \eta)$ satisfies the following relations.*

$$(1) \quad \begin{aligned} Q_{ij}v(a) &= (a_j - 1)v(a + 1_i - 1_j), \\ \frac{\partial}{\partial \zeta_{ip}}v(a) &= (a_p - 1)v(a + 1_i - 1_p), \\ \sum_i \zeta_{ip} Q_{i\ell}v(a) &= (a_\ell - 1)v(a + 1_p - 1_\ell), \\ \sum_i \zeta_{ip} \frac{\partial}{\partial \zeta_{iq}} &= (a_q - 1)v(a + 1_p - 1_q), \end{aligned}$$

$$(2) \quad \begin{aligned} \frac{1}{2-\delta_{j\ell}} \frac{\partial}{\partial \eta_{j\ell}}v(a) &= (a_{n+1} - 1)v(a + 1_j + 1_\ell - 1_{n+1}), \\ \sum_j \frac{\zeta_{jq}}{2-\delta_{ij}} \frac{\partial}{\partial \eta_{ij}}v(a) &= (a_{n+1} - 1)v(a + 1_i + 1_q - 1_{n+1}), \\ \sum_{i \geq j} \zeta_{ip} \zeta_{jq} \frac{\partial}{\partial \eta_{ij}}v(a) &= (a_{n+1} - 1)v(a + 1_p + 1_q - 1_{n+1}), \end{aligned}$$

$$(3) \quad \begin{aligned} \sum_{i,j} \eta_{ij}(Q_{i\ell} Q_{jm} - \delta_{j\ell} Q_{im})v(a) &= (a_m - 1)(a_\ell - 1 - \delta_{m\ell})v(a - 1_m - 1_\ell + 1_{n+1}), \\ \sum_{i,j} \eta_{ij} \frac{\partial}{\partial \zeta_{jp}} Q_{i\ell}v(a) &= (a_\ell - 1)(a_p - 1)v(a - 1_\ell - 1_p + 1_{n+1}), \\ \sum_{ij} \eta_{ij} \frac{\partial^2}{\partial \zeta_{ip} \partial \zeta_{jq}}v(a) &= (a_p - 1)(a_q - 1 - \delta_{pq})v(a - 1_p - 1_q + 1_{n+1}), \end{aligned}$$

where i, j, ℓ, m run from 1 to k and p, q from $k+1$ to n .

The sets of equations (1), (2) and (3) correspond to the operators A , B and C , respectively. Combinations of equations among (2) and combinations of equations in (1) and (2) with

which is equal to $-\delta_{jm}\delta_{il} - \delta_{im}\delta_{jl}$ when $g = I_k$. We see further the identity

$$\frac{\partial^2 v}{\partial z_{il}\partial z_{jm}} = (-\delta_{jl}P_{im} + P_{il}P_{jm})v.$$

Then, defining the operator $L_{jm,il}$ by

$$L_{jm,il}v = \frac{\partial^2 u}{\partial z_{il}\partial z_{jm}} \Big|_{g=I_k},$$

we get a relation

$$(4.14) \quad \begin{aligned} L_{jm,il} &= \delta_{jm}\delta_{il} + \delta_{im}\delta_{jl} - \delta_{jm}P_{il} - \delta_{il}P_{jm} - \delta_{jl}P_{im} + P_{il}P_{jm} \\ &= Q_{il}Q_{jm} - \delta_{jl}Q_{im}. \end{aligned}$$

Next, because of the identity

$$\begin{aligned} \frac{\partial^2 u}{\partial z_{ip}\partial z_{jl}} &= \frac{\partial}{\partial z_{ip}} \left(-(\det g)^{-2}C_{j\ell}v + (\det g)^{-1}\frac{\partial v}{\partial z_{jl}} \right) \\ &= -(\det g)^{-2}C_{j\ell}\frac{\partial v}{\partial z_{ip}} + (\det g)^{-1}\frac{\partial^2 v}{\partial z_{ip}\partial z_{jl}}, \end{aligned}$$

we have

$$\frac{\partial^2 u}{\partial z_{ip}\partial z_{jl}} \Big|_{g=I_k} = M_{ip,j\ell}v$$

where

$$(4.15) \quad \begin{aligned} M_{ip,j\ell} &= -\delta_{j\ell}\frac{\partial}{\partial \zeta_{ip}} + \frac{\partial}{\partial \zeta_{ip}}P_{j\ell} \\ &= \frac{\partial}{\partial \zeta_{ip}}Q_{j\ell}. \end{aligned}$$

Finally, we see

$$(4.16) \quad \frac{\partial^2 u}{\partial z_{iq}\partial z_{jp}} \Big|_{g=I_k} = \frac{\partial^2 v}{\partial \zeta_{iq}\partial \zeta_{jp}}.$$

As for the second set of equations of (III), we have $\partial^2 u / \partial w_{ij}\partial w_{\ell m} = \partial^2 v / \partial \eta_{ij}\partial \eta_{\ell m}$ when $g = I_k$. Hence, we have seen that the system (III) on the space $GL_k \backslash X'(k, n)$ can be written as follows:

$$(4.17) \quad \left\{ \begin{array}{l} L_{jm,il}v = L_{im,j\ell}v \quad \text{for } 1 \leq i, j, \ell, m \leq k, \\ M_{ip,j\ell}v = M_{jp,il}v \quad \text{for } 1 \leq i, j, \ell \leq k, \quad k+1 \leq p \leq n, \\ \frac{\partial^2 v}{\partial \zeta_{iq}\partial \zeta_{jp}} = \frac{\partial^2 v}{\partial \zeta_{ip}\partial \zeta_{jq}} \quad \text{for } 1 \leq i, j \leq k, \quad k+1 \leq p, q \leq n, \\ (2 - \delta_{il})(2 - \delta_{jm})\frac{\partial^2 v}{\partial \eta_{ij}\partial \eta_{m\ell}} = (2 - \delta_{ij})(2 - \delta_{m\ell})\frac{\partial^2 v}{\partial \eta_{il}\partial \eta_{jm}} \\ \quad \text{for } 1 \leq i, j, \ell, m \leq k. \end{array} \right.$$

Define the vector field $P_{j\ell}$ by

$$(4.8) \quad P_{j\ell} = - \sum_p \zeta_{\ell p} \frac{\partial}{\partial \zeta_{jp}} - \eta_{\ell j} \frac{\partial}{\partial \eta_{jj}} - \sum_m \eta_{\ell m} \frac{\partial}{\partial \eta_{jm}}.$$

Then we get

$$\begin{aligned} \left. \frac{\partial v}{\partial z_{j\ell}} \right|_{g=I_k} &= - \sum_{i,p} \zeta_{\ell p} \delta_{ij} \frac{\partial v}{\partial \zeta_{ip}} - \sum_{i \geq m} (\delta_{ij} \eta_{\ell m} + \eta_{i\ell} \delta_{jm}) \frac{\partial v}{\partial \eta_{im}} \\ &= P_{j\ell} v. \end{aligned}$$

This means that $\partial/\partial z_{j\ell} = P_{j\ell}$ on the space of functions of (ζ, η) . The equivariance condition (I) is written as follows:

$$(4.9) \quad P_{jj} v = a_j v,$$

$$(4.10) \quad \sum_j \zeta_{jp} \frac{\partial v}{\partial \zeta_{jp}} = (a_p - 1)v,$$

$$(4.11) \quad \sum_{i \geq j} \eta_{ij} \frac{\partial v}{\partial \eta_{ij}} = (a_{n+1} - 1)v.$$

For later use, it is convenient to introduce the operator

$$(4.12) \quad Q_{jk} = -\delta_{jk} + P_{jk}.$$

Then, (4.9) can be rewritten as

$$(4.13) \quad Q_{jj} v = (a_j - 1)v.$$

The condition (II) turns out to be equal to the identity $\partial/\partial z_{j\ell} = P_{j\ell}$. Note that the conditions (4.9)-(4.11) are not independent; namely, (4.11) follows from (4.9) and (4.10) in view of (4.8).

Let us next examine the second-order derivatives. We consider the first set of equations of (III). By (4.5) we have

$$\begin{aligned} \frac{\partial^2 u}{\partial z_{i\ell} \partial z_{jm}} &= - \frac{\partial}{\partial z_{i\ell}} \{ (\det g)^{-2} C_{jm} \} v - (\det g)^{-2} C_{jm} \frac{\partial v}{\partial z_{i\ell}} \\ &\quad - (\det g)^{-2} C_{i\ell} \frac{\partial v}{\partial z_{jm}} + (\det g)^{-1} \frac{\partial^2 v}{\partial z_{i\ell} \partial z_{jm}}, \end{aligned}$$

where

$$\frac{\partial}{\partial z_{i\ell}} \{ (\det g)^{-2} C_{jm} \} = -2(\det g)^{-3} C_{i\ell} C_{jm} + (\det g)^{-2} \frac{\partial C_{jm}}{\partial z_{i\ell}},$$

Let us consider the coordinate change

$$(z, w) \longrightarrow (g, \zeta, \eta)$$

defined by

$$(4.1) \quad (z_1 \quad z_2) = g(I_k \quad \zeta),$$

$$(4.2) \quad w = g\eta g^T,$$

where $g = (g_i^j)$ is a $k \times k$ -matrix and I_k is the $k \times k$ identity matrix; so, ζ is a $k \times (n - k)$ -matrix. The component-wise expression is

$$(4.3) \quad \begin{aligned} z_{ij} &= g_i^j, & z_{ip} &= \sum_j g_i^j \zeta_{jp}, & w_{ij} &= \sum_{m,\ell} g_i^m \eta_{m\ell} g_j^\ell, \\ g_i^j &= z_{ij}, & \zeta_{ip} &= \sum_j G_i^j z_{jp}, & \eta_{ij} &= \sum_{m,\ell} G_i^m w_{m\ell} G_j^\ell, \end{aligned}$$

where $G = (G_i^j)$ is the inverse matrix of g . In the following, i, j, ℓ, m are reserved for the index range from 1 to k , and p, q from $k + 1$ to n unless otherwise stated. With this change the function u transforms as

$$(4.4) \quad u(a; z, w) = (\det g)^{-1} v(a; \zeta, \eta),$$

by thus defining v . It is easy to see

$$\begin{aligned} \frac{\partial}{\partial z_{j\ell}} &= \frac{\partial}{\partial g_j^\ell} - \sum_{i,p} \zeta_{\ell p} G_i^j \frac{\partial}{\partial \zeta_{ip}} - \sum_{i \geq m} (G_i^j \eta_{\ell m} + \eta_{i\ell} G_m^j) \frac{\partial}{\partial \eta_{im}}, \\ \frac{\partial}{\partial z_{jp}} &= \sum_i G_i^j \frac{\partial}{\partial \zeta_{ip}}, \\ \frac{\partial}{\partial w_{jj}} &= \sum_{i \geq \ell} G_i^j G_\ell^j \frac{\partial}{\partial \eta_{i\ell}}, \\ \frac{\partial}{\partial w_{ij}} &= \sum_{\ell \geq m} (G_\ell^j G_m^i + G_\ell^i G_m^j) \frac{\partial}{\partial \eta_{\ell m}} \quad \text{for } i \neq j. \end{aligned}$$

Let $(C_{j\ell})$ denote the cofactor matrix of g . Then we have

$$(4.5) \quad \frac{\partial u}{\partial z_{j\ell}} = -(\det g)^{-2} C_{j\ell} v + (\det g)^{-1} \frac{\partial v}{\partial z_{j\ell}},$$

$$(4.6) \quad \frac{\partial u}{\partial z_{jp}} = (\det g)^{-1} \sum_i G_i^j \frac{\partial v}{\partial \zeta_{ip}},$$

$$(4.7) \quad \frac{\partial u}{\partial w_{ij}} = (\det g)^{-1} \frac{2 - \delta_{ij}}{2} \sum_{\ell \geq m} (G_\ell^j G_m^i + G_\ell^i G_m^j) \frac{\partial v}{\partial \eta_{\ell m}}.$$

Let us consider the case where the matrix z has the form

$$z = \begin{pmatrix} & & \zeta_{1 \ k+1} & \cdots & \zeta_{1 \ n+1} \\ & I_k & & \cdots & \\ & & \zeta_{k \ k+1} & \cdots & \zeta_{k \ n+1} \end{pmatrix}.$$

Then it is easy to see that the rank of the matrix M in this case is $k(k+1)/2$. More precisely, we can solve (3.10):

$$(3.11) \quad \lambda_{ij} = - \sum_{k < p \leq n} \zeta_{ip} \zeta_{jp} \lambda_{pp} - \sum_{k < p < q \leq n} (\zeta_{ip} \zeta_{jq} + \zeta_{iq} \zeta_{jp}) \lambda_{pq} - \sum_{1 \leq p \leq k < q \leq n} (\delta_{ip} \zeta_{jq} + \delta_{jp} \zeta_{iq}) \lambda_{pq}.$$

For a general $z = (z_1 \ z_2)$ where z_1 is a $k \times k$ non-singular matrix, we need only to put $\zeta = (z_1)^{-1} z_2$. Thus, when the rank of z is k , the dimension of the space $\{(\lambda_{pq})\}$ is $n(n+1)/2 - k(k+1)/2$ and the dimension of $\mathbb{C}\{B_{pq}\}$ is $k(k+1)/2$.

The identities (3.11) and (3.4)-(3.6) imply the relations that hold among u with distinct parameters.

Proposition 3.2. *Let $z = (z_1 \ z_2)$ be general in the sense that z_1 is non-singular and put $\zeta = (z_1)^{-1} z_2$. Then the integral u satisfies the following difference equations relative to the parameters a_i .*

$$(S1) \quad u(a + 2_p - 1_{n+1}) = \sum_{1 \leq i, j \leq k} \zeta_{ip} \zeta_{jp} u(a + 1_i + 1_j - 1_{n+1}) \quad \text{for } k < p \leq n,$$

$$(S2) \quad u(a + 1_p + 1_q - 1_{n+1}) = \sum_{1 \leq i, j \leq k} (\zeta_{ip} \zeta_{jq} + \zeta_{iq} \zeta_{jp}) u(a + 1_i + 1_j - 1_{n+1}) \quad \text{for } k < p < q \leq n,$$

$$(S3) \quad u(a + 1_i + 1_p - 1_{n+1}) = 2 \sum_{1 \leq j \leq k} \zeta_{jp} u(a + 1_i + 1_j - 1_{n+1}) \quad \text{for } 1 \leq i \leq k < p \leq n.$$

We call these relations (S1)-(S3) the *Gauss relations*.

§4. Reduction of the system $Q(k, n)$

Because of the symmetry under the group $G(k, n)$, the system $Q(k, n)$ can be pushed down onto the space $G(k, n) \backslash X(k, n)$. In this section, we derive an intermediate system that is the reduction of $Q(k, n)$ on the quotient space $GL_k \backslash X(k, n)$ and, in the next section, we treat $Q(3, 4)$ more concretely. To be precise, the space $X(k, n)$ is a little too large for our purpose; instead, we consider the subspace $X'(k, n)$ consisting of (z_1, z_2) such that z_1 is nonsingular.

Referring to the expression (2.2), we easily see that

$$(3.4) \quad A_{pq}u(a; z, w) = (a_q - 1)u(a + 1_p - 1_q; z, w),$$

$$(3.5) \quad B_{pq}u(a; z, w) = (a_{n+1} - 1)u(a + 1_p + 1_q - 1_{n+1}; z, w),$$

$$(3.6) \quad C_{pq}u(a; z, w) = (a_p - 1)(a_q - 1 - \delta_{pq})u(a - 1_p - 1_q + 1_{n+1}; z, w).$$

Here and in what follows we use the abbreviation such as $a + 1_p - 1_q = (a_1, \dots, a_p + 1, \dots, a_q - 1, \dots, a_{n+1})$ and $a + 2_p = (a_1, \dots, a_p + 2, \dots, a_{n+1})$. We call the identities (3.4)-(3.6) the *contiguity relations* of the integral u . The operators has the following property.

- Lemma 3.1.** (1) $B_{pq} = B_{qp}$ and $C_{pq} = C_{qp}$.
(2) $\{A_{pq}; 1 \leq p, q \leq n\}$ and $\{C_{pq}; 1 \leq p \leq q \leq n\}$ are linearly independent over \mathbb{C} for generic z and w .
(3) $\{B_{pq}; 1 \leq p \leq q \leq n\}$ span a vector space of dimension $k(k+1)/2$ when the matrix z is of maximal rank.
(4) The space generated by A_{pq} and B_{pq} and the space generated by A_{pq} and C_{pq} are Lie algebras by the respective rules:

$$(3.7) \quad [A_{pq}, A_{rs}] = \delta_{qr}A_{ps} - \delta_{sp}A_{rq},$$

$$(3.8) \quad [A_{pq}, B_{rs}] = \delta_{qr}B_{ps} + \delta_{qs}B_{pr}, \quad [B_{pq}, B_{rs}] = 0,$$

$$(3.9) \quad [A_{pq}, C_{rs}] = -\delta_{ps}C_{qr} - \delta_{pr}C_{qs}, \quad [C_{pq}, C_{rs}] = 0.$$

- (5) The subalgebra $\mathbb{C}\{A_{pq}\}$ is isomorphic to \mathfrak{gl}_n .

Proof. The property (1) follows from the definition, the property (2) is seen easily, and the bracket relations in (4) are easy consequences of the definition. The isomorphism in (5) is given by corresponding A_{pq} to the matrix E_{pq} with 1 at the (pq) -th component and 0 elsewhere. The property (3) will be checked in the following.

Assume that $\{B_{pq}\}$ has a relation

$$\sum_{p,q=1}^n \lambda_{pq}B_{pq} = 0 \quad \text{where} \quad \lambda_{pq} = \lambda_{qp} \in \mathbb{C}.$$

This relation is equivalent to the condition

$$\sum_{p,q=1}^n \lambda_{pq}z_{ip}z_{jq} = 0 \quad \text{for any} \quad i, j.$$

If we denote by (ij) and (pq) the unordered pairs of i, j and p, q , then the condition is

$$(3.10) \quad \sum_{(pq)} M_{(ij)(pq)} \lambda_{(pq)} = 0,$$

where

$$M_{(ij)(pq)} = \begin{cases} z_{ip}z_{jp} & \text{for } q = p \\ z_{ip}z_{jq} + z_{iq}z_{jp} & \text{for } q \neq p. \end{cases}$$

that $a_1 = \cdots = a_m = 0$ and $a_{m+1} \neq 0, \dots, a_k \neq 0$ for some m . Then $\mu_i(2 - \delta_{ij})c_i c_j = 0$ for $1 \leq j \leq m$. Hence $c_j = 0$ and we may assume that $c_1 = \cdots = c_m = \cdots = c_\ell = 0$, $c_{\ell+1} \neq 0, \dots, c_k \neq 0$ without loss of generality; here, $\ell \geq m$. If $\ell < k$ and $i, j, j' \geq \ell + 1$, then by (2.10) we have

$$-\mu_i(2 - \delta_{ij})c_i c_j / a_j = \sum_p z_{ip} b_p = -\mu_i(2 - \delta_{ij'})c_i c_{j'} / a_j.$$

In particular, for $j' = i$, we have $(2 - \delta_{ij})c_j / a_j = c_i / a_i$. Hence, if $i \neq j$, this leads to $2c_j / a_j = c_i / a_i$; then, by reversing the role of i and j , we see $c_i = 0$. This means that we must have $\ell + 1 = k$ in case $c \neq 0$. However, by (2.5), this leads to a contradiction. Namely, $c = 0$ and $\eta = 0$. Now (2.10) reduces to

$$a_i \sum_p z_{jp} b_p = 0.$$

Then the argument in [GG] can be repeated for getting $a = 0$. In fact, if we assume $a_i \neq 0$ for some i , then we get $\sum z_{jp} b_p = 0$ for any j . Since every $k \times k$ -minor of z is nonsingular, the number of the nonzero components of b is not smaller than k . Then we have at least k equations such as $\sum z_{ip} a_i = 0$, which implies $a = 0$. Hence $\xi = 0$ and this completes the proof.

Remark. As for the dimension of the space of solutions, called the rank of the system, we know the following: If one of the exponents a_j is not an integer, then by the pure co-dimensionality ([KN], [C]) of twisted cohomology groups associated to the integral (1.1), the dimension r of the unique non-vanishing cohomology group of degree $k - 1$ is equal to $(-1)^{k-1}$ -times of the Euler number of the complement of $\{q \prod_{j=1}^n l_j = 0\}$ in \mathbb{P}^{k-1} , which can be known easily by induction; we have

$$r = 1 + \binom{n}{k-1}.$$

Thus the rank of the system $Q(k, n)$ is equal to or greater than r ; we believe that it is exactly equal to r , though we have no proof yet in general. In §5 we show that this is the case when $k = 3$ and $n = 4$.

§3. Contiguity relations and Gauss relations

We define three kinds of differential operators on the space $X(k, n)$ by

$$(3.1) \quad A_{pq} = \sum_{j=1}^k z_{jp} \frac{\partial}{\partial z_{jq}}, \quad 1 \leq p, q \leq n,$$

$$(3.2) \quad B_{pq} = \sum_{i,j=1}^k \frac{z_{ip} z_{jq}}{2 - \delta_{ij}} \frac{\partial}{\partial w_{ij}}, \quad 1 \leq p, q \leq n,$$

$$(3.3) \quad C_{pq} = \sum_{i,j=1}^k w_{ij} \frac{\partial^2}{\partial z_{ip} \partial z_{jq}}, \quad 1 \leq p, q \leq n.$$

we obtain differential identities

$$(III) \quad \left\{ \begin{array}{l} \frac{\partial^2 u}{\partial z_{jq} \partial z_{ip}} = \frac{\partial^2 u}{\partial z_{iq} \partial z_{jp}} \quad \text{for } 1 \leq p, q \leq n, \\ (2 - \delta_{j\ell})(2 - \delta_{im}) \frac{\partial^2 u}{\partial w_{\ell m} \partial w_{ij}} = (2 - \delta_{\ell m})(2 - \delta_{ij}) \frac{\partial^2 u}{\partial w_{j\ell} \partial w_{im}} \\ \quad \text{for } 1 \leq i, j, \ell, m \leq n. \end{array} \right.$$

Thus, we obtain a system $Q(k, n)$: (I), (II), (III).

Theorem 2.1. *The system $Q(k, n)$ is holonomic.*

Proof. Let V be the characteristic variety of the system, that is, the set of common zeros of the principal symbols of the equations (I) to (III) and the equations that are derived recursively by these equations. The holonomicity means that the dimension of V is one half of the dimension of T^*X . Instead of V , we consider the variety $Z \subset T^*X$ that is defined as the set of common zeros of the principal symbols of (I) to (III). It is the set of $(z, w; \xi, \eta) \in T^*X$ which satisfy the following equations:

$$(2.4) \quad \sum_i z_{ip} \xi_{ip} = 0 \quad 1 \leq p \leq n,$$

$$(2.5) \quad \sum_{i,j} w_{ij} \eta_{ij} = 0,$$

$$(2.6) \quad \sum_p z_{mp} \xi_{\ell p} + 2 \sum_i w_{mi} \eta_{\ell i} = 0 \quad 1 \leq \ell, m \leq k,$$

$$(2.7) \quad \xi_{jp} \xi_{iq} - \xi_{ip} \xi_{jq} = 0,$$

$$(2.8) \quad (2 - \delta_{ij})(2 - \delta_{\ell m}) \eta_{im} \eta_{j\ell} - (2 - \delta_{im})(2 - \delta_{\ell j}) \eta_{ij} \eta_{\ell m} = 0.$$

Since $\dim X \leq \dim V \leq \dim Z$, it suffices to prove that $\dim Z = \dim X$.

First, from (2.8), the rank of ξ is not greater than 1. Hence it is written as $\xi_{ip} = \alpha_i \beta_p$ for a k -vector α and an n -vector β . From (5) similarly, $\eta_{ij} = (2 - \delta_{ij}) \gamma_i \gamma_j$ for a k -vector γ . Second, since w is symmetric, we may assume that it is diagonal: instead of $(z, w; \xi, \eta)$, consider $(gz, gwg^T; g\xi, g\eta g^T)$ so that gwg^T is diagonal for some matrix g . Then, for $w = \text{diag}(\mu_1, \dots, \mu_k)$, the equations (2.4) and (2.6) are written as follows:

$$(2.9) \quad \beta_p \sum_i z_{ip} \alpha_i = 0,$$

$$(2.10) \quad \alpha_i \sum_p z_{jp} \beta_p + \mu_j (2 - \delta_{ij}) \gamma_i \gamma_j = 0.$$

The equation (2.5) that is written as $\text{tr}(w\eta) = 0$ is a consequence of (2.4) and (2.6). Let U be the subset of X consisting of points (z, w) where any $k \times k$ -minor of z and the matrix w are nonsingular. We prove that $\xi = \eta = 0$ when $(z, w) \in U$. For this purpose, we assume

acts on this space as follows:

$$\begin{aligned}
(\mathbb{C}^*)^{n+1} \ni \lambda &= (\lambda_1, \dots, \lambda_n, \lambda_{n+1}) : \\
(z, w) &= (z_1, \dots, z_n, w) \longrightarrow \lambda(z, w) = (\lambda_1 z_1, \dots, \lambda_n z_n, \lambda_{n+1} w), \\
GL_k \ni g &: (z, w) \longrightarrow g(z, w) = (gz, gw g^T).
\end{aligned}$$

This action can be naturally extended to the functions u . The first action induces the transformation rule

$$1^\circ \quad u(a; \lambda(z, w)) = \prod_{p=1}^{n+1} (\lambda_p)^{a_p-1} u(a; z, w)$$

and the second action yields

$$\begin{aligned}
2^\circ \quad u(a; g(z, w)) &= \int \prod \langle t, g z_p \rangle^{a_p-1} q(g w g^T, t)^{a_{n+1}-1} dt \\
&= \int \prod \langle t g, z_p \rangle^{a_p-1} q(w, t g)^{a_{n+1}-1} dt \\
&= (\det g)^{-1} u(a; z, w).
\end{aligned}$$

These two kinds of rules imply that the infinitesimal generators of the action define differential operators on the space of the functions u . Let us first differentiate 1° relative to λ ; then we obtain

$$(I) \quad \begin{cases} \sum_{j=1}^k z_{jp} \frac{\partial u}{\partial z_{jp}} = (a_p - 1)u & \text{for } 1 \leq p \leq n, \\ \sum_{i \geq j} w_{ij} \frac{\partial u}{\partial w_{ij}} = (a_{n+1} - 1)u. \end{cases}$$

Next note that we have $(gz_p)_j = \sum g_j^i z_{ip}$ and $(gw g^T)_{ij} = \sum_{\ell, m} g_i^\ell w_{\ell m} g_j^m$ for an element $g = (g_j^i) \in GL_k$; then the differentiation of 2° relative to g shows

$$\frac{\partial u}{\partial g_m^\ell} = \sum_p z_{\ell p} \frac{\partial u}{\partial z_{mp}} + \sum_{j \leq m} \sum_i \frac{\partial u}{\partial w_{mj}} w_{\ell i} g_j^i + \sum_{i \geq m} \sum_j \frac{\partial u}{\partial w_{im}} g_i^j w_{j\ell}.$$

Setting $g = I_k$, we have

$$(II) \quad \sum_{p=1}^n z_{\ell p} \frac{\partial u}{\partial z_{mp}} + w_{\ell m} \frac{\partial u}{\partial w_{mm}} + \sum_{j=1}^k w_{\ell j} \frac{\partial u}{\partial w_{mj}} = -\delta_{\ell m} u.$$

Since

$$(2.3) \quad \begin{aligned} \frac{\partial u}{\partial z_{jp}} &= \int (a_p - 1) t^j \varphi / \ell_p dt & \text{for } 1 \leq p \leq n, \\ \frac{\partial u}{\partial w_{ij}} &= \int (2 - \delta_{ij})(a_{n+1} - 1) t^i t^j \varphi / q dt, \end{aligned}$$

A1 of Appendix explains how a K3 surface S arises when $k = 3$, $n = 4$, and $a_i = 1/2$. In Section A2, we explicitly clarify the lattice structure of the second homology group of the surface S and, in Section A3, construct 7 transcendental cycles. These cycles define the period mapping associated with the integral (1.1), which is explained in Section A4.

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§2. The system of differential equations

In this section we derive the system of differential equations associated with the integral (1.1). Let ℓ_p , $1 \leq p \leq n$, be linear forms and q a quadratic form both defined on the complex vector space $\mathbb{C}^k(t)$:

$$q = q(w, t) = \sum w_{ij} t^i t^j = t w t^T; \quad w_{ij} = w_{ji}, \quad t = (t^1, \dots, t^k),$$

$$\ell_p = \sum z_{jp} t^j = \langle t, z_p \rangle, \quad 1 \leq p \leq n; \quad z_p = (z_{1p}, \dots, z_{kp})^T,$$

where \cdot^T means transposition. Here $z = (z_{ip})$ is a $k \times n$ -matrix and $w = (w_{ij})$ is a symmetric $k \times k$ -matrix. Set

$$\varphi(z, w; t) = q^{a_{n+1}-1} \prod_{p=1}^n \ell_p^{a_p-1},$$

where $a = (a_1, \dots, a_{n+1})$ is a set of complex parameters, and let

$$dt = \sum_{i=1}^k (-1)^{i-1} t^i dt^1 \wedge \dots \wedge dt^{i-1} \wedge dt^{i+1} \wedge \dots \wedge dt^k$$

be a $(k-1)$ -form. We assume that $k \leq n$ and that the parameter satisfies a relation

$$(2.1) \quad a_1 + \dots + a_n + 2a_{n+1} = n - k + 2$$

so that the total degree of the form φdt relative to t is zero, which means that this form is defined on the projective space $\mathbb{P}^{k-1}(t)$. We further assume that $k \geq 3$ in the following. Then consider the integral

$$(2.2) \quad u(a; z, w) = \int \varphi(z, w; t) dt$$

over a certain cycle in $\mathbb{P}^{k-1}(t)$. The set $X = X(k, n) = \{(z, w)\}$ parametrizes the set of ℓ_p and q ; the integral $u(a; z, w)$ is a function on the space X . The space X has a natural symmetry. In fact, the group

$$G = G(k, n) = (GL_k \times (\mathbb{C}^*)^{n+1}) / \mathbb{C}^*$$

On the system of differential equations associated with a quadric and hyperlanes

Keiji Matsumoto and Takeshi Sasaki

§1. Introduction

Consider the integral

$$(1.1) \quad \int q^{a_{n+1}-1} \prod_{i=1}^n \ell_i^{a_i-1} \omega_k,$$

where ℓ_i are linear forms and q is a quadratic form on the projective space $\mathbb{P}^{k-1}(t)$ of dimension $k-1$; $\omega_k = \sum_{i=1}^k (-1)^{i-1} t^i dt^1 \wedge \cdots \wedge dt^{i-1} \wedge dt^{i+1} \cdots \wedge dt^k$, where $t = (t^1, \dots, t^k)$, denotes the homogeneous coordinates and $a_1 + \cdots + a_n + 2a_{n+1} = n - k + 2$. The integral is a function of the coefficients of ℓ_i and q . In this paper, we derive a system of differential equations which annihilate the integral and prove that the system is holonomic. We also investigate the structure of contiguity operators.

The integral (1.1) can be thought of a generalization of Aomoto's integral representation of the so-called Appell hypergeometric function F_4 ; refer to §6, [A].

The integral (1.1) can be also thought of a generalization of the following integral

$$(1.2) \quad \int \ell_1^{a_1-1} \cdots \ell_n^{a_n-1} \omega_k, \quad \sum a_j = n - k,$$

which is studied by many authors from various aspects ([G], [KY], [MY], [MSY2], [MSTY], [T]). Especially when $k = 3, n = 6, a_j = 1/2$, the integral gives a period of the K3 surface which is the double cover of the projective plane ramifying along the 6 lines $\{l_j = 0\}$; the period map is deeply studied in [MSY1]. Refer also to [M] and [GGR1,2], [S], [SU]. So we are naturally interested in the K3 surface which is the double cover of the projective plane ramifying along four lines and a conic in general position; the structure of algebraic and transcendental lattices are clarified in Appendix. For surfaces obtained from degenerate arrangements of four lines and a conic, see [Sh].

The content is as follows. In Section 2, we derive the system associated with the integral (1.1); we denote it by $Q(k, n)$. The system is defined on the coarse configuration space, which we denote by X (see §2), and the group $G = (GL_k \times (\mathbb{C}^*)^{n+1})/\mathbb{C}^*$ acts on this space. Hence, the system has a symmetry, which is sometimes called the contiguity relations. In Section 3, we compute the symmetry and also the Gauss relations that are difference equations relative to the complex parameters. In Section 4, we derive the system induced on the quotient space $GL_k \backslash X$ and, in Section 5, we pay special attention to the system $Q(3, 4)$ and compute the system induced on the quotient space $G \backslash X$. In Section 6, we calculate an explicit solution of the system $Q(3, 4)$ that is given by a power-series and discuss the relation with classical hypergeometric functions. In Section 7, we briefly mention a few analogous integrals and how to derive the associated systems. The section