# The uniformizing differential equation of the complex hyperbolic structure on the moduli space of marked cubic surfaces

By Takeshi SASAKI Department of Mathematics, Kobe University, Kobe 657 Japan
and Masaaki YOSHIDA, Department of Mathematics, Kyushu University 1, Fukuoka 810 Japan

**Abstract:** We find the uniformizing equation, governing the developing map, of a complex hyperbolic structure on the(4-dimensional) moduli space of marked cubic surfaces. Our equation is invariant under the action of the Weyl group of type $E_6$
**Keywords:** uniformizing differential equation, cubic surface, moduli space, Schwarzian derivative, complex hyperbolic structure, developing map, Weyl group

## 0. Introduction

For any hermitian locally symmetric space $M$, its developing map $f$ from $M$ to the model space can be given by solutions of a (system of) linear differential equation(s) $E$ on $M$, which is called the uniformizing differential equation. When the model space is a projective space or a quadratic hypersurface, several interesting examples are known ([?], [?]); in most cases, the spaces $M$ are moduli spaces of algebraic varieties, and their uniformizing equations are the so-called generalized hypergeometric equations.

The original example is the elliptic modular case: $M = \mathbf{P}^1 - \{0, 1, \infty\}$ is the moduli space of the elliptic curves $t^2 = s(s-1)(s-x)$ with parameter $x$, the developing map $f : M \to \mathbf{H} \subset \mathbf{P}^1$ is given by the ratio of two linearly independent solutions of the hypergeometric equation

$$x(1-x)v'' + (1-2x)v' - v/4 = 0,$$

where $\mathbf{H}$ is the upper half plane, and its monodromy group is conjugate to the elliptic modular group $\Gamma(2)$ inducing the isomorphism $f : M \overset{\cong}{\to} \mathbf{H}/\Gamma(2)$.

In this paper, we find the uniformizing equation $E$ of the moduli space $M$ of marked cubic surfaces, which is known to be 4-dimensional and to carry a complex hyperbolic structure ([?]). Its monodromy group is a discontinuous group acting on the complex 4-dimensional ball $\mathbf{B}^4$, and solutions of $E$ give the developing map from $M$ to $\mathbf{B}^4$, which induces the equivalence between $M$ and the quotient of the ball under the monodromy group.

The space $M$ admits a bi-regular action of the Weyl group of type $E_6$, and can be identified with a Zariski open subset of $\mathbf{C}^4$. Our $E$ is a system of differential equations in 4 variables of rank 5 defined on $M$, and is invariant under this group. The system $E$ is unknown so far, though its restriction to a(ny) singular locus turns out to be the Appell-Lauricella hypergeometric system.

## 1. Moduli space of marked cubic surfaces

We recall a description of the moduli space of marked cubic surfaces. (Refer to [?] and [?].) Since any nonsingular cubic surface can be obtained by blowing up the projective plane $\mathbf{P}^2$ at six points, the moduli space $M$ of such surfaces can be parametrized by $3\times 6$-matrices of which columns give homogeneous coordinates of the six points; in order to get a smooth cubic surface from six points, no three points are assumed to be collinear and the six points are assumed to be not lying on any conic. Killing ambiguity of homogeneous coordinates on $\mathbf{P}^2$ by left action of $GL_3$ and right action of the diagonal subgroup ($\cong (\mathbf{C}^\times)^6$) of $GL_6$, we get the following expression

$$x = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x^1 & x^2 \\ 0 & 0 & 1 & 1 & x^3 & x^4 \end{pmatrix}.$$

The cubic surface obtained by blowing up the six points represented by this matrix $x$ is non-singular if and only if the quantity

$$\begin{aligned} D(x) := \prod_{i=1}^{4} x^i(x^i-1) \times \{x^1x^4 - x^2x^3\} \\ \times(x^1-x^2)(x^1-x^3)(x^2-x^4)(x^3-x^4) \\ \times\{(x^1-1)(x^4-1) - (x^2-1)(x^3-1)\} \\ \times\{x^1(x^2-1)(x^3-1)x^4 - (x^1-1)x^2x^3(x^4-1)\} \end{aligned}$$

does not vanish. Thus we can identify the moduli space $M$ and the affine open set

$$\{x=(x^1,\dots,x^4)\in\mathbf{C}^4 \mid D(x)\neq 0\}.$$

## 2. Hyperbolic structure on the moduli space

Let $CS(x)$ be the cubic surface corresponding to $x\in M$ and $TC(x)$ the triple cover of $\mathbf{P}^3$ branching along $CS(x)$. Let $\omega$ be a primitive cube root of unity, and put $\mathcal{E}=\mathbf{Z}[\omega]$. Following is known ([?], [?]):

(i) The 3-fold $TC(x)$ has five $\mathcal{E}$-independent periods (integrals of a 3-form along 3-cycles); the (multi-valued) period map for an appropriate choice of periods $v_1(x),\dots,v_5(x)$,

$$f: M\ni x\longmapsto v(x)=v_1(x):\cdots:v_5(x)\in\mathbf{P}^4$$

has its image in the ball

$$\begin{aligned}\mathbf{B}^4= \quad &\{v_1:\cdots:v_5\in\mathbf{P}^4 \mid \\ &{}^t\bar{v}hv:=|v_1|^2-|v_2|^2-\cdots-|v_5|^2>0\}.\end{aligned}$$

(ii) The projective monodromy group of $f$ is the principal congruence subgroup

$$\Gamma(1-\omega):=\{g\in\Gamma \mid g\equiv I_5 \bmod (1-\omega)\}/\text{center},$$

with level $(1-\omega)$, of the modular group

$$\Gamma:=\{g\in GL_5(\mathcal{E}) \mid {}^t\bar{g}hg=h\}/\text{center}.$$

Moreover the isomorphism

$$\mathcal{E}/(1-\omega)\mathcal{E}\cong\mathbf{F}_3$$

(the field with three elements) induces the isomorphisms

$$\begin{aligned}\Gamma/\Gamma(1-\omega) \quad &\cong \{g\in GL_5(\mathbf{F}_3) \mid {}^tghg=h\}/\text{center}\\ &\cong W(E_6),\end{aligned}$$

the Weyl group of type $E_6$.

(iii) $\Gamma(1-\omega)$ is a reflection group; let $\mathcal{H}$ be the union of the mirrors (inside the ball) of the reflections. Then $f$ induces the isomorphism

$$M\xrightarrow{\cong}(\mathbf{B}^4-\mathcal{H})/\Gamma(1-\omega).$$

This isomorphism gives a hyperbolic structure on the moduli space $M$.

## 3. Uniformizing differential equations

Since the functions $v_i(x)$ are defined by the integals, they should satisfy a system of differential equations defined on $M$ of rank (= dimension of local solutions at a(ny) generic point) 5. The aim of this paper is to announce its explicit form.

Our recipe is the following: (In this section here after, we freely use the properties of the Schwarzian derivatives stated in §4.) Since $M$ is covered by the ball, and $f$ is the developing map, we apply Schwarzian derivatives

$$S^k_{ij}\{f;x\}=:S^k_{ij}(x),\quad i,j,k=1,\dots,4$$

to the map $f$ with respect to the coordinates $x=(x^1,\dots,x^4)$. The map $f$ can be recovered (up to multiplying a function) by getting linearly independent solutions of the system

$$E:\quad \frac{\partial^2 v}{\partial x^i\partial x^j}=\sum_{k=1}^4 S^k_{ij}\frac{\partial v}{\partial x^k}+S^0_{ij}v,\quad (1\le i,j,k\le 4)$$

where the coefficients $S^0_{ij}$ are polynomials in $S^k_{ij}$ and their derivatives. Thanks to $PGL_5$-invariance of the Schwarzian derivatives, $S^k_{ij}(x)$ are single-valued, and so they are rational functions with poles only along $\{D=0\}$. The local behavior and the integrability condition would determine the system $E$, since Mostow rigidity does not allow the existence of extra parameters. Instead of computing directly the integrability condition, we take advantage of the invariance of $E$ under the action of a subgroup $G\cong W(E_6)$ of $Aut(M)$.

## 4. Schwarzian derivatives

In general when $n\ge 2$ (in our case $n=4$), for a non-degenerate map (Jacobian $\not\equiv 0$) $x=(x^1,\dots,x^n)\mapsto z=(z^1,\dots,z^n)$, the Schwarzian derivatives are defined as

$$S^k_{ij}\{z;x\}=\begin{pmatrix}k\\ij\end{pmatrix}-\frac{\delta^k_i}{n+1}\sum_q\begin{pmatrix}q\\qj\end{pmatrix}-\frac{\delta^k_j}{n+1}\sum_q\begin{pmatrix}q\\qi\end{pmatrix},$$

$1\le i,j,k\le n$, where $\delta$ is the Kronecker symbol and

$$\begin{pmatrix}k\\ij\end{pmatrix}=\sum_p\frac{\partial^2 z^p}{\partial x^i\partial x^j}\frac{\partial x^k}{\partial z^p}.$$

They have the properties (cf. [?])

(j) (projective invariance)

$$S^k_{ij}\{Az;x\}=S^k_{ij}\{z;x\}\quad\text{for}\quad A\in PGL_{n+1}.$$

(jj) (connection formula) For a change of coordinates from $x$ to $y$,

$$S^k_{ij}\{z;y\}=S^k_{ij}\{x;y\}+\sum_{p,q,r}S^r_{pq}\{z;x\}\frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}\frac{\partial y^k}{\partial x^r}.$$

(jjj) (local behavior along ramifying singularities) If $z = z(x)$ is ramified along $\{x^1 = 0\}$ with exponent $\alpha$, that is,

$$z^1(x) = (x^1)^\alpha v^1, \quad z^2(x) = v^2, \dots, z^n(x) = v^n,$$
$$|\partial z/\partial x| = (x^1)^{\alpha-1}u,$$

where $v^j$ $(1 \le j \le n)$ and $u$ are holomorphic functions not divisible by $x^1$, then

$$S^k_{ij}\{z;x\}, \quad S^k_{1j}\{z;x\} + \delta^k_j \tfrac{1}{n+1}\tfrac{\alpha-1}{x^1},$$
$$(x^1)^{-1}S^1_{ij}\{z;x\}, \quad S^1_{1j}\{z;x\},$$
$$x^1 S^k_{11}\{z;x\}, \quad S^1_{11}\{z;x\} - \tfrac{n-1}{n+1}\tfrac{\alpha-1}{x^1}$$

are holomorphic for $2 \le i,j,k \le n$.

(jv) (relation with differential equations) Put $S^k_{ij} = S^k_{ij}(x) = S^k_{ij}\{z;x\}$ $(1 \le i,j,k \le n)$ and

$$S^0_{ij} = -\frac{\partial S^k_{ij}}{\partial x^k} - \sum_t S^t_{ij}S^k_{kt} + \frac{\partial S^k_{kj}}{\partial x^i} + \sum_t S^t_{kj}S^k_{it},$$

where $k \ne i$ (recall the assumption $n \ge 2$). Then the system

$$E_n: \quad \frac{\partial^2 u}{\partial x^i \partial x^j} = \sum_k S^k_{ij}(x)\frac{\partial u}{\partial x^k} + S^0_{ij}(x)u,$$

$(1 \le i,j \le n)$ is of rank $n+1$ satisfying the (normality) condition

$$\sum_k S^k_{kj}(x) = 0 \quad (1 \le j \le n).$$

Let $u_0, \dots, u_n$ be linearly independent solutions of $E_n$. Then $u_1/u_0, \dots, u_n/u_0$ are projectively related to $z_1, \dots, z_n$.

## 5. Automorphisms of the moduli space $M$

Let us define after [?] six bi-rational transformations $s_1, \dots, s_6$ in $x = (x^1, \dots, x^4)$ :

$$\begin{aligned}
s_1 : x &\to \left(\frac{1}{x^1}, \frac{1}{x^2}, \frac{x^3}{x^1}, \frac{x^4}{x^2}\right),\\
s_2 : x &\to (x^3, x^4, x^1, x^2),\\
s_3 : x &\to \left(\frac{x^1 - x^3}{1 - x^3}, \frac{x^2 - x^4}{1 - x^4}, \frac{x^3}{x^3 - 1}, \frac{x^4}{x^4 - 1}\right),\\
s_4 : x &\to \left(\frac{1}{x^1}, \frac{x^2}{x^1}, \frac{1}{x^3}, \frac{x^4}{x^3}\right),\\
s_5 : x &\to (x^2, x^1, x^4, x^3),\\
s_6 : x &\to \left(\frac{1}{x^1}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}\right).
\end{aligned}$$

If $M$ is regarded as the configuration space of six points in $\mathbf{P}^2$, the transformation $s_1$, for example, corresponds to the interchange of the two points labelled 1 and 2. Each $s_i$ turns out to be a bi-regular involution on $M$, and they form a group $G$ isomorphic to the Weyl group of type $E_6$; relation of the generators are given by the Coxeter graph

$$\begin{array}{ccccccccc}
s_1 & — & s_2 & — & s_3 & — & s_4 & — & s_5\\
 & & & & | & & & & \\
 & & & & s_6 & & & &
\end{array}$$

If $M$ is regarded as the moduli space of cubic surfaces, each transformation $s_i$ takes cubic surfaces to the isomorphic ones but changes linearly their chosen cycles defining the period map $f$. Thanks to the projective invariance of the Schwarzian derivatives (see §4 (j)) our system $E$ with coefficients $S^k_{ij}\{f;x\} =: S^k_{ij}(x) \in \mathbf{C}(x^1, \dots, x^4)$ is invariant under the action of $G$. The invariance under $s \in G$ implies

$$S^k_{ij}(y) = S^k_{ij}\{x;y\} + \sum_{p,q,r=1}^{4} S^r_{pq}(x)\frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}\frac{\partial y^k}{\partial x^r},$$

where $y = sx$, because the right-hand side equals $S^k_{ij}\{f;y\}$ (see §4 (jj)), and the left hand side is the pull-back $s^*S^k_{ij}(x)$. The transformations $s = s_2, t = s_5$ and $u = ts$ give the identities:

$$\begin{aligned}
S^2_{ij}(x) &= S^1_{t(i),t(j)}(tx), \quad t: 1 \leftrightarrow 2, 3 \leftrightarrow 4,\\
S^3_{ij}(x) &= S^1_{s(i),s(j)}(sx), \quad s: 1 \leftrightarrow 3, 2 \leftrightarrow 4,\\
S^4_{ij}(x) &= S^1_{u(i),u(j)}(ux), \quad u: 1 \leftrightarrow 4, 2 \leftrightarrow 3.
\end{aligned}$$

These identities and the local property (jjj) completely determine all the coefficients of the system $E$. The details of procedures getting the coefficients and further study on our system $E$ will appear in our forthcoming papers. We just state the result in the next section.

## 6. Result: Explicit form of the system $E$

The period map (developing map) $f: M \to \mathbf{B}^4$ can be given by five solutions of the system $E$ with the following coefficients $S^k_{ij} = S^k_{ij}(x^1, x^2, x^3, x^4)$. Since the coefficients $S^k_{ij}$ $(k = 0, 2, 3, 4)$ can be expressed by $S^1_{ij}$ (see §§4 and 5), we give only $S^1_{ij}$.

$$S^1_{23} = \frac{x^1(x^1 - 1)P^1_{23}}{J},$$

$$
\begin{aligned}
S^1_{24} &= \frac{x^1(x^1-1)P^1_{24}}{(x^2-x^4)J},\\
S^1_{34} &= S^1_{24}(x^1,x^3,x^2,x^4),\\
S^1_{44} &= \frac{x^1(x^1-1)P^1_{44}}{x^4(x^4-1)(x^3-x^4)(x^2-x^4)J},\\
S^1_{22} &= \frac{x^1(x^1-1)P^1_{22}}{x^2(x^2-1)(x^1-x^2)(x^2-x^4)J},\\
S^1_{33} &= S^1_{22}(x^1,x^3,x^2,x^4),
\end{aligned}
$$

where

$$
\begin{aligned}
J &= (x^1x^4-x^2x^3)\\
&\quad\times\{(x^1-1)(x^4-1)-(x^2-1)(x^3-1)\}\\
&\quad\times\{x^1x^4(x^2-1)(x^3-1)\\
&\qquad -x^2x^3(x^1-1)(x^4-1)\},\\
P^1_{23} &= -\frac{1}{3}x^4(x^4-1)(x^1-x^3)(x^1-x^2),\\
P^1_{24} &= \frac{1}{3}(x^1-x^3)\{2x^4x^2x^3x^1-x^3x^1x^2\\
&\quad +x^3(x^2)^2-(x^4)^2x^3x^1+(x^4)^2x^2x^3\\
&\quad -2x^4x^3x^2-(x^4)^2x^2x^1+(x^4)^2x^1\\
&\quad -(x^2)^2(x^3)^2+x^2(x^3)^2\},\\
P^1_{44} &= -\frac{1}{3}\{2x^4x^2x^3x^1-x^3x^1x^2+x^3(x^2)^2\\
&\quad -(x^4)^2x^3x^1+(x^4)^2x^2x^3-2x^4x^3x^2\\
&\quad -(x^4)^2x^2x^1+(x^4)^2x^1-(x^2)^2(x^3)^2\\
&\quad +x^2(x^3)^2\}^2,\\
P^1_{22} &= \frac{1}{3}(x^4-x^2)J-\frac{1}{3}x^2(x^2-1)x^4(x^4-1)\\
&\quad\times(x^1-x^2)(x^3-x^4)(x^1-x^3)^2.
\end{aligned}
$$

$$
\begin{aligned}
S^1_{11} &= -\frac{2}{5}\frac{1}{x^1}-\frac{2}{5}\frac{1}{x^1-1}-\frac{1}{15}\frac{1}{x^1-x^2}\\
&\quad -\frac{1}{15}\frac{1}{x^1-x^3}+R^1_{11},\\
S^1_{12} &= \frac{2}{15}\frac{1}{x^2}+\frac{2}{15}\frac{1}{x^2-1}+\frac{1}{5}\frac{1}{x^1-x^2}\\
&\quad +\frac{2}{15}\frac{1}{x^2-x^4}+R^1_{12},\\
S^1_{13} &= S^1_{12}(x^1,x^3,x^2,x^4),\\
S^1_{14} &= \frac{2}{15}\frac{1}{x^4}+\frac{2}{15}\frac{1}{x^4-1}-\frac{2}{15}\frac{1}{x^2-x^4}\\
&\quad -\frac{2}{15}\frac{1}{x^3-x^4}+R^1_{14},
\end{aligned}
$$

where (in the following, we put $x^1=x, x^2=y, x^3=z, x^4=w$)

$$
\begin{aligned}
R^1_{11} &= -(-3zw^2x+2zw^2y-zywx-3w^2xy\\
&\quad +4y^2z^2+2w^3x^2z+2w^3yx^2-wy^3z^2\\
&\quad -y^2z^3w+2w^2x^2+y^3z^3+6z^2xwy^2\\
&\quad -6z^2xw^2y-2zx^2w^2y-6w^2y^2zx\\
&\quad +3y^2w^2z^2+2yw^3zx-3z^2y^2x\\
&\quad -2zx^2w^2+2zyx^2w-w^3yz+y^3zw\\
&\quad -3wy^2z^2+3w^2y^2x-2wyz^2+wyz^3\\
&\quad -2wy^2z-2w^2yx^2-y^3z^2-y^2z^3\\
&\quad +11zyw^2x-2w^3x^2-3yw^3x+3xw^3\\
&\quad +3xw^2z^2-3xw^3z)/15J,
\end{aligned}
$$

$$
\begin{aligned}
R^1_{12} &= (4zw^2x-3zywx+y^2z^2+2w^3x^2z\\
&\quad +y^2z^3w-3w^2x^2-2w^3x^3-z^2xwy^2\\
&\quad +2z^2xw^2y-4z^3xwy-2zx^2w^2y\\
&\quad +2z^2x^2wy+z^3xy^2-z^2x^2w^2\\
&\quad +2zx^3w^2-2x^3zw-x^2z^2y-z^2y^2x\\
&\quad +4zx^2w+z^2xy-4zx^2w^2+z^3xy\\
&\quad -wy^2z^2-w^2yz^2+2w^2x^3+wyz^2\\
&\quad +wyz^3+w^2yx^2-y^2z^3+3z^2xyw\\
&\quad +y^2xzw+2w^3x^2-yz^3-4xwz^2\\
&\quad -2xw^3z+2wz^3x)/15J,
\end{aligned}
$$

$$
\begin{aligned}
R^1_{14} &= (-7zywx-2y^2z^2+3y^3xz^2\\
&\quad -x^2yw+4w^2x^2-3y^3z^3+2z^2xwy^2\\
&\quad -4zx^2w^2y-6ywx^3z-2z^2x^2wy\\
&\quad -2y^2wx^2z+4yx^3w^2+x^2y^2w-x^3yw\\
&\quad +3z^3xy^2+4zx^3w^2+3x^3yz-x^3zw\\
&\quad -11z^2y^2x-6x^2yz-zx^2w+6z^2xy\\
&\quad +6y^2xz+x^2z^2y^2-4zx^2w^2\\
&\quad +z^2x^2w-3z^3xy-3y^3xz+17zyx^2w\\
&\quad -wy^2z^2-4w^2x^3-4w^2yx^2\\
&\quad +3y^3z^2+3y^2z^3+4zyw^2x+x^3w)/15J.
\end{aligned}
$$

## 7. Restriction of $E$ along singular loci

It is known that the configuration space of six points in the projective line can be uniformized by the 3-ball with the Appell-Lauricella hypergeometric system $E_D(a;b_1,b_2,b_3;c)$, defined below, as the uniformizing equation (see e.g. [?]). The most symmetric uniformization comes from the family of curves

$$t^3=s(s-1)(s-y^1)(s-y^2)(s-y^3),$$

and the uniformizing equation is equivalent to $E_D(2/3; 1/3, 1/3, 1/3; 4/3)$.

On the other hand, the hypersurface of $\mathbf{C}^4$, defined by the factor

$$x^1(x^2-1)(x^3-1)x^4-(x^1-1)x^2x^3(x^4-1)$$

of $D(x)$, represents six points lying on a conic. So this locus identifies with the configuration space above. In this way, recalling that every singular locus is equivalent under the action of the group $G$, we naturally expect that the restriction of $E$ along a(ny) singular locus is equivalent to the Appell-Lauricella hypergeometric system.

Without loss of generality, we restrict our system $E$ to the divisor $\{x^4=0\}$. We express solutions $v$ of $E$ as

$$v=(x^4)^\lambda(w(x^1,x^2,x^3)+w_1(x^1,x^2,x^3)x^4+\cdots)$$

and find the exponent $\lambda$ and the system of differential equations satisfied by $w$. From the equations

$$\frac{\partial^2 v}{\partial x^i \partial x^4}=\sum S^k_{i4}\frac{\partial v}{\partial x^k}+S^0_{i4}v$$

in $E$, we get $\lambda=2/15$, and from the rest of $E$, we find that $w$ satisfies

$$\frac{\partial^2 w}{\partial x^i \partial x^j}=\sum_{k=1}^{3} T^k_{ij}\frac{\partial w}{\partial x^k}+T^0_{ij}w, \quad 1\le i,j\le 3$$

where

$$T^k_{ij}=S^k_{ij}|_{x^4=0}, \quad T^0_{ij}=\lambda(S^4_{ij}/x^4)|_{x^4=0}+S^0_{ij}|_{x^4=0}.$$

Introduce the new variables $y=(y^1,y^2,y^3)$ by

$$y^1=\frac{x^1}{x^3}, \quad y^2=\frac{1}{x^3}, \quad y^3=\frac{(x^1-x^2)}{x^3(1-x^2)}$$

and the new unknown $u$ by multiplying the factor

$$\begin{aligned}&(y^1(y^1-1)y^2(y^2-1)y^3(y^3-1))^{-2/15}(y^2)^{3/5}\\ &\times(y^2-y^3)^{4/15}(y^1-y^3)^{-1/5}(y^1-y^2)^{-1/3}\end{aligned}$$

to the old unknown $w$. Then the system with the new unknown $u$ and the new variable $y$ is exactly the Appell-Lauricella hypergeometric system $E_D(2/3; 1/3, 1/3, 1/3; 4/3)$ in three variables, where $E_D(a; b_1, \ldots, b_n; c)$ is the system annihilating the Appell-Lauricella hypergeometric series $F_D(a; b_1, \ldots, b_n; c \mid y^1, \ldots, y^n)$:

$$\sum_{m_1=0,\cdots}^{\infty}\frac{(a,m_1+\cdots)(b_1,m_1)\cdots}{(c,m_1+\cdots)m_1!\cdots}(y^1)^{m_1}\cdots,$$

where $(a,n)=a(a+1)\cdots(a+n-1)$ (cf. [?]). Note that the integrals

$$\begin{aligned}\int & s^{b_1+b_2+b_3-c}(s-1)^{c-a-1}\\ &\times(s-y^1)^{-b_1}(s-y^2)^{-b_2}(s-y^3)^{-b_3}ds\end{aligned}$$

give solutions of the system.

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