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T.Sasaki
Department of Mathematics
Faculty of Science
Kobe University
Kobe 657 , Japan

T.Yagi
Graduate School of Science
Kobe University
Kobe 657 , Japan

Proposition 3 implies

$$\begin{aligned}
(\alpha)\qquad & \psi_{p1n}\psi_{n1n}-\psi_{p11}\psi_{nnn}=O(d^{-2}),\\
(\beta)\qquad & \psi_{p1n}\psi_{q1n}-\psi_{p11}\psi_{qnn}=O(d^{-1}),\\
(\gamma)\qquad & \psi_{p12}\psi_{p12}-\psi_{p11}\psi_{p22}=O(1),\\
(\delta)\qquad & \psi_{p12}\psi_{q12}-\psi_{p11}\psi_{q22}=O(d^{-1}).
\end{aligned}$$

Proposition 4 shows

$$\begin{aligned}
(\alpha)\qquad & h_{11}h_{nn}-h_{1n}^2=\frac{1}{4^2}d^{-3}+O(d^{-2})\text{ and }h^{pn}=O(d^2),\\
(\beta)\qquad & h_{11}h_{nn}-h_{1n}^2=\frac{1}{4^2}d^{-3}+O(d^{-2})\text{ and }h^{pq}=O(d),\\
(\gamma)\qquad & h_{11}h_{22}-h_{12}^2=\frac{1}{4^2}d^{-2}+O(d^{-1})\text{ and }h^{pn}=O(d),\\
(\delta)\qquad & h_{11}h_{22}-h_{12}^2=\frac{1}{4^2}d^{-2}+O(d^{-1})\text{ and }h^{pn}=O(d^2).
\end{aligned}$$

Since ψ is of the same order as $d(x,\partial\Omega)$, the equation (4.4) proves the result.

Remark 3. Repeating the arguments in Appendix B of [S1], we can see that (Ω,ω) and (Ω,κ) are complete Riemannian manifolds of asymptotically negative constant curvature. Roughly speaking, along any divergent geodesic $\gamma(t)$, the sectional curvature is

$$K_{ij}(\gamma(t))\sim -1+ce^{-c't},$$

for some positive constants c and c'. Therefore the integral

$$\int^{+\infty}|K_{ij}(\gamma(t))+1|dt$$

is finite. Refer to [S1] for the precise definition and the property of such a Riemannian metric.

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Propsition 3. Assume $n \geq 2$. Then the function ψ is C^1-differentiable on $\bar{\Omega}$ and the derivatives of ψ up to third order have finite continuous values on $\partial\Omega$ except ψ_{nn},ψ_{11n},ψ_{1nn} that are of order $d(x,\partial\Omega)^{-1/2}$ at most and ψ_{nnn} that is of order $d(x,\partial\Omega)^{-3/2}$ at most. If $n \geq 4$, then the function ψ is C^2-differentiable on $\bar{\Omega}$ and the derivatives of ψ up to third order have finite continuous values on $\partial\Omega$ except ψ_{nnn} that is of order $d(x,\partial\Omega)^{-1/2}$ at most. If $n \geq 6$, then the function ψ is C^3-differentiable on $\bar{\Omega}$.

Propsition 4. The fundamental tensor h_{ij} has the form

$$\frac{1}{4}\begin{pmatrix} d(x,\partial\Omega)^{-1} & & & O(d(x,\partial\Omega)^{-1}) \\ & \ddots & O(1) & \vdots \\ O(1) & & & \\ & & d(x,\partial\Omega)^{-1} & O(d(x,\partial\Omega)^{-1}) \\ O(d(x,\partial\Omega)^{-1}) & \cdots & O(d(x,\partial\Omega)^{-1}) & d(x,\partial\Omega)^{-2} \end{pmatrix},$$

whose inverse (h^{pq}) is equal to

$$4\begin{pmatrix} d(x,\partial\Omega) & & & O(d(x,\partial\Omega)^{2}) \\ & \ddots & O(d(x,\partial\Omega)^{2}) & \vdots \\ O(d(x,\partial\Omega)^{2}) & & & \\ & & d(x,\partial\Omega) & O(d(x,\partial\Omega)^{2}) \\ O(d(x,\partial\Omega)^{2}) & \cdots & O(d(x,\partial\Omega)^{2}) & d(x,\partial\Omega)^{2} \end{pmatrix}.$$

With these propositions we have the next theorem.

Theorem 4. Assume $n \geq 2$. Then the cuvature tensor of the metric κ tends to that of constant negative sectional curvature -1 at the boundary.

Proof. The sectional curvature $\tilde{K}_{ij} = \tilde{K}(\partial/\partial x^i, \partial/\partial x^j)$ is given by

$$\tilde{K}_{ij} = \frac{\tilde{R}_{ijij}}{h_{ii}h_{jj} - (h_{ij})^2}.$$

We devide the consideration into four cases:

$$\begin{aligned}
&(\alpha) \qquad (i,j) = (1,n) \text{ and } q = n, \\
&(\beta) \qquad (i,j) = (1,n) \text{ and } q \neq n, \\
&(\gamma) \qquad (i,j) = (1,2) \text{ and } p = q \neq n, \\
&(\delta) \qquad (i,j) = (1,2) \text{ and } \{p \neq q \text{ or } p = q = n\}.
\end{aligned}$$

Proof. The sectional curvature $K_{ij} = K(\partial/\partial x^i, \partial/\partial x^j)$ is given by

$$K_{ij} = \frac{R_{ijij}}{g_{ii}g_{jj} - (g_{ij})^2}.$$

Proposition 1 implies

$$\varphi_{p12}\varphi_{q12} - \varphi_{p11}\varphi_{q22} = O(1),$$
$$\varphi_{p1n}\varphi_{q1n} - \varphi_{p11}\varphi_{qnn} = O(d(x, \partial\Omega)^{-1}),$$

and Proposition 2 shows

$$g_{11}g_{22} - (g_{12})^2 = \frac{1}{4^2} d(x, \partial\Omega)^{-2} + O(d(x, \partial\Omega)^{-1}),$$
$$g_{11}g_{nn} - (g_{1n})^2 = \frac{1}{4^2} d(x, \partial\Omega)^{-3} + O(d(x, \partial\Omega)^{-2}),$$
$$g^{pq} = O(d(x, \partial\Omega)).$$

Since φ is of the same order as $d(x, \partial\Omega)$, the equation (4.1) proves the result.

Next we give a sketch of the calculation for the metric κ. Set

$$\kappa = \sum_{i,j=1}^{n} h_{ij} dx^i dx^j$$

and define the auxiliary function ψ by

$$\psi = -k^{\frac{1}{n+1}}.$$

The curvature tensor $\tilde{R}_{ijkl}$ is given by

$$\tilde{R}_{ijkl} = -(h_{il}h_{jk} - h_{ik}h_{jl}) + \sum_{p,q=1}^{n} \frac{h^{pq}}{2\psi^2}(\psi_{pil}\psi_{qjk} - \psi_{pik}\psi_{qjl}). \tag{4.4}$$

By definition we have

$$\tilde{c}_i(\beta'; \beta_n + 1) = \frac{2n + |\beta'| + 2\beta_n + 2 - i}{2} \tilde{c}_i(\beta'; \beta_n), \tag{4.5}$$

(4.6) $\tilde{c}_0(\beta_1, \ldots, \beta_i + 2, \ldots, \beta_{n-1}; \beta_n) =$

$$\frac{(2n + |\beta'| + 2\beta_n + 2)(\beta_i + 1)}{4} \tilde{c}_0(\beta_1, \ldots, \beta_i, \ldots, \beta_{n-1}; \beta_n).$$

By a similar calculation, we have the following propositions.

Propsition 2. The fundamental tensor g_{ij} has the form

$$\frac{1}{4}\begin{pmatrix} d(x,\partial\Omega)^{-1} & & O(1) & O(d(x,\partial\Omega)^{-1}) \\ & \ddots & & \vdots \\ O(1) & & & \\ & & d(x,\partial\Omega)^{-1} & O(d(x,\partial\Omega)^{-1}) \\ O(d(x,\partial\Omega)^{-1}) & \cdots & O(d(x,\partial\Omega)^{-1}) & d(x,\partial\Omega)^{-2} \end{pmatrix},$$

whose the inverse (g^{pq}) is equal to

$$4\begin{pmatrix} d(x,\partial\Omega) & & O(d(x,\partial\Omega)^{2}) & O(d(x,\partial\Omega)^{2}) \\ & \ddots & & \vdots \\ O(d(x,\partial\Omega)^{2}) & & & \\ & & d(x,\partial\Omega) & O(d(x,\partial\Omega)^{2}) \\ O(d(x,\partial\Omega)^{2}) & \cdots & O(d(x,\partial\Omega)^{2}) & d(x,\partial\Omega)^{2} \end{pmatrix}.$$

Proof. For simplicity we set $d = d(x,\partial\Omega)$. Since

$$g_{ij} = \frac{1}{n+1}\left\{\frac{\chi_{ij}}{\chi} - \frac{n+2}{n+1}\frac{\chi_i}{\chi}\frac{\chi_j}{\chi}\right\},$$

Lemma 3 imlies

$$\begin{aligned} g_{11} &= \tfrac{1}{4}d^{-1} + O(1), & g_{12} &= O(1), \\ g_{1n} &= -\tfrac{1}{2(n+1)}c_1 d^{-1} + O(1), & g_{nn} &= \tfrac{1}{4}d^{-2} + O(d^{-1}). \end{aligned}$$

From the fact that $\det(g_{ij}) = 4^{-n}d^{-n-1}$, we have

$$\begin{aligned} g^{11} &= (4^{n-1}d^{n})(4^{n}d^{n+1}) = 4d, \\ g^{12} &= (4^{n-1}O(d^{n-1}))(4^{n}d^{n+1}) = O(d^2), \\ g^{1n} &= (4^{n-1}O(d^{n-1}))(4^{n}d^{n+1}) = O(d^2), \\ g^{nn} &= (4^{n-1}d^{n-1})(4^{n}d^{n+1}) = 4d^2. \end{aligned}$$

This proves the proposition.

With these propositions we can conclude the next theorem.

Theorem 3. Assume $n \geq 2$. Then the cuvature tensor of the metric ω tends to that of constant negative sectional curvature -1 at the boundary.

Lemma 3. Assume $n \geq 2$. Then we have

$$\frac{\chi_1}{\chi} = c_1 + O(d), \qquad \frac{\chi_n}{\chi} = \frac{n+1}{2} d^{-1} + O(1),$$
$$\frac{\chi_{11}}{\chi} = \frac{n+1}{4} d^{-1} + O(1), \qquad \frac{\chi_{12}}{\chi} = O(1),$$
$$\frac{\chi_{1n}}{\chi} = \frac{n+1}{2} c_1 d^{-1} + O(1), \qquad \frac{\chi_{nn}}{\chi} = \frac{(n+1)(n+3)}{4} d^{-2} + O(d^{-1}),$$
$$\frac{\chi_{111}}{\chi} = O(d^{-1}), \qquad \frac{\chi_{112}}{\chi} = O(d^{-1}),$$
$$\frac{\chi_{123}}{\chi} = O(d^{-1}), \qquad \frac{\chi_{11n}}{\chi} = \frac{(n+1)(n+3)}{8} d^{-2} + O(d^{-1}),$$
$$\frac{\chi_{12n}}{\chi} = O(d^{-1}), \qquad \frac{\chi_{1nn}}{\chi} = \frac{(n+1)(n+3)}{4} c_1 d^{-2} + O(d^{-1}),$$
$$\frac{\chi_{nnn}}{\chi} = \frac{(n+1)(n+3)(n+5)}{8} d^{-3} + O(d^{-2}).$$

When $n \geq 4$, we have estimates of higher order:

$$\frac{\chi_n}{\chi} = \frac{n+1}{2} d^{-1} - c_2 + O(d)$$
$$\frac{\chi_{nn}}{\chi} = \frac{(n+1)(n+3)}{4} d^{-2} - (n+1) c_2 d^{-1} + O(1)$$
$$\frac{\chi_{nnn}}{\chi} = \frac{(n+1)(n+3)(n+5)}{8} d^{-3} - \frac{3}{4}(n+1)(n+3) c_2 d^{-2} + O(d^{-1}).$$

These estimates imply the boundary regularity of the function φ as follows.

Propsition 1. Assume $n \geq 2$. Then the function φ is C^2-differentiable on $\bar{\Omega}$ and the derivatives of φ up to third order have finite continuous values on $\partial\Omega$ except φ_{nnn} that is of order $d(x, \partial\Omega)^{-1}$ at most. If $n \geq 4$, then the function φ is C^3-differentiable on $\bar{\Omega}$.

Proof. By definition of φ, we see that

$$-\frac{(n+1)\varphi_i}{2\varphi} = \frac{\chi_i}{\chi},$$
$$-\frac{(n+1)\varphi_{ij}}{2\varphi} = \frac{\chi_{ij}}{\chi} - \frac{n+3}{n+1}\frac{\chi_i}{\chi}\frac{\chi_j}{\chi},$$
$$-\frac{(n+1)\varphi_{ijk}}{2\varphi} = \frac{\chi_{ijk}}{\chi} - \frac{n+3}{n+1}\left(\frac{\chi_{ij}}{\chi}\frac{\chi_k}{\chi} + \frac{\chi_{jk}}{\chi}\frac{\chi_i}{\chi} + \frac{\chi_{ik}}{\chi}\frac{\chi_j}{\chi}\right) + \frac{2(n+2)(n+3)}{(n+1)^2}\frac{\chi_i}{\chi}\frac{\chi_j}{\chi}\frac{\chi_k}{\chi}.$$

Since φ has the same order as $d(x, \partial\Omega)$, we have only to see that the right hand sides are at most of order $d(x, \partial\Omega)^{-1}$ when $n \geq 4$, and that they are at most of order $d(x, \partial\Omega)^{-1}$ except φ_{nnn} when $n \geq 2$. We can examine this fact by making use of Lemma 3. The values at the boundary are dependent only on y and define smooth functions on $\partial\Omega$. (see Proof of Theorem 1.)

Now the boundary estimates of (g_{ij}) and the inverse (g^{pq}) follow from Proposition 1:

Moreover, we need not to make any distinction among the first $(n-1)$ components; we use the abbreviation $(p;q)$ for denoting $\beta = (p,0,\ldots,0;q)$. By Lemma 1 we see that

$$\begin{aligned}
c_0(0;1) &= \tfrac{n+1}{2}c_0(0), & c_0(0;2) &= \tfrac{(n+1)(n+3)}{4}c_0(0),\\
c_0(0;3) &= \tfrac{(n+1)(n+3)(n+5)}{8}c_0(0), & c_0(2;0) &= \tfrac{n+1}{4}c_0(0),\\
c_0(2;1) &= \tfrac{(n+1)(n+3)}{8}c_0(0), & c_1(1;1) &= \tfrac{n+1}{2}c_1(1;0),\\
c_1(1;2) &= \tfrac{(n+1)(n+3)}{4}c_1(1;0), & c_2(0;1) &= \tfrac{n-1}{2}c_2(0),\\
c_2(0;2) &= \tfrac{(n-1)(n+1)}{4}c_2(0), & c_2(0;3) &= \tfrac{(n-1)(n+1)(n+3)}{8}c_2(0).
\end{aligned}$$

On the other hand, making use of the expansion in Lemma 1, we get the following expansions:

$$\chi = d^{-\frac{n+1}{2}}\big\{c_0(0) + O(d)\big\},$$

$$\begin{aligned}
\chi_1 &= d^{-\frac{n+1}{2}}\big\{c_1(1;0)+O(d)\big\}, & \chi_n &= d^{-\frac{n+3}{2}}\big\{c_0(0;1)+c_2(0;1)d+O(d^2)\big\},\\
\chi_{11} &= d^{-\frac{n+3}{2}}\big\{c_0(2;0)+c_2(2;0)d+O(d^2)\big\}, & \chi_{12} &= O(d^{-\frac{n+1}{2}}),\\
\chi_{1n} &= d^{-\frac{n+3}{2}}\big\{c_1(1;1)+O(d)\big\}, & \chi_{nn} &= d^{-\frac{n+5}{2}}\big\{c_0(0;2)+c_2(0;2)d+O(d^2)\big\},\\
\chi_{111} &= O(d^{-\frac{n+3}{2}}), & \chi_{112} &= O(d^{-\frac{n+3}{2}}),\\
\chi_{123} &= O(d^{-\frac{n+3}{2}}), & \chi_{11n} &= d^{-\frac{n+5}{2}}\big\{c_0(2;1)+O(d)\big\},\\
\chi_{12n} &= O(d^{-\frac{n+3}{2}}), & \chi_{1nn} &= d^{-\frac{n+5}{2}}\big\{c_1(1;2)+O(d)\big\},
\end{aligned}$$

$$\chi_{nnn} = d^{-\frac{n+7}{2}}\big\{c_0(0;3)+c_2(0;3)d+O(d^2)\big\}.$$

The χ_i for $2 \le i \le n-1$ is not listed because it has the same exapnsion as χ_1; similarly for χ_{ij}, χ_{in}, and so on. When $n \ge 4$, we have a more precise estimate:

$$\chi = d^{-\frac{n+1}{2}}\big\{c_0(0)+c_2(0)d+O(d^2)\big\}.$$

Hence, by setting

$$c_1 = \frac{c_1(1;0)}{c_0(0)}, \qquad c_2 = \frac{c_2(0)}{c_0(0)},$$

we get the following lemma.

Remark 2. The coefficient $\tilde{c}_i(\beta)$ is a smooth function on the closure of Ω and its value at the boundary is determined by the local geometric data of the boundary in the sense stated in Remark 1.

§4. Sectional curvature

We have defined two metrics ω and κ in Introduction. The aim of this section is to study the boundary behavior of the curvature tensor of these metrics. We carry out the calculation for the case ω. Set

$$\omega = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j$$

and define an auxiliary function φ by

$$\varphi = -\chi^{\frac{2}{n+1}}.$$

Then

$$\omega = -\frac{1}{\sqrt{-\varphi}} d^2\left(\sqrt{-\varphi}\right), \; g_{ij} = \frac{1}{2}\left(-\frac{\varphi_{ij}}{\varphi} + \frac{\varphi_i\varphi_j}{2\varphi^2}\right).$$

The curvature tensor R_{ijkl} of the metric of this form is given by

$$R_{ijkl} = -(g_{il}g_{jk} - g_{ik}g_{jl}) + \sum_{p,q=1}^{n} \frac{g^{pq}}{2\varphi^2}(\varphi_{pil}\varphi_{qjk} - \varphi_{pik}\varphi_{qjl}); \tag{4.1}$$

see [S1]. Here the matrix (g^{pq}) is the inverse of the matrix (g_{ij}) and φ_{ijk} denote the third derivatives of φ. So it is necessary to obtain the boundary estimate of derivatives of φ. For this purpose, the explicit calculation of the second term of the expansion of the χ_β is necessary. Fixing a boundary point y and choosing coordinates as before, we continue the calculation in § 2.

We write

$$\chi_i = \frac{\partial\chi(x)}{\partial x^i}, \quad \chi_{ij} = \frac{\partial^2\chi(x)}{\partial x^i\partial x^j}, \quad \chi_{ijk} = \frac{\partial^3\chi(x)}{\partial x^i\partial x^j\partial x^k}, \text{ and } \; d = d(x, \partial\Omega),$$

and assume that $n \geq 2$. In the following estimate, the first $(n-1)$ components of β and the last component β_n play the different roles; we write $\beta = (\beta'; \beta_n)$. By (2.11) we have

$$c_i(\beta'; \beta_n + 1) = \frac{n + |\beta'| + 2\beta_n + 1 - i}{2} c_i(\beta'; \beta_n), \tag{4.2}$$

$$c_0(\beta_1, \ldots, \beta_i + 2, \ldots, \beta_{n-1}; \beta_n) = \frac{(n + |\beta'| + 2\beta_n + 1)(\beta_i + 1)}{4} c_0(\beta_1, \ldots, \beta_i, \ldots, \beta_{n-1}; \beta_n). \tag{4.3}$$

where $a_{k,m} = \sum_{p+q=k} \lambda_{q,m}(\Theta)\mu_p$ whose degree modulo 2 is equal to k and $a_{0,0} = 1$.

Furthermore, we can write

$$B_{\beta'}(t) = \sum_{i\geq 0} t^{\frac{2n+|\beta'|+i}{2}} \int \lambda_i f'^{\beta'} d\Theta + O(t^{N_2}),$$

where

$$\lambda_i = \frac{1}{2c_0(0)} \sum_{k+2m=i} a_{k,m} B\left(\frac{n+3}{2} + m, \frac{n+|\beta'|-1+k}{2}\right).$$

The summation ranges up to certain finite number so that the remaining term is of order N_2 with respect to t. Note that the degree of λ_i modulo 2 is equal to i and $\lambda_0 = 1$.

Now the argument in §2 can be repeated to yield the next lemma.

Lemma 2. The derivatives of the kernel function have the following expansions on Ω.

$$k_\beta(x) = \sum_{k=0}^{2N_2-1} \tilde{c}_k(\beta) d(x,\partial\Omega)^{-\frac{2n+|\beta'|+2\beta_n+2}{2}+\frac{k}{2}} + O(d(x,\partial\Omega)^{-\frac{2n+|\beta'|+2\beta_n+2}{2}+N_2}),$$

where

$$\begin{aligned}\tilde{c}_{2p+q} =& (-1)^{|\beta'|} \frac{\Gamma\left(\frac{2n+|\beta'|+2}{2} + \frac{2p+q}{2}\right)\Gamma\left(\frac{2n+|\beta'|+2}{2} + \beta_n - \frac{2p+q}{2}\right)}{\Gamma\left(\frac{2n+|\beta'|+2}{2} - \frac{2p+q}{2}\right)} \\ &\times \sum_{j=0}^{p} \frac{\Gamma\left(\frac{2n+|\beta'|+2}{2} - \frac{2j+q}{2}\right)}{\Gamma(p-j+1)} \int \lambda_{2j+q} f'^{\beta'} d\Theta.\end{aligned}$$

Note that q takes the value 0 or 1.

A similar argument depending on the parity of β as in §2 implies the following theorem.

Theorem 2. Assume that the domain is strictly convex and the boundary is smooth. Then the derivatives k_β of the kernel function have the following expansions according to the parity of the index β that is defined in §2.

$$\text{case }(a)\quad k_\beta(x) = d(x,\partial\Omega)^{-\frac{2n+|\beta'|+2\beta_n+2}{2}} \left(\sum_{i=0}^{N_2-1} \tilde{c}_{2i}(\beta) d(x,\partial\Omega)^i + O(d(x,\partial\Omega)^{N_2})\right),$$

$$\text{case }(b)\quad k_\beta(x) = d(x,\partial\Omega)^{-\frac{2n+|\beta'|+2\beta_n+1}{2}} \left(\sum_{i=0}^{N_2-1} \tilde{c}_{2i+1}(\beta) d(x,\partial\Omega)^i + O(d(x,\partial\Omega)^{N_2-\frac{1}{2}})\right),$$

$$\text{case }(c)\quad k_\beta(x) = d(x,\partial\Omega)^{-\frac{2n+|\beta'|+2\beta_n}{2}} \left(\sum_{i=0}^{N_2-2} \tilde{c}_{2i+2}(\beta) d(x,\partial\Omega)^i + O(d(x,\partial\Omega)^{N_2-1})\right).$$

Let us put

$$P_{q,m} = \sum_{|\alpha|=q} a^6_{\alpha,m} \zeta'^{\alpha},$$

where $P_{q,m}(\zeta')$ is a homogeneous polynomial of ζ' of degree q and $P_{0,0} = 1$. Then the estimate of $B_{\beta'}(t)$ is obtained by computing the integral

$$\frac{1}{c_0(0)} \int_{\Omega^* \cap \{0 \le l(\zeta') \le t\}} \zeta'^{\beta'} P_{q,m}(\zeta') |t - \ell(\zeta')|^{m+\frac{n+1}{2}} d\zeta'.$$

Relative to the polar coordinates $\zeta_i = r f_i(\Theta)$ we have

$$\ell(\zeta') = r^2 \Big(1 + \sum_{p \ge 1} \epsilon_p r^p + O(r^{2N_0})\Big),$$

as we did before. Note that N_0 is a sufficiently large integer greater than N_2. Further we set

$$\ell(\zeta') = tu^2,$$

introducing a new variable u. Then we have

$$r = \sqrt{t}u \Big(1 + \sum_{p \ge 1} \nu_p (\sqrt{t}u)^p + O((\sqrt{t}u)^{N_0})\Big),$$

where ν_p is determined by ϵ_p; its degree with respect to f_i is $p+2$. From the fact that

$$dr = \sqrt{t}\Big(1 + \sum_{p \ge 1} (p+1)\nu_p (\sqrt{t}u)^p + O((\sqrt{t}u)^{N_0})\Big) du,$$

we can get

$$r^{q+|\beta'|+n-2} dr = \Big(\sqrt{t}u\Big)^{q+|\beta'|+n-2} \sqrt{t}\Big(1 + \sum_{p \ge 1} (p+1)\mu_p (\sqrt{t}u)^p + O((\sqrt{t}u)^{N_0})\Big) du,$$

where the degree of μ_p with respect to f_i is also $p+2$. With this notation, the integral is equal to

$$\sum_{p \ge 1} \frac{1}{2c_0(0)} B\left(\frac{n+3}{2} + m, \frac{n + |\beta'| - 1 + q + p}{2}\right) t^{\frac{2n+|\beta'|+q+p+2m}{2}} \int \lambda_{q,m}(\Theta) \mu_p f'^{\beta'} d\Theta,$$

where $\lambda_{q,m}(\Theta) = P_{q,m}((f'(\Theta))$. Hence the summation implies

$$B_{\beta'}(t) = \frac{1}{2c_0(0)} \sum_{k,m} \int a_{k,m} f'^{\beta'} B\left(\frac{n+3}{2} + m, \frac{n + |\beta'| - 1 + k}{2}\right) t^{\frac{2n+|\beta'|+k+2m}{2}} d\Theta + O(t^{N_2}),$$

§3. The boundary estimates of the derivatives of the function k_Ω

We can obtain similarly the estimate of k_β as follows. Fix $y \in \Omega$ and choose the coordinates as before. For a point $x = ky$, we have

$$\begin{aligned} k_\beta(x) &= \int_{\Omega^*} (-1)^{|\beta|}(2n+|\beta|+1)!\, \xi'^{\beta'}(\ell-1)^{\beta_n}(k\ell+(1-k))^{-2n-2-|\beta|}\chi_{\Omega^*}(\xi)^{-1}d\xi' d\ell \\ &= \int_0^b (-1)^{|\beta|}(2n+|\beta|+1)!(kt+(1-k))^{-2n-|\beta|-2}(t-1)^{\beta_n}B_{\beta'}(t)dt, \end{aligned}$$

where

$$B_{\beta'}(t) = \int_{\Omega^*\cap\{\ell=t\}} \xi'^{\beta'}\chi_{\Omega^*}(\xi)^{-1}d\xi'.$$

Choose ξ with $t = \ell(\xi)$ sufficiently small so that $d(\xi, \partial\Omega^*)$ is attained by a unique point $\zeta \in \partial\Omega^*$. The point $\zeta = (\zeta', \zeta_n)$ is given by

$$\zeta_i = \xi_i + (t-\ell(\zeta'))\frac{\partial \ell}{\partial \xi_i}(\zeta'), \quad \zeta_n = \ell(\xi').$$

Then, by a simple argument, we get the following approximate identities:

$$\begin{aligned} |\xi-\zeta| &= |t-\ell(\zeta')|(1+\sum_{|\alpha|\geq 2} a^1_\alpha \zeta'^\alpha + O(|\zeta'|^{2N_1})), \\ d\xi' &= d\zeta'(1+\sum_{|\alpha|,m\geq 1} a^2_{\alpha,m}\zeta'^\alpha|t-\ell(\zeta')|^m + O(|\zeta'|^{2N_1}) + O(|t-\ell(\zeta')|^{N_1})), \\ \xi'^{\beta'} &= \zeta'^{\beta'}(1+\sum_{m\geq 1} a^3_m|t-\ell(\zeta')|^m + O(|t-\ell(\zeta')|^{N_1})), \\ \gamma(\zeta) &= \gamma(0) + \sum_{|\alpha|\geq 1} a^4_\alpha\zeta'^\alpha + O(|\zeta'|^{2N_1}), \end{aligned}$$

where $\gamma(\zeta)$ denotes the Gauss curvature of $\partial\Omega^*$ at ζ and N_1 is a sufficiently large integer greater than the following integer N_2. The characteristic function $\chi(\xi)$ has the asymptotic expansion:

$$\frac{2^{\frac{n-1}{2}}c_0(0)}{\sqrt{\gamma(\zeta)}}d(\xi,\zeta)^{-\frac{n+1}{2}}(1+\sum_{m\geq 1} a^5_m d(\xi,\zeta)^m + O(d(\xi,\zeta)^{N_2})),$$

where $c_0(0) = \Gamma(\frac{1}{2})^{n-1}\Gamma(\frac{n+1}{2})$ and $N_2 = [\frac{n}{2}]$; see [S2]. Since $\gamma(0) = 2^{n-1}$ by (2.0), we can see that $\xi'^{\beta'}\chi(\xi)^{-1}d\xi'$ is equal to

$$\frac{1}{c_0(0)}\zeta'^{\beta'}|t-\ell(\zeta')|^{\frac{n+1}{2}}(1+\sum_{|\alpha|,m\geq 1} a^6_{\alpha,m}\zeta'^\alpha|t-\ell(\zeta')|^m + O(|\zeta'|^{2N_2}) + O(|t-\ell(\zeta')|^{N_2}))d\zeta'.$$

Therefore, by (2.11) we see that

$$
\begin{aligned}
&\text{case } (a) \quad c_0(\beta) = \Gamma\left(\frac{n+|\beta'|+1}{2}\right)^{-1} \times \prod_{i=1}^{n-1} \Gamma\left(\frac{\beta_i+1}{2}\right),\ c_{2i+1}(\beta) = 0,\\
&\text{case } (b) \quad c_{2i}(\beta) = 0,\\
&\text{case } (c) \quad c_0(\beta) = 0,\ c_{2i+1}(\beta) = 0.
\end{aligned}
$$

We have proved the following theorem.

Theorem 1. Assume that the domain is strictly convex and the boundary is smooth. Then the derivatives χ_β of the characteristic function have the following expansions near the boundary according to the parity of the index β defined above.

$$
\begin{aligned}
&\text{case } (a) \quad \chi_\beta(x) = d(x,\partial\Omega)^{-\frac{n+|\beta'|+2\beta_n+1}{2}} \sum_{i=0}^{\gamma(\beta,n)} c_{2i}(\beta) d(x,\partial\Omega)^i + E_{\beta,n}(d(x,\partial\Omega)),\\
&\text{case } (b) \quad \chi_\beta(x) = d(x,\partial\Omega)^{-\frac{n+|\beta'|+2\beta_n}{2}} \sum_{i=0}^{\gamma(\beta,n)} c_{2i+1}(\beta) d(x,\partial\Omega)^i + E_{\beta,n}(d(x,\partial\Omega)),\\
&\text{case } (c) \quad \chi_\beta(x) = d(x,\partial\Omega)^{-\frac{n+|\beta'|+2\beta_n-1}{2}} \sum_{i=0}^{\gamma(\beta,n)-1} c_{2i+2}(\beta) d(x,\partial\Omega)^i + E_{\beta,n}(d(x,\partial\Omega)).
\end{aligned}
$$

The constant $\gamma(\beta,n)$ and the function $E_{\beta,n}(d(x,\partial\Omega))$ is given by

$$
\gamma(\beta,n) = \left[\frac{n+|\beta'|}{2}\right] + \beta_n
$$

$$
E_{\beta,n}(d(x,\partial\Omega)) = \begin{cases} O(1) & \text{if } n+|\beta'| \text{ is even,}\\ -b_{n+|\beta'|+2\beta_n+1}(\beta)\log d(x,\partial\Omega) + O(1) & \text{if } n+|\beta'| \text{ is odd.} \end{cases}
$$

Remark 1. Each coefficient $c_p(\beta)$ is a smooth function around the boundary and its value at the boundary is a polynomial of the coefficients a_α given in (2.2); hence, by the star mapping, $c_p(\beta)$ can be written as a polynomial of the coefficients b_α when the boundary is written as the graph

$$
y = 1 - \sum_{i=1}^{n-1}(x^i)^2 + \sum_{|\alpha|\geq 3} b_\alpha x^\alpha + O(|x|^{2N_0}).
$$

As for the number N_0 given in (2.2), which is not fixed yet explicitly, it is enough to take $\gamma(\beta,n)+1$ for each β.

Lemma 1. The derivative $\chi_\beta(x)$ has the following expansion on Ω.

$$\chi_\beta(x) = \sum_{k=0}^{n+|\beta'|+2\beta_n} c_k(\beta) d(x,\partial\Omega)^{-\frac{n+|\beta'|+2\beta_n+1}{2}+\frac{k}{2}} - b_{n+|\beta'|+2\beta_n+1}(\beta)\log d(x,\partial\Omega) + O(1), \tag{2.9}$$

where $c_k(\beta)$ is given by

$$c_k(\beta) = \sum_{i+2j=k} b_i(\beta) e_{i,j} B(M_i+1, N-M_i). \tag{2.10}$$

By a simple calculation, we see

$$\begin{aligned} c_{2p+q} =& (-1)^{|\beta'|} \frac{\Gamma\left(\frac{n+|\beta'|+1}{2}+\frac{2p+q}{2}\right)\Gamma\left(\frac{n+|\beta'|+1}{2}+\beta_n-\frac{2p+q}{2}\right)}{\Gamma\left(\frac{n+|\beta'|+1}{2}-\frac{2p+q}{2}\right)} \\ &\times \sum_{j=0}^{p} \frac{\Gamma\left(\frac{n+|\beta'|+1}{2}-\frac{2j+q}{2}\right)}{\Gamma(p-j+1)} \Delta_{2j+q}, \end{aligned} \tag{2.11}$$

where q takes the value 0 or 1.

Let us compute the coefficients $c_p(\beta)$ more concretely. Note that the value of Δ_p depends on the parity of β_i because of the integral formula:

$$\int \prod_{i=1}^{n-1} (f_i')^{k_i} d\Theta = \begin{cases} \frac{n+|k'|-1}{\Gamma(\frac{n+|k'|+1}{2})} \prod_{i=1}^{n-1} \Gamma(\frac{k_i+1}{2}) & \text{when } k_i, 1 \le i \le n-1, \text{ are even;} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we devide the consideration into three cases:

(a) every β_i $(i=1,\dots,n-1)$ is even,

(b) at least one of β_i $(i=1,\dots,n-1)$ is odd, and $|\beta'|$ is odd,

(c) at least one of β_i $(i=1,\dots,n-1)$ is odd, and $|\beta'|$ is even.

Then we get

case (a) $\Delta_0 = \Gamma\left(\frac{n+|\beta'|+1}{2}\right)^{-1} \times \prod_{i=1}^{n-1} \Gamma\left(\frac{\beta_i+1}{2}\right)$, $\Delta_{2i+1} = 0$,

case (b) $\Delta_{2i} = 0$,

case (c) $\Delta_0 = 0$, $\Delta_{2i+1} = 0$.

and set

$$f'^{\beta'} = \prod_{i=1}^{n-1} (f_i)^{\beta_i}.$$

Then

$$\begin{aligned} A_{\beta'}(t) &= \int d\Theta \int_0^{r(\Theta)} r^{|\beta'|} f'^{\beta'} r^{n-2} dr \\ &= \sum_{p \geq 0} t^{\frac{n+|\beta'|-1+p}{2}} \int \frac{\delta_p f'^{\beta'}}{n+|\beta'|-1} d\Theta + O(t^{N_0}). \end{aligned}$$

Note that $\delta_0 = 1$ and δ_p $(p \geq 1)$ is a polynomial of f_i and its degree is $p+2$. We set

(2.6)
$$\Delta_p = \int \frac{\delta_p f'^{\beta'}}{n+|\beta'|-1} d\Theta$$

and

(2.7)
$$b_i(\beta) = (-1)^{|\beta'|}(n+|\beta|)! \sum_{p+2j=i} (-1)^j \binom{\beta_n}{j} \Delta_p.$$

Then

$$\chi_\beta(x) = \sum_{i \geq 0} b_i(\beta) \int_0^b (kt+(1-k))^{-N-1} t^{M_i} dt + \int_0^b (kt+(1-k))^{-N-1} O(t^{\frac{n+|\beta'|-1+N_0}{2}}) dt,$$

where

$$N = n + |\beta|, \quad M_i = \frac{n+|\beta'|-1+i}{2}.$$

Because of $d(x, \partial\Omega) = d(x,y) = 1-k$, a simple calculation shows

$$\begin{aligned} &\int_0^b (kt+(1-k))^{-N-1} t^{M_i} dt \\ &= \begin{cases} B(M_i+1, N-M_i) \sum_{j=0}^{\gamma_i} e_{i,j} d(x,\partial\Omega)^{M_i-N+j} + O(1) & \text{if } N > M_i, \\ -\log d(x,\partial\Omega) + O(1) & \text{if } N = M_i, \\ O(1) & \text{if } N < M_i, \end{cases} \end{aligned}$$

where $B(p,q)$ denotes the beta function and

(2.8)
$$e_{i,j} = \begin{cases} 1 & (j=0) \\ \frac{1}{j!} \prod_{k=1}^{j} (M_i+k) & (j \neq 0) \end{cases}, \qquad \gamma_i = [N - M_i - 1].$$

Hence we have the following lemma.

We use notations $x' = (x^1, \ldots, x^{n-1})$, $\xi' = (\xi_1, \ldots, \xi_{n-1})$, and $\beta' = (\beta_1, \ldots, \beta_{n-1})$. Set $\ell(\xi) = \ell_y(\xi) = 1 + \langle y, \xi \rangle = 1 + \xi_n$ and we use the coordinates (ξ', t) instead of ξ where $t = 1 + \xi_n = \ell(\xi)$. Let $x = (0, \ldots, 0, k) = ky$, so that $\ell_x(\xi) = k\ell(\xi) + (1-k)$. Then

$$\chi_\beta(x) = \int_0^b (-1)^{|\beta|}(n+|\beta|)!(t-1)^{\beta_n}(kt + (1-k))^{-n-1-|\beta|} A_{\beta'}(t)dt,$$

where $b = \max_{\xi \in \Omega^*}(\ell(\xi))$ and

$$A_{\beta'}(t) = \int_{\Omega^* \cap \{\ell = t\}} \xi'^{\beta'} d\xi'.$$

We expand ℓ into a Taylor series

$$\ell = \sum_{i=1}^{n-1} (\xi_i)^2 + \sum_{|\alpha| \geq 3} a_\alpha \xi^\alpha + O(|\xi|^{2N_0}). \tag{2.1}$$

Here N_0 is a sufficiently large integer which may depend on β'; it will be specified later. Let $(r, \Theta) = (r, \theta_1, \ldots, \theta_{n-2})$ be the polar coordinate system for ξ':

$$\xi_i = r f_i(\Theta) \qquad 1 \leq i \leq n-1, \tag{2.2}$$

where $f_i(\Theta)$ is specified as follows:

$$\begin{aligned} f_1(\Theta) &= \cos\theta_1, \\ f_i(\Theta) &= \left(\prod_{j=1}^{i-1} \sin\theta_j\right) \cos\theta_i \qquad (i = 2, \ldots, n-2), \\ f_{n-1}(\Theta) &= \prod_{j=1}^{n-2} \sin\theta_j. \end{aligned} \tag{2.3}$$

Then the boundary surface $\partial\Omega^* \cap \{\ell = t\}$ can be written as

$$r = r(\Theta) = t^{\frac{1}{2}}\Big(1 + \sum_{p \geq 1} \sigma_p t^{\frac{p}{2}} + O(t^{N_0})\Big), \tag{2.4}$$

where the coefficient σ_p is a polynomial of f_i and a_α; the degree relative to f_i is $p+2$. Define δ_p by

$$r(\Theta)^{n+|\beta'|-1} = t^{\frac{n+|\beta'|-1}{2}}\Big(\sum_{p \geq 0} \delta_p t^{\frac{p}{2}} + O(t^{N_0})\Big) \tag{2.5}$$

We remark that each of the metrics is identical with the Hilbert metric when the domain is a ball. Refer to [S2] for details.

In §2 and §3 we study the boundary behavior of derivatives of the functions χ_Ω and k_Ω. The calculation of the curvature tensor in §4 is done similarly to that used in [K], [S1].

§2. Boundary estimates of derivatives of the function χ_Ω

We fix once and for all a euclidean structure on $\mathbf{R}^n$. Let $\{x^1, \dots, x^n\}$ be a coordinate system of $\mathbf{R}^n$ and $\{\xi_1, \dots, \xi_n\}$ the dual coordinate system of $\mathbf{R}_n$ so that $\langle \xi, x\rangle = \sum \xi_i x^i$. Set $\ell_x(\xi) = 1 + \langle \xi, x\rangle$. For the multiindex $\beta = (\beta_1, \dots, \beta_n)$, we write

$$\chi_\beta = \frac{\partial^{|\beta|}}{\partial x^\beta}\chi_\Omega(x), \quad k_\beta = \frac{\partial^{|\beta|}}{\partial x^\beta}k_\Omega(x), \quad \text{and} \quad \xi^\beta = \prod_{i=1}^n \xi_i^{\beta_i}.$$

Then

$$\chi_\beta(x) = \int_{\Omega^*} (-1)^{|\beta|}\,(n+|\beta|)!\,\xi^\beta\, l_x(\xi)^{-n-|\beta|-1}\,d\xi,$$
$$k_\beta(x) = \int_{\Omega^*} (-1)^{|\beta|}\,(2n+|\beta|+1)!\,\xi^\beta\, l_x(\xi)^{-2n-|\beta|-2}\,\chi_{\Omega^*}(\xi)^{-1}\,d\xi.$$

We will calculate the boundary asymptotics of χ_β and k_β. For this purpose we choose a special coordinate system as follows. Assume that the domain Ω is bounded and strictly convex in $\mathbf{R}^n$ and that the boundary $\partial\Omega$ is smooth. In terms of the function $H_\Omega(\xi) = \sup\{\langle \xi, x\rangle; x \in \Omega\}$, called the support function, the dual domain is written as $\Omega^* = \{\xi \in \mathbf{R}_n; H_\Omega(-\xi) < 1\}$. Since Ω is strictly convex, there exists for any ξ a unique point $y(\xi) \in \partial\Omega$ with the property $H_\Omega(\xi) = \langle \xi, y(\xi)\rangle$. Thus we get the correspondence from $\partial\Omega^*$ to $\partial\Omega$ sending ξ to $y(\xi)$. This is the inverse of the star mapping from $\partial\Omega$ to $\partial\Omega^*$; recall that the star mapping is defined on Ω as the mapping sending $x \in \Omega$ to

$$x^* = -\mathrm{grad}\chi_\Omega(x)((n+1)\chi_\Omega(x) + \langle \mathrm{grad}\chi_\Omega(x), x\rangle)^{-1} \in \Omega^*$$

and that this mapping is smoothly extended to the closure $\overline{\Omega}$.

Fix a point $y \in \partial\Omega$ and choose coordinates $(x^1, \dots, x^n)$ so that $y = (0, \dots, 0, 1)$ and $d(ky, \partial\Omega) = d(ky, y) = 1-k$ for $k \in (0,1)$ sufficiently near 1. Let $\eta = y^*$ denote the image of y by the star mapping and choose coordinates $(\xi_1, \dots, \xi_n)$ so that $\eta = (0, \dots, 0, 1)$ and the boundary $\partial\Omega^*$ around η is written as

$$\xi_n = -1 + \sum_{i=0}^{n-1}(\xi_i)^2 + O(|\xi'|^3). \tag{2.0}$$

Sectional curvature of projective invariant metrics on a strictly convex domain

Takeshi Sasaki and Takeshi Yagi

§1. Introduction

In the paper [S2] we defined two projectively invariant metrics on a convex domain. These metrics bear a strong resemblance to the so-called Blaschke metric that is realized as the affine metric on a hyperbolic affine hypersphere (refer to [C]); in fact, they coincide with the Blaschke metric when the domain is projectively homogeneous. In this paper we deal with the case where the domain is bounded and strictly convex and the boundary is smooth and show that the sectional curvatures of both metrics tend to -1 at the boundary, which is compared with the fact that the sectional curvature of the Blaschke metric on the ball is -1 everywhere.

Let us recall the definition of the metrics. We consider an open convex domain Ω in the affine space $\mathbf{R}^n$ and assume that it contains no straight line. Let $\mathbf{R}_n$ denote the dual affine space. Then the dual domain Ω^* is defined as the interior of the set $\{\xi \in \mathbf{R}_n; 1 + \langle \xi, x\rangle \geq 0 \quad \text{for any} \quad x \in \Omega\}$. The dual of the dual domain Ω^* is projectively equivalent to Ω. Then the characteristic function of Ω is by definition

$$\chi_\Omega(x) = \int_{\Omega^*} n!(1 + \langle \xi, x\rangle)^{-n-1} d\xi \tag{1.1}$$

and the kernel function of Ω is

$$k_\Omega(x) = \int_{\Omega^*} (2n+1)!(1 + \langle \xi, x\rangle)^{-2(n+1)} \chi_{\Omega^*}(\xi)^{-1} d\xi. \tag{1.2}$$

This definition of the characteristic function is a generalization of that of the characteristic function on convex cones; see [V].

When the boundary $\partial\Omega$ is smooth, we know that $\chi_\Omega(x) \sim d(x, \partial\Omega)^{-(n+1)/2}$ and $k_\Omega(x) \sim d(x, \partial\Omega)^{-n-1}$ around the boundary, where $d(x, \partial\Omega)$ denotes the distance function to the boundary relative to a fixed euclidean structure on $\mathbf{R}^n$. So both functions $v = \chi_\Omega^{-1/(n+1)}$ and $w = k_\Omega^{-1/(2n+2)}$ have the order $d(x, \partial\Omega)^{-1/2}$. We define the metrics ω and κ by

$$\omega = -\frac{1}{v}\sum \frac{\partial^2 v}{\partial x^i \partial x^j} dx^i dx^j \quad \text{and} \quad \kappa = -\frac{1}{w}\sum \frac{\partial^2 w}{\partial x^i \partial x^j} dx^i dx^j, \tag{1.3}$$

relative to a euclidean coordinate system $\{x^1, \ldots, x^n\}$. They are Riemannian metrics that are invariant under projective transformations and complete when the boundary is smooth.