Dimension formula of solution spaces of $\mathcal{A}$-hypergeometric differential-difference systems

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Let $A = (a_{ij})$ be a matrix in $M(d, n, \mathbf{Z}_{\geq 0})$. We suppose that the set of the column vectors of A spans $\mathbf{Z}^d$. We denote by $S_i : f(s_i) \mapsto f(s_i - 1)$ the difference operator with respect to a variable s_i.

Definition 1. We call the following differential-difference system $\mathbf{H}_A$ an *$\mathcal{A}$-hypergeometric differential-difference system*:

$$\begin{aligned} \left(\sum_{j=1}^{n} a_{ij} x_j \partial_j - s_i \right) \bullet f &= 0, \qquad (i = 1, \dots, d) \\ \left(\partial_j - \prod_{i=1}^{d} S_i^{a_{ij}} \right) \bullet f &= 0. \qquad (j = 1, \dots, n) \end{aligned}$$

Definition 2. Let I be a left ideal of the ring of differential-difference operators

$$D = \mathbf{C}(x_1, \dots, x_n, s_1, \dots, s_d)\langle \partial_1, \dots, \partial_n, S_1, \dots, S_d, S_1^{-1}, \dots, S_d^{-1} \rangle.$$

We define the *rank* of I by

$$\mathrm{rank}(I) = \dim_{\mathbf{C}(x,s)} D/I,$$

where D/I is a vector space over $\mathbf{C}(x, s)$.

Theorem 1. *The rank of $\mathbf{H}_A$ agrees with the normalized volume of A.*

Let a_i be the i-th column vector of the matrix A and $F(\beta, x)$ the integral

$$F(\beta, x) = \int_C \exp\left(\sum_{i=1}^{n} x_i t^{a_i} \right) t^{-\beta - 1} dt, \qquad t = (t_1, \dots, t_d).$$

The integral $F(\beta, x)$ satisfies an $\mathcal{A}$-hypergeometric differential system "formally". Moreover, $F(s, x)$ satisfies an $\mathcal{A}$-hypergeometric differential-difference system "formally". We use the word "formally" because, there is no general and rigorous description about the cycle C. However, the integral representation gives an intuitive figure of what are solutions of $\mathcal{A}$-hypergeometric differential-difference system.

Rank theories of $\mathcal{A}$-hypergeometric differential system have been developed since Gel'fand, Zelevinsky and Kapranov. In the end of 1980's, under the condition that the points lie on a same hyperplane, they proved that the rank of $\mathcal{A}$-hypergeometric differential system $H_A(\beta)$ agrees with the normalized volume of A if the toric ideal I_A has the Cohen-Macaulay property. After their result had been gotten, many people proved theorems for equivalence of the rank and the volume under various conditions. In particular, Matusevich, Miller and Walther proved that I_A does not have the Cohen-Macaulay property if there exists a parameter β such that the rank of $H_A(\beta)$ is greater than the volume of A. ([2])

We proved Theorem 1 utilizing theorems above, uniform convergence of series solutions, and Mutsumi Saito's results for contiguity relations. Finally, we note that, for studying $\mathcal{A}$-hypergeometric differential-difference system, we wrote a program "yang" ([3], [4]) on a computer algebra system Risa/Asir and did experimentations on computers.

Example 1. Put $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$. Then the rank of $\mathcal{A}$-hypergeometric differential-difference system $\mathbf{H}_A$ is 3. By using Gröbner bases of $\mathbf{H}_A$, we have the following explicit difference Pfaffian system for $F = {}^t(f,\ S_1 \bullet f,\ S_2 \bullet f)$:

$$S_1 F = A_1(x, s) F, \qquad S_2 F = A_2(x, s) F$$

Here $A_1(x, s)$ and $A_2(x, s)$ are matrices of rational forms. We do not show the explicit expression of A_1, A_2 in this paper because they are more complicated. But at the point $x = (1, 1, 0, 1)$ they are

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{3(6s_1-2s_2-3)(3s_1-s_2)}{31} & \frac{85s_1-31s_2-58}{31} & \frac{2((3s_1-s_2)^2-1)}{31} \\ \frac{3s_1-s_2}{2} & -\frac{3}{2} & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ \frac{3s_1-s_2}{2} & -\frac{3}{2} & 0 \\ -\frac{31s_1-13s_2}{4(3s_1-s_2+2)} & \frac{31}{4(3s_1-s_2+2)} & \frac{3(3s_1-s_2+1)}{2(3s_1-s_2+2)} \end{pmatrix}$$

References

[1] K. Aomoto and M. Kita, *Hypergeometric Functions*, Springer-Verlag Tokyo, 1994. (in Japanese)

[2] L. F. Matusevich, E. Miller, and U. Walther, Homological Methods for Hypergeometric Families, 2004, arXiv:math.AG/0406383

[3] K. Ohara, Risa/Asir Package for Non-commutative Gröbner Bases and its Applications, RIMS Kôkyûroku **1395** (2004), 45–49. (in Japanese)

[4] The OpenXM project, `http://www.OpenXM.org/`

[5] M.Saito, B.Sturmfels, and N.Takayama, *Gröbner Deformations of Hypergeometric Differential Equations*, Springer, 2000.