Generalized Hypergeometric Functions

By

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§1. Introduction

The first author has been developing the theory concerning linear systems of ordinary differential equations of the form

$$(tI - B)dx/dt = Ax,$$

where t is a complex variable, x is a complex column vector, A and B are constant square matrices and I is the identity matrix ([3]). Hypergeometric differential equations such as Gauss' Hypergeometric equation, Jordan-Pochhammer equation, the generalized hypergeometric equation and Kummer's confluent hypergeometric equation can be reduced to systems of this form. We note that the form of this system is invariant under differentiation in t, namely, by the substitution $y(t) = dx(t)/dt$, the system is transformed to $(tI - B)dy/dt = (A - I)y$. The purpose of this note is to make a global study of the generalized hypergeometric differential equation by reducing it to a system of this form.

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The generalized hypergeometric function ${}_nF_{n-1}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_{n-1};t)$ is defined by the series

$$\sum_{m=0}^{\infty} \frac{(\alpha_1,m)\ldots\ldots(\alpha_n,m)}{(\beta_1,m)\ldots(\beta_{n-1},m)(1,m)}\, t^m$$

which is convergent for $|t|<1$, where (a,m) denotes the factorial function

$$(a,m) = a(a+1)\ldots.(a+m-1) = \Gamma(a+m)/\Gamma(a).$$

In the case $n=2$, it reduces to Gauss' hypergeometric function. The function $u(t) = {}_nF_{n-1}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_{n-1};\, t)$ satisfies the linear differential equation

$$[t(\delta+\alpha_n)\ldots.(\delta+\alpha_1) - \delta(\delta+\beta_{n-1}-1)\ldots(\delta+\beta_1-1)]\, u(t) = 0,$$

$$\delta = td/dt,$$

which is called the generalized hypergeometric differential equation ([1]).

Set

$$\begin{aligned} x^1(t) &= u(t),\\ x^2(t) &= (\delta + \beta_1 - 1)\, x^1(t),\\ &\vdots\\ x^n(t) &= (\delta + \beta_{n-1} - 1)\, x^{n-1}(t) \end{aligned}$$

and denote by $x(t)$ the column vector ${}^t(x^1(t),\ldots,x^n(t))$. Then the generalized hypergeometric differential equation is transformed to the system

$$\text{(E)} \qquad (tI - B)dx/dt = Ax,$$

where

$$B = \mathrm{diag}[0,\ldots\ldots,0,1]$$

and A is an n by n constant matrix. For the sake of simplicity, we suppose the minor matrix $(a_{ij})_{1\le j,k\le n-1}$ of $A = (a_{ij})_{1\le j,k\le n}$ is similar to a diagonal matrix. Then we can suppose without loss of generality that A is of the form

$$A = \begin{pmatrix} a_1 & & O & c_1 \\ \vdots & \ddots & & \vdots \\ \vdots & O & a_{n-1}, & c_{n-1} \\ b_1, & \ldots\ldots, & b_{n-1}, & a_n \end{pmatrix}$$

We investigate in this note system (E) where $B = \mathrm{diag}[0,\ldots\ldots,0,1]$ and A is of the form.

Let us denote by $\rho_1,\ldots\ldots,\rho_n$ the characterixtic roots of the matrix A. Then system (E) is of Fuchsian type with regular singular points at $t=0,1,\infty$ whose characteristic exponents are

$$\begin{array}{ll} a_1,\ldots\ldots,a_{n-1},\ 0, & \text{at } t=0 \\ 0,\ldots\ldots,0,\ a_n, & \text{at } t=1 \\ -\rho_1,\ldots\ldots\ldots\ldots,-\rho_n, & \text{at } t=\infty. \end{array}$$

In Section 2, we give some preliminary propositions concerning local solutions. In Section 3, we prove an important identity which states a global structure of solutions of (E). We call the identity a generalized Gauss' formula because we can easily derive from it the well known Gauss' formula

$$\lim_{t\to 1,\, 0<t<1} F(\alpha,\beta,\gamma;t) = \Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)/[\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)].$$

In Section 4, the monodromy group of system (E) is computed. We make use of the fundamental relations of $\pi_1(P^1(C)-\{0,1,\infty\})$ as Riemann did in computing the group of Gauss' hypergeometric differential equation ([4]). However, the relations are not sufficient to eliminate all undetermined constants in the case $n>2$, namely, some global information is necessary. We use the identity given in §3 as such one.

We remark last that the monodromy group of the generalized hypergeometric differential equation was first computed by A.H.M. Levelt([2]).

§2. Local solutions

In this section, we shall study the local solutions of

$$\text{(E)}' \qquad (tI - B)dx/dt = (A + \mu I)x$$

at $t = 0, 1$. Here B and A are the same matrices as in (E), μ is a parameter. We denote by D_0 and D_1 the open disks

$$D_0 = \{t\in C \ ; \ |t|<1\}, \quad D_1 = \{t\in C \ ; \ |t-1|<1\},$$

and by ε_j, $1\le j\le n$, the constant vector ${}^t(0,\ldots,0,\overset{j}{1},0,\ldots,0)$.

Proposition 1. Given $1\le j\le n-1$, suppose $a_j \neq -1-\mu, -2-\mu, \ldots$ and $a_j - a_k \neq -1,-2,\ldots,$ for any $k \neq j$, $1\le k\le n-1$. Then (E)' has a unique solution $x_j(t,\mu)$ of the form

$$(2.1) \qquad t^{a_j+\mu} \sum_{m=0}^{\infty} g_j(m,\mu)t^m$$

convergent in D_0 and satisfying

$$(2.2) \qquad g_j(o,\mu) = \varepsilon_j.$$

Proof. It is known that every formal power series solution converges at a regular singular point. Therefore, we have only to prove the proposition formally. Let us find a recursion formula to determine $g_j(m,\mu), m\ge 0$. By substituting (2.1) in (E)' and identifying the coefficients of like powers of t, we have

$$(2.3) \qquad (a_j+\mu+m+1)Bg_j(m+1,\mu) = (a_j+m-A)g_j(m,\mu), \qquad m \geq 0,$$

or by denoting the k-th component of $g_j(m,\mu)$ by $g_j^k(m,\mu)$,

$$(2.4) \qquad (a_j-a_k+m)g_j^k(m,\mu) = c_k g_j^n(m,\mu), \qquad 1 \leq k \leq n-1,$$

$$(a_j+\mu+m+1)g_j^n(m+1,\mu) = -\sum_{k=1}^{n-1} b_k g_j^k(m,\mu) + (a_j-a_n+m)g_j^n(m,\mu),$$

for $m\geq 0$. It is easy to see that, under the assumptions in the proposition, $g_j(m,\mu)$, $m\geq 0$, are successively and uniquely determined by the initial condition (2.2) and the equations (2.4).

In a similar way, we get

Proposition 2. Suppose $a_n \neq -1-\mu, -2-\mu, \ldots$. Then (E)' has a unique solution $x_n(t,\mu)$ of the form

$$(t-1)^{a_n+\mu} \sum_{m=0}^{\infty} g_n(m,\mu)(t-1)^m \tag{2.5}$$

convergent in D_1 and satisfying

$$g_n(0,\mu) = \varepsilon_n. \tag{2.6}$$

Proof. We only note that $g_n(m,\mu), m\geq 0$, are determined by (2.6) and

$$(a_n+\mu+m+1)(B-I)g_n(m+1,\mu) = (a_n+m-A)g_n(m,\mu), \quad m\geq 0. \tag{2.7}$$

Next we shall study the existence of solutions at $t=0$ and $t=1$ corresponding to the characteristic exponent μ.

Proposition 3. Suppose $\mu \neq -1,-2,\ldots$, and $a_j \neq 0,1,2,\ldots$, for any $1\leq j\leq n-1$. Then (E)' has a unique solution $x_n^*(t,\mu)$ of the form

$$t^{\mu} \sum_{m=0}^{\infty} g_n^*(m,\mu)\, t^m \tag{2.8}$$

convergent in D_0 such that

$$g_n^{*n}(0,\mu) = 1, \tag{2.9}$$

where $g_n^{*n}(0,\mu)$ denotes the n-th component of $g_n^*(0,\mu)$. The other components of $g_n^*(0,\mu)$ must be

(2.10) $$g_n^{*k}(0,\mu) = -c_k/a_k, \qquad 1\leq k\leq n-1.$$

Proof. By observing the formulas

(2.11) $$(-a_k+m)g_n^{*k}(m,\mu) = c_k g_n^{*n}(m,\mu), \qquad 1\leq k\leq n-1,$$

$$(\mu+m+1)g_n^{*n}(m+1,\mu) = -\textstyle\sum_{k=1}^{n-1} b_k g_n^{*k}(m,\mu) + (-a_n+m)g_n^{*}(m,\mu)$$

we can easily show the proposition.

Proposition 4. Suppose $\mu \neq -1,-2,\ldots.$ and $a_n \neq 0,1,2,\ldots$ Then (E)' has unique solutions $x_j^*(t,\mu)$, $1\leq j\leq n-1$, of the form

(2.12) $$(t-1)^{\mu} \textstyle\sum_{m=0}^{\infty} g_j^*(m,\mu)(t-1)^m, \qquad 1\leq j\leq n-1,$$

convergent in D_1 such that

(2.13) $$g_j^{*k}(0,\mu) = \delta_{jk} \qquad 1\leq j,k\leq n-1,$$

where δ_{jk} is Kronecker's delta. In particular, $g_j^*(0,\mu)$ must be

(2.14) $$g_j^*(0,\mu) = {}^t(0,\ldots,0,1,0,\ldots,0,-b_j/a_n) \qquad 1\leq j\leq n-1.$$

These n-1 solutions are linearly independent.

Proof. $g_j^*(m,\mu)$, $m\geq 0$, $1\leq j\leq n-1$, are determined by

(2.15) $$(\mu+m+1)g_j^{*k}(m+1,\mu) = -(-a_k+m)g_j^{*k}(m,\mu) + c_k g_j^{*n}(m,\mu), \quad 1\leq k\leq n-1$$

$$(-a_n+m)g_j^{*n}(m,\mu) = \textstyle\sum_{k=0}^{n-1} b_k g_j^{*k}(m,\mu).$$

The linear independence of the solutions follows from (2.14).

§3. Generalized Gauss' formula

We shall first prove a formula given in the following Theorem 1 and next show how to derive Gauss' formula from it.

We first note

Proposition 5 (Fuchs' relation).

$$\sum_{j=1}^{n} a_j = \sum_{k=1}^{n} \rho_k .$$

Proof. Since $\rho_1,\ldots,\rho_n$ are the characteristic roots of A, the relation follows from the invariance of the trace of A.

Let us assume $a_j - a_k \neq \pm 1, \pm 2, \ldots,$ for any $j \neq k$, $1 \leq j,k \leq n-1$, $a_j \neq -1, -2, \ldots,$ for any $1 \leq j \leq n$. Then from Propositions 1 and 3, system (E) has unique solutions $x_1(t),\ldots,x_{n-1}(t)$ in D_0 and $x_n(t)$ in D_1 of the form

$$(3.1) \qquad x_j(t) = t^{a_j} \sum_{m=0}^{\infty} g_j(m) t^m, \quad g_j(0) = \varepsilon_j, \quad 1 \leq j \leq n-1,$$

$$x_n(t) = (t-1)^{a_n} \sum_{m=0}^{\infty} g_n(m)(t-1)^m, \quad g_n(0) = \varepsilon_n .$$

Let D be the simply connected domain in $C - \{0,1\}$

$$D = D_0 \cap D_1 = \{t \in C \; ; \; |t|<1, \; |t-1|<1\},$$

then we have

Theorem 1. Suppose $a_j - a_k \neq \pm 1, \pm 2, \ldots,$ for any $j \neq k$, $1 \leq j,k \leq n-1$ and $a_j, \rho_k \neq -1, -2, \ldots,$ for any $1 \leq j,k \leq n$. Then we have the identity

$$(3.2) \qquad \det (x_1(t),\ldots,x_n(t))$$

$$= t^{\sum_{j=1}^{n-1} a_j} (t-1)^{a_n} \Pi_{j=1}^{n} \Gamma(a_j+1) / [\Pi_{k=1}^{n} \Gamma(\rho_k+1)]$$

which is valid for $t \in D$. Here we take the branches of $x_1(t), \ldots,$ $x_n(t)$ in D determined by $\arg t = 0$, $\arg(t-1) = \pi$, for $0<t<1$.

Proof. It is sufficient to show (3.2) in a nonempty simply connected compact subset D' of D.

For any integer $\mu \geq 0$, from Propositions 1 and 3 and from the assumptions $a_j - a_k \neq \pm 1, \pm 2, \ldots,$ $j \neq k$, $1 \leq j,k \leq n-1$, $a_j \neq -1, -2, \ldots,$ $1 \leq j \leq n$, there exist unique solutions $x_1(t,\mu), \ldots, x_n(t,\mu)$ of (E)' in D such that $x_1(t,\mu), \ldots, x_{n-1}(t,\mu)$ are of the form (2.1) with (2.2) at $t=0$ and $x_n(t,\mu)$ is of the form (2.5) with (2.6) at $t=1$. Set

$$X(t,\mu) = (x_1(t,\mu), \ldots, x_n(t,\mu))$$

$$w(t,\mu) = \det X(t,\mu).$$

We shall first obtain a relation between $w(t,\mu)$ and $w(t,0)$. Let μ be a positive integer. Then $y_j(t,\mu) = dx_j(t,\mu)/dt$, $1 \leq j \leq n$, are solutions of

$$(t - B)dy/dt = (A + \mu - 1)y$$

and they are of the form

$$y_j(t,\mu) = t^{a_j+\mu-1} \textstyle\sum_{m=0}^{\infty} (a_j+\mu+m) g_j(m,\mu) t^m, \qquad 1 \leq j \leq n-1,$$

$$y_n(t,\mu) = (t-1)^{a_n+\mu-1} \textstyle\sum_{m=0}^{\infty} (a_n+\mu+m) g_n(m,\mu) (t-1)^m.$$

Hence, by the uniqueness stated in Propositions 1 and 3, we have

$$y_j(t,\mu) = (a_j+\mu) x_j(t,\mu-1), \qquad 1 \leq j \leq n.$$

These relations are written by

$$(t-B)X(t,\mu-1)\mathrm{diag}[a_1+\mu,\ldots,a_n+\mu] = (A+\mu)X(t,\mu).$$

Hence, noting the assumption $\rho_k \neq -1,-2,\ldots$ $(1\le k\le n)$, we get the equation

$$w(t,\mu) = w(t,\mu-1)t^{n-1}(t-1)\Pi_{j=1}^{n}(a_j+\mu)/\Pi_{k=1}^{n}(\rho_k+\mu)$$

for $t\in D'$ and positive integer μ. Therefore, by using the equation μ times, we obtain

$$(3.3)\qquad w(t,\mu) = w(t,0)t^{\mu(n-1)}(t-1)^{\mu}\Pi_{k=1}^{n}\Gamma(\rho_k+1)\Pi_{j=1}^{n}\Gamma(a_j+\mu+1)$$
$$\times[\Pi_{j=1}^{n}\Gamma(a_j+1)\Pi_{k=1}^{n}\Gamma(\rho_k+\mu+1)]^{-1}$$

We shall next investigate the asymptotic behabior of $w(t,\mu)$ as $\mu>0$ tends to ∞. The coefficients $g_j(m,\mu), m\ge 0, 1\le j\le n$, are determined by (2.2),(2.3) for $1\le j\le n-1$, and by (2.6),(2.7) for $j=n$. Let us substitute by

$$g_j(m,\mu)$$
$$= h_j(m,\mu)\Gamma(a_j+\mu+1)\Gamma(a_j+m+1)/[\Gamma(a_j+\mu+m+1)\Gamma(a_j+1)],\quad 1\le j\le n.$$

Then $h_j(m,\mu), 1\le j\le n$, are transformed to

$$(a_j+m+1)Bh_j(m+1,\mu) = (a_j+m-A)h_j(m,\mu),\qquad 1\le j\le n-1,$$

$$(a_n+m+1)(B-I)h_n(m+1,\mu) = (a_n+m-A)h_n(m,\mu)$$

with

$$h_j(0,\mu) = \varepsilon_j,\qquad\qquad 1\le j\le n.$$

Therefore, from the uniqueness properties in Propositions 1,3, it follows that

$$h_j(m,\mu) = g_j(m), \qquad m\geq 0,\ 1\leq j\leq n,$$

hence

$$(3.4) \quad g_j(m,\mu)= g_j(m)\Gamma(a_j+\mu+1)\Gamma(a_j+m+1)/[\Gamma(a_j+\mu+m+1)\Gamma(a_j+1)].$$

Now we estimate $\sum_{m=1}^{\infty} g_j(m,\mu)t^m$, $1\leq j\leq n-1$ and $\sum_{m=1}^{\infty} g_n(m,\mu)(t-1)^m$. We can easily verify the inequalities

$$(3.5) \qquad \left|\frac{\Gamma(a_j+\mu+1)\Gamma(a_j+m+1)}{\Gamma(a_j+\mu+m+1)\Gamma(a_j+1)}\right| \leq 2(|a_j|+1)\mu^{-1}, \qquad 1\leq j\leq n,$$

for any $m\geq 1$, $\mu\geq 2(|a_j|+1)>0$, since the left side of (3.5) is

$$\left|\frac{a_j+m}{a_j+m+\mu}\right|\left|\frac{a_j+m-1}{a_j+m-1+\mu}\right|\cdots\cdots\left|\frac{a_j+2}{a_j+2+\mu}\right|\left|\frac{a_j+1}{a_j+1+\mu}\right| .$$

Let $\|\ \|$ be the norm defined by $\|g\| = \max\{|g^1|,\ldots,|g^n|\}$. Then it follows from (3.4),(3.5) that

$$\|\sum_{m=1}^{\infty} g_j(m,\mu)t^m\| \leq 2(|a_j|+1)\mu^{-1}\sum_{m=1}^{\infty}\|g_j(m)\|\,|t|^m, \quad 1\leq j\leq n-1,$$

$$\|\sum_{m=1}^{\infty} g_n(m,\mu)(t-1)^m\| \leq 2(|a_n|+1)\mu^{-1}\sum_{m=1}^{\infty}\|g_n(m)\|\,|t-1|^m,$$

for $t\in D'$, $\mu\geq\max_{1\leq j\leq n}\{2(|a_j|+1)\}$. Since D' is a compact subset of D, $\sum_{m=1}^{\infty}\|g_j(m)\|\,|t|^m$, $1\leq j\leq n-1$ and $\sum_{m=1}^{\infty}\|g_n(m)\|\,|t-1|^m$ are uniformly convertent on D' and hence they are bounded on D'. Therefore, we have,

$$x_j(t,\mu) = t^{a_j+\mu}\{\varepsilon_j + O(1/\mu)\}, \qquad 1\leq j\leq n-1,$$

$$x_n(t,\mu) = (t-1)^{a_n+\mu}\{\varepsilon_n + O(1/\mu)\}$$

for $t\in D'$, which implies

$$(3.6) \qquad w(t,\mu) = t^{\sum_{j=1}^{n-1}a_j+\mu(n-1)}(t-1)^{a_n+\mu}(1 + O(1/\mu)).$$

Here O denotes Landau's symbol.

By combining (3.3) and (3.6), we have

$$w(t,0) = t^{\sum_{j=1}^{n-1} a_j}(t-1)^{a_n}(1 + O(1/\mu)) \tag{3.7}$$

$$\times \Pi_{j=1}^{n}\Gamma(a_j+1)\Pi_{k=1}^{n}\Gamma(\rho_k+\mu+1)$$

$$\times [\Pi_{k=1}^{n}\Gamma(\rho_k+1)\Pi_{j=1}^{n}\Gamma(a_j+\mu+1)]^{-1}$$

for $t \in D'$, $\mu \geq \max_{1\leq j\leq n}\{2(|a_j|+1)\}$. By virtue of Fuchs' relation (Proposition 5) and Stirling's formula, it holds that

$$\lim_{\mu\to\infty,\mu>0} \Pi_{k=1}^{n}\Gamma(\rho_k+\mu+1)/\Pi_{j=1}^{n}\Gamma(a_j+\mu+1) = 1 .$$

Therefore, by taking the limit as μ goes to $+\infty$ in (3.7), we obtain the identity (3.2) for $t \in D'$, which completes the proof of Theorem 1.

Corollary(Gauss' formula). Assume that γ, $\gamma-\alpha-\beta$, $\gamma-\alpha$, $\gamma-\beta \neq 0, -1, -2, \ldots.$ and $\mathrm{Re}(\gamma-\alpha-\beta)>0$. Then we have

$$\lim_{t\to 1, 0<t<1} F(\alpha,\beta,\gamma;t) = \Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)/[\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)]. \tag{3.8}$$

Proof. By definition

$$F(\alpha,\beta,\gamma;t) = {}_2F_1(\alpha,\beta;\gamma;t) = \sum_{m=0}^{\infty}\frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)}\, t^m.$$

Consider system (E) in the case where

$$A = \begin{pmatrix} \gamma-1, & \alpha \\ -\beta\ , & \gamma-\alpha-\beta-1 \end{pmatrix}$$

The characteristic roots of A are $\gamma-\alpha-1$, $\gamma-\beta-1$. Then the assumptions in Theorem 1 are satisfied. Let $x_1(t)$ and $x_2(t)$ be the solutions stated in Theorem 1, namely,

$$x_1(t) = t^{\gamma-1}\sum_{m=0}^{\infty} g_1(m)t^m, \quad g_1(0) = \varepsilon_1,$$

$$x_2(t) = (t-1)^{\gamma-\alpha-\beta-1}\sum_{m=0}^{\infty} g_2(m)(t-1)^m, \; g_2(0) = \varepsilon_2,$$

then, we have

$$\text{(3.10)} \qquad \det(x_1(t),x_2(t)) = t^{\gamma-1}(t-1)^{\gamma-\alpha-\beta-1} \times \Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)/[\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)]$$

for $t\in D$, where $\arg t = 0$, $\arg(t-1) = \pi$ for $0<t<1$. We can verify that the first component of $g_1(m)$ is $(\alpha,m)(\beta,m)/[(\gamma,m)(1,m)]$, that is, the first component of $x_1(t)$ is $t^{\gamma-1}F(\alpha,\beta,\gamma;t)$.

There exists a solution $x_1^*(t)$ and a constant c such that

$$x_1(t) = x_1^*(t) + cx_2(t), \qquad t\in D.$$

From (3.10), we have

$$\text{(3.11)} \qquad \det(x_1^*(t),x_2(t)) = \det(x_1(t),x_2(t))$$

$$= t^{\gamma-1}(t-1)^{\gamma-\alpha-\beta-1}\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)/[\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)].$$

Since the first component of $x_2(t)$ is $(t-1)^{\gamma-\alpha-\beta}O(1)$ and the second component of $x_2(t)$ is $(t-1)^{\gamma-\alpha-\beta-1}(1 + O(1))$, we can derive from (3.11) that the first component of $x_1^*(t)$ is

$$\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)/[\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)] + O(|t-1|^{1-\varepsilon}).$$

Here ε is a constant $0\leq\varepsilon<1$, which is equal to 0 if $x_1^*(t)$ has no logarithmic term. Therefore we have

$$t^{\gamma-1}F(\alpha,\beta,\gamma;t)$$

$$= \Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)/[\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)] + O(|t-1|^{1-\varepsilon}) + (t-1)^{\gamma-\alpha-\beta}O(1).$$

By taking the limit as $t \to 1$, $0<t<1$, we obtain (3.8).

§4. Monodromy group

In this section, we shall calculate the monodromy group of system (E). Let us fix a base point t_0 in D. Let ℓ_0(or ℓ_1) be a closed curve in C $-$ $\{0,1\}$,beginning and ending at $t=t_0$, encircling only the point $t=0$(or $t=1$) once in the positive direction. Let $Y(t)$ denote a branch in D of a fundamental matrix of system (E). If we continue $Y(t)$ analytically along the curve ℓ_j, then we have a different branch in D which can be written as $Y(t)N_j$ where $N_j \in$ $Gl(n,C)$. We call N_0(or N_1) the circuit matrix of $Y(t)$ around $t=0$ (or $t=1$). The subgroup of $GL(n,C)$ generated by N_0 and N_1 is called the monodromy group of (E) with respect to $Y(t)$.

Set

$$e_j = \exp(2\pi\sqrt{-1}a_j), \qquad 1\leq j\leq n,$$

$$f_k = \exp(2\pi\sqrt{-1}\rho_k), \qquad 1\leq k\leq n,$$

then, we have

Theorem 2. Assume that none of $a_j(1\leq j\leq n)$, $a_j-a_k(j\neq k,1\leq j,k\leq n-1)$, $a_j-\rho_k(1\leq j\leq n-1,\ 1\leq k\leq n)$ are integers and none of $\rho_k(1\leq k\leq n)$ are negative integers. Then the group generated by

$$N_0 = \begin{pmatrix} e_1 & & & & e_1-1 \\ & \ddots & & 0 & \vdots \\ & & \ddots & & \vdots \\ & 0 & & e_{n-1} & e_{n-1}-1 \\ & & & & 1 \end{pmatrix}$$

$$N_1 = \begin{pmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & \ddots & & \\ & 0 & & 1 & \\ d_1(e_n-1) & \cdots & \cdots & d_{n-1}(e_n-1) & e_n \end{pmatrix}$$

is the monodromy group of system (E). Here

$$d_j = -\Pi_{k=1}^{n}(f_k-e_j)/[e_j(e_j-1)(e_n-1)\Pi_{k\neq j,1\le k\le n-1}(e_k-e_j)]$$

$$= -\Pi_{k=1}^{n}\sin(\rho_k-a_j)\pi/[\sin a_j\pi \sin a_n\pi\ \Pi_{k\neq j,1\le k\le n-1}\sin(a_k-a_j)\pi],$$

$1\le j\le n-1$.

Proof. Let $x_1(t),\ldots,x_n(t)$ be solutions stated in Theorem 1. Then, under our assumptions, $\det(x_1(t),\ldots,x_n(t)) \neq 0$. We shall first compute the circuit matrices of the fundamental matrix $X(t) = (x_1(t),\ldots,x_n(t))$.

Let $\xi_1,\ldots,\xi_{n-1}$ be constants and $x_n^*(t)$ be a solution of (E) such that

$$x_n(t) = \textstyle\sum_{j=1}^{n-1} \xi_j x_j(t) + x_n^*(t), \qquad t\in D.$$

We can verify by Proposition 3 that $x_n^*(t)$ is holomorphic at $t=0$. Let η_j be a constant and $x_j^*(t)$ be a solution of (E) such that

$$x_j(t) = x_j^*(t) + \eta_j x_n(t), \qquad t\in D,\ 1\le j\le n-1.$$

From Proposition 4, $x_j^*(t)$ is holomorphic at $t=1$. Set

$$X_0(t) = (x_1(t),\ldots\ldots,x_{n-1}(t),x_n^*(t)),$$

$$X_1(t) = (x_1^*(t),\ldots\ldots,x_{n-1}^*(t),x_n(t)),$$

then the above relations are written by

$$X(t) = X_0(t)C_0, \quad X(t) = X_1(t)C_1$$

where

$$C_0 = \begin{pmatrix} 1 & & & \xi_1 \\ & \ddots & 0 & \vdots \\ & 0 & 1, & \xi_{n-1} \\ & & & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & & & \\ & \ddots & 0 & \\ 0 & & 1 & \\ \eta_1, & \ldots\ldots, & \eta_{n-1}, & 1 \end{pmatrix}.$$

Since $X(t)$ is a fundamental matrix, $X_0(t)$ and $X_1(t)$ are also fundamental matrices.

From the local expressions (3.1) of $x_1(t),\ldots,x_{n-1}(t)$ at $t=0$ and from the holomorphy of $x_n^*(t)$ at $t=0$, the circuit matrix E_0 of $X_0(t)$ around $t=0$ is of the form

$$E_0 = \mathrm{diag}[e_1,\ldots\ldots,e_{n-1},1].$$

Similarly, it follows that the circuit matrix E_1 of $X_1(t)$ around $t=1$ is of the form

$$E_1 = \mathrm{diag}[1,\ldots\ldots,1,e_n].$$

Therefore, the curcuit matrices M_0 and M_1 of $X(t)$ around $t=0$ and $t=1$ are

$$M_0 = C_0^{-1}E_0C_0 = \begin{pmatrix} e_1 & & & \xi_1(e_1-1) \\ & \ddots & 0 & \vdots \\ & 0 & e_{n-1}, & \xi_{n-1}(e_{n-1}-1) \\ & & & 1 \end{pmatrix}$$

$$M_1 = C_1^{-1}E_1C_1 = \begin{pmatrix} 1 & & & \\ & \ddots & 0 & \\ 0 & & 1 & \\ \eta_1(e_n-1), & \ldots\ldots, & \eta_{n-1}(e_n-1), & e_n \end{pmatrix}$$

next

We shall determine the values of $d_j=\xi_j\eta_j$, $1\leq j\leq n-1$.

Since M_0M_1 is the inverse circuit matrix of $X(t)$ around $t=\infty$, the characteristic roots of $(M_0M_1)^{-1}$ are the characteristic multipliers of system (E) around $t=\infty$. On the other hand, since $\rho_1,\dots,\rho_n$ are the characteristic roots of the matrix A, the characteristic multipliers of system (E) around $t=\infty$ are $f_1^{-1} = \exp(-2\pi\sqrt{-1}\rho_1),\dots,$ $f_n^{-1} = \exp(-2\pi\sqrt{-1}\rho_n)$. Thus we have the identity in λ

$$\det(M_0M_1 - \lambda) = \Pi_{k=1}^{n}(f_k - \lambda).$$

By direct computation, we have that the components r_{jk}, $1\le j,k\le n$ of the matrix M_0M_1 are

$$r_{jk} = \delta_{jk}e_j + \xi_j\eta_k(e_j-1)(e_n-1), \qquad 1\le j,k\le n-1$$

$$r_{jn} = \xi_j(e_j-1)e_n, \qquad 1\le j\le n-1$$

$$r_{nk} = \eta_k(e_n-1), \qquad 1\le k\le n-1$$

$$r_{nn} = e_n.$$

Hence

$$\det(M_0M_1 - \lambda) = \begin{vmatrix} e_1-\lambda & & & O & \xi_1(e_1-1)\lambda \\ & \ddots & & & \vdots \\ & O & & e_{n-1}-\lambda & \xi_{n-1}(e_{n-1}-1)\lambda \\ \eta_1(e_n-1), & \dots\dots, & & \eta_{n-1}(e_n-1), & e_n-\lambda \end{vmatrix}$$

$$= \Pi_{q=1}^{n}(e_q-\lambda) - (e_n-1)(\textstyle\sum_{q=1}^{n-1} d_q(e_q-1)\Pi_{p\neq q,1\le p\le n-1}(e_p-\lambda))\lambda.$$

Thus we obtain

(4.1) $\qquad \Pi_{q=1}^{n}(e_q-\lambda) - (e_n-1)(\sum_{q=1}^{n-1}d_q(e_q-1)\Pi_{p\neq q,1\le p\le n-1}(e_p-\lambda))\lambda$

$$= \Pi_{k=1}^{n}(f_k - \lambda).$$

By putting the special value $\lambda = e_j$, $1 \leq j \leq n-1$ in (4.1), we immediately have

$$(4.2) \qquad d_j = -\Pi_{k=1}^{n}(f_k - e_j)/[e_j(e_j - 1)(e_n - 1)\Pi_{k \neq j, 1 \leq k \leq n-1}(e_k - e_j)]$$

$$= -\Pi_{k=1}^{n}\sin(\rho_k - a_j)\pi/[\sin a_j\pi \sin a_n\pi \, \Pi_{k \neq j, 1 \leq k \leq n-1}\sin(a_k - a_j)\pi]$$

$1 \leq j \leq n-1$. The second equation in (4.2) is shown by Fuchs' relation (proposition 5) and the identity

$$(2\sqrt{-1})^{-1}(\exp(\pi\sqrt{-1}\theta) - \exp(-\pi\sqrt{-1}\theta)) = \sin \pi\theta.$$

We note that $d_j = \xi_j \eta_j \neq 0$, $1 \leq j \leq n-1$ under the assumption that none of $a_j - \rho_k$, $1 \leq j \leq n-1$, $1 \leq k \leq n$ are integers.

Finally, set

$$Y(t) = X(t)\mathrm{diag}[\xi_1, \ldots\ldots, \xi_{n-1}, 1].$$

Then the circuit matrices N_0 and N_1 of $Y(t)$ around $t=0$ and $t=1$ are of the form given in the theorem. Thus we have completed the proof of Theorem 2.

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