Chapter 3 MONODROMY REPRESENTATIONS

3.1. Local Canonical set.

A matrix solution $X(t)$ of the system

$$(t-B)x' = Ax$$

is a local canonical set when it is a fundamental set of solutions and when $X(t)$ is continued analytically around some particular regular singularity $t=\lambda_k$ $X(t)$ is transformed linearly:

$$X(t) \to X(t)C_{\lambda_k}$$

with a diagonal matrix C_{λ_k}.

The above definition is somewhat restrictive when we admit of logarithmic solutions into our system. In general, C_{λ_k} be in Jordan canonical form. But in this chapter, too, we assume the conditions (i)-(v) of the preceding chapter, so that there is no logarithmic solution and the matrices A has diagonal blocks at each singular point.

Let us fix a particular fundamental set of solutions $X(t)$ of (3.1). Then for any path starting from a fixed point λ and ending at the same point in $\mathbb{P}^1-\overline{S}$ where $\overline{S}$ is the union of $S=[\lambda_1,\lambda_2,\dots,\lambda_d]$ and $[t=\infty]$:

$$\overline{S} = [\lambda_1,\dots,\lambda_d,\infty]$$

$X(t)$ is transformed into another fundamental set $X^*(t)$

$$X(t) \to X^*(t) = X(t)C \qquad (C\varepsilon GL(d,C))$$

which induces a representation of the fundamental group $\pi^1(\mathbb{P}-\overline{S})$ into $GL(d,C)$.

We call this representation, the monodromy representation of the solutions of (3.1) with respect to the fundamental set X(t). Clearly, this representation into a subgroup of GL(d,C) is completely determined by computing the generators at every $\lambda_1,\ldots,\lambda_k$, since the group is finitely generated by transformations M_k at $t=\lambda_k$ and their product $M_1M_2\ldots M_d$, if the domain should properly chosen by introducing cuts, becomes the inverse of the generator at infinity.

Although we can not give a logical explanation as to how we come to choose our particular fundamental set, the sole subtlety of the present investigation lies in the choice of X(t) as the set defined in the preceding chapter whose wronskian has an explicit representation with the choice of induced local canonical sets .

Let X(t) be the set of solutions in the matrix form:

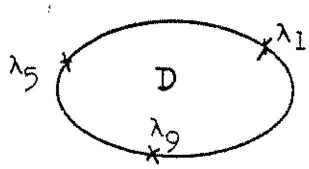
$$X(t) = [\ x_1(t),\ldots\ldots, x_d(t)\ ] \tag{1.1}$$

where for each $k=1,2,\ldots,d$, the column vector $x_k(t)$ is the solution with the specified singular behavior:

$$x_k(t) = (t-\lambda_k)^{a_{kk}} \sum_{m=0}^{\infty} g_k(m)(t-\lambda_k)^m \tag{1.2}$$

(cf. eq.(2.1) of the preceding chapter).

Let D be the simply connected domain obtained by joining the vertices $\lambda_1,\ldots,\lambda_d$ by simple arcs counterclockwise. Here we assume that these arcs enclose a bounded domain D by renumbering the vertices $\lambda_1,\ldots,\lambda_d$, if necessary.



And we count multiple roots only once for a class. See the figure, for an example, with d=9 $\lambda_1=\lambda_2=\lambda_3=\lambda_4, \lambda_5=\lambda_6=\lambda_7$ and $\lambda_8=\lambda_9$, and with the label L(S)=432.

As one particular local solution is continued analytically in D near to another singular point, say $x_k(t)$ continued to $t=\lambda_j$ $(\lambda_j \neq \lambda_k)$, $x_k(t)$ is expressed as a linear combination of the set of singular solutions and one definite holomorphic solution:

$$x_k(t) = \sum_{\lambda_p=\lambda_j} t_{k,p}\, x_p(t) + x^*_{k(j)}(t) \tag{1.3}$$

We define the local canonical set of solutions to be the matrix:

$$X_j(t) = [\ x^*_1(t),\ldots,x^*_{m_1}(t),\ldots\ldots,x_j(t),\ldots\ldots,x_{j+m}(t),\ldots\ldots\ldots,x^*_d(t)\ ] \tag{1.4}$$

where for the sake of simplicity, we omitted the trivial index j of $x^*_{k(j)}(t)$ in (1.3) for we are concerned with the local solution at the singular point $t=\lambda_j=\ldots$ $\ldots=\lambda_{j+m}$. Every solution with an asterist (*) is an holomorphic solution, and a solution without it is a singular solution with the specified behavior by (1.2).

Example. With $d=9, \lambda_1=\lambda_2=\lambda_3=\lambda_4$, $\lambda_5=\lambda_6=\lambda_7$, and $\lambda_8=\lambda_9$, we have four sets of solutions

$$X(t) = [\ x_1(t),x_2(t),x_3(t),x_4(t),x_5(t),x_6(t),x_7(t),x_8(t),x_9(t)\ ]$$

$$X_1(t) = [\ x_1(t),x_2(t),x_3(t),x_4(t),x^*_5(t),x^*_6(t),x^*_7(t),x^*_8(t),x^*_9(t)\ ]$$

$$X_5(t) = [\ x^*_1(t),x^*_2(t),x^*_3(t),x^*_4(t),x_5(t),x_6(t),x_7(t),x^*_8(t),x^*_9(t)\ ]$$

$$X_8(t) = [\ x^*_1(t),x^*_2(t),x^*_3(t),x^*_4(t),x^*_5(t),x^*_6(t),x^*_7(t),x_8(t),x_9(t)\ ]$$

Naturally, by our convention, the holomorohic solution $x^*_1(t)$ in $X_5(t)$ and $x^*_1(t)$ in $X_8(t)$ are solutions defined in different neighborhoods, and are not the same solutions.

Proposition 1.1.

The set $X_j(t)$ is a local canonical set in the sense,

$$X_j(t) \dashrightarrow X_j(t)C_j \tag{1.5}$$

as $X_j(t)$ is continued analytically around the singular point $t=\lambda_j$ counterclockwise, where C_j is a diagonal matrix of the form:

$$C_j = \begin{pmatrix} I(m^*) & O(m^*,m) & O(m^*,m^{**}) \\ O(m,m^*) & E_j & O(m,m^{**}) \\ O(m^{**},m^*) & O(m^{**},m) & I(m^{**}) \end{pmatrix} \quad (m+m^*+m^{**}=d) \tag{1.6}$$

where $I(p)$ is an identity matrix of size p, $O(p,q)$ is a null matrix of the size p by q and where E_j is the diagonal matrix:

$$E_j = \begin{pmatrix} e_j & & & \\ & e_{j+1} & & 0 \\ 0 & & \ddots & \\ & & & e_{j+m} \end{pmatrix} \tag{1.7}$$

with

$$e_p = \exp(2\pi i a_{p,p}) \qquad (p=j,j+1,\ldots,j+m) \tag{1.8}$$

$$m = \text{multiplicity of the eigenvalue } \lambda_j \text{ of the matrix B} \tag{1.9}.$$

Proof. Trivial.

Remark. $X_j(t)$ is a fundamental set of solutions, for the linear relation with the fundamental set is given by:

$$X(t) = X_j(t)\left[\, I + \begin{pmatrix} O(m^*,m^*) & O(m^*,m) & O(m^*,m^{**}) \\ T^* & O(m,m) & T^{**} \\ O(m^{**},m^*) & O(m^{**},m) & O(m^{**},m^{**}) \end{pmatrix} \right] \tag{1.10}$$

where T^* and T^{**} are the matrices of the respective sizes m by m^* and m by m^{**} with elements $t_{p,q}$ such that $\lambda_p=\lambda_j$ and $\lambda_q\neq\lambda_j$ in the expression (1.3).
We abbreviate eq.(1.10) as,

$$X(t) = X(t)T_j = X_j(t)[\, I + P_j\,] \tag{1.11}$$

The matrix T_j has the determinant 1 because it is an identity matrix plus an idempotent matrix. We can write down the inverse T_j^{-1} immediately as follows.

$$T_j^{-1} = [\, I + P_j]^{-1} = I - P_j + P_j^2 - P_j^3 + \ldots\ldots$$

$$= I - P_j = I - \begin{pmatrix} O(m^*,m^*) & O(m^*,m) & O(m^*,m^{**}) \\ T^* & O(m,m) & T^{**} \\ O(m^{**},m^*) & O(m^{**},m) & O(m^{**},m^{**}) \end{pmatrix} \tag{1.12}$$

When we refer to the condition (iii) of the preceding chapter, and ,in particular, the blockwise partition of the matrix A, we see that sum of the all T_j,counting only once for a class of identical eigenvalues $\lambda_j=\lambda_k$, we can build up a matrix of connection coefficients:

$$Q = \left(\begin{array}{c|c|c|c} O_{1,1} & T_{1,2}, & & T_{1,r} \\ \hline T_{2,1} & O_{2,2} & & T_{2,r} \\ \hline & \cdots\cdots & \cdots\cdots & \\ & \cdots\cdots & \cdots\cdots & \\ \hline T_{r,1} & T_{r,2}, & \cdots\cdots\cdots & O_{r,r} \end{array}\right) \tag{1.13}$$

In the next section, we will show that if we can compute all the elements of this matrix, then we can determine the monodromy representation.

3.2. Generators.

Theorem 1.

The generators of the monodromy representation with respect to the fundamental set $X(t)$ are given by:

$$M_j = I + \begin{pmatrix} O(m^*,m^*) & O(m^*,m) & O(m^*,m^{**}) \\ (E_j-I)T^* & E_j-I & (E_j-I)T^{**} \\ O(m^{**},m^*) & O(m^{**},m) & O(m^{**},m^{**}) \end{pmatrix} \quad (j=1,2,\dots,r) \tag{2.1}$$

where

$$m^* = m_1+\dots+m_{j-1}, \qquad m^{**} = m_{j+1} + \dots + m_r \ ; \ m = m_j \tag{2.2}$$

and where E_j is the diagonal matrix of the size m_j by m_j:

$$E_j = \mathrm{diag}[\ \exp(2\pi i a_{m^*+1,m^*+1}),\dots\dots,\ \exp(2\pi i a_{m^*+m_j,m^*+m_j})\ ] \tag{2.3}$$

Proof.

When we continue the fundamental set $X(t)$ analytically around the j-th regular singular point $t=\lambda$ with multiplicity m_j, we have the following diagram.

$$\begin{array}{ccc} X(t) & \dashrightarrow & X(t)M_j \\ \| & & \| \\ X_j(t)T_j & \dashrightarrow & X_j(t)C_jT_j = X_j(t)T_j(T_j^{-1}C_jT_j) \end{array} \tag{2.4}$$

We only have to compute $M_j=T_j^{-1}C_jT_j$ from the expressions (1.6),(1.10) and (1.12). The expression (2.1) is a direct consequence of an easy computation.

Remark, The number of unknown constants contained in the generator M_j is equal to $m_j(d-m_j)$ since the diagonal matrix $(E_j - I)$ in the middle is known from the diagonal elements $(a_{1,1},\ldots\ldots,a_{d,d})$. Consequently, to determine the monodromy group which is a free group generated by $M_1,\ldots\ldots,M_r$, we have to know

$$\sum_{j=1}^{r} m_j(d-m_j) = d^2 - \sum_{j=1}^{r} m_j^2 \qquad (2.5)$$

constants $t_{p,k}$. But if we were to be content with a group isomorphic with this group G^*, we may use a diagonal matrix D to use another fundamental set of solutions $X(t)D$. This fundamental set consists of solutions $x_1(t),\ldots\ldots,x_d(t)$ multiplied by some constant multipliers, and hence the corresponding local canonical set is again a local canonical set:

$$X(t)D \dashrightarrow X(t)D(D^{-1}M_jD) \qquad D^{-1}C_jD = C_j \qquad (2.6)$$

which introduces more (d-1) degree of freedom.

Theorem 2.(Riemann-Fuchs)

The eigenvalues of the matrix M_∞ defined by the product:

$$M_\infty = M_1M_2\cdot\cdots\cdots M_r \qquad (2.7)$$

where r is the number of distinct classes of eigenvalues $S=[\lambda_1,\lambda_2,\ldots\ldots,\lambda_d]$, are

$$f_k = \exp(2\pi i\rho_k) \qquad (k=1,2,\ldots\ldots,d)$$

where ρ_k's are the eigenvalues of the matrix A.

Proof. From the construction of our domain D, the successive product of the paths encircling the distinct singularities $\lambda_1,\ldots,\lambda_d$, is the inverse of the path encircling the singularity $t=\infty$ in the negative sense. This is the only one topological property of the set $\mathbb{P}^1-\overline{S}$ we use. If we were to deal with problems on Riemann surfaces, we have to consider other fundamental relations. At the singular point $t=\infty$, we have a set of singular solutions:

$$x^k(t) = t^{\rho_k}\sum_{s=o}^{\infty} h(s)t^{-s} \qquad (k=1,2,\ldots,d) \tag{2.8}$$

which are linearly independent and free of logarithmic terms under the condition (ii) of the preceding chapter II. Hence, the matrix solution

$$X_\infty(t) = [\ x^1(t),x^2(t),\ldots\ldots,x^d(t)\ ] \tag{2.9}$$

has the circuit matrix

$$M^\infty = \mathrm{diag}[\ \exp(2\pi i\rho_1),\ldots\ldots\ldots,\exp(2\pi i\rho_d)\ ]$$

at $t=\infty$. While any two fundamental set has a definite connection matrix if we specify the domain properly, so we define T^∞ be the connection matrix across the simple arc connecting λ_d and λ_1 as a part of the boundary of the simply connected domain D, then we have

$$X(t)=X^\infty(t)T^\infty \quad \text{-----}> \ X\,(t)\cdot(T^\infty)^{-1}M^\infty T^\infty = X(t)M_1M_2\cdots M_r \tag{2.10}$$

Since eigenvalues are invariants of linear automorphisms, we complete the proof.

3.3. Decomposition into generalized reflections.

According to Shephard,G.C.,([18]) a reflection in a unitary space is a linear transformation whose matrix has all but one of the eigenvalues one, and the remaining eigenvalue equal to a primitive root of unity. For the sake of convenience, we slightly overgeneralize this concept and call a linear transformation G, a generalized reflection if all but one eigenvalues of the matrix are one and the remaining eigenvalue is equal to

$$\exp(2\pi i a) = e \tag{3.1}$$

for some real number a.

Theorem 3. Every generator of our monodromy representation is a product of generalized reflections.

Proof. We prove the theorem only for the first generator M_1, because by a suitable permutation of dependent variables, we may bring the generator in question into the type of M_1,i.e.,

$$M_1 = I_d + \begin{pmatrix} E-I_m & T \\ O_1 & O_2 \end{pmatrix} \tag{3.2}$$

where I_d and I_m are identity matrices of respective dimensions d and m, E is a diagonal matrix of size m of the form:

$$E = \begin{pmatrix} e_1 & & & 0 \\ & e_2 & & \\ & & \ddots & \\ 0 & & & e_m \end{pmatrix} \tag{3.3}$$

and where T is a matrix of the size m by (d-m). We denote by $t_{j,k}$ the (j,k) element of the matrix T.

Now we define a set of generalized reflections by

$$G_j = \begin{pmatrix} 1 & 0 & & \cdots\cdots & & & 0 \\ 0 & 1 & & & & & 0 \\ 0 & \cdots & e_j & 0 \cdots & t_{j,1} & \cdots\cdots & t_{j,d-m} \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & 0 \\ 0 & & \cdots\cdots & & & 0 & 1 \end{pmatrix} \qquad (j=1,2,\ldots,m) \tag{3.4}$$

All the diagonal elements are one except for the j-th element which is $e_j = \exp(2\pi i a_{j,j})$. All the off-diagonal elements are zero except for the j-th row whose last (d-m) elements are those of the j-th row vector of the matrix T.

We claim that:

$$M_1 = G_1 G_2 \cdot \cdots \cdot G_m \tag{3.5}$$

We prove our claim by induction on m. For m=1, M_1 is trivially a generalized reflection.

Let us assume (3.5) to be true for m-1, and we write:

$$G_1 G_2 \cdots\cdots G_{m-1} = \begin{pmatrix} e_1 & & & 0 & t_{1,1} & \cdots\cdots & t_{1,d-m} \\ & e_2 & \cdots & 0 & & & \\ & & e_{m-1} & 0 & t_{m-1,1} & \cdots\cdots & t_{d-m} \\ & & & 1 & 0 & \cdots\cdots & 0 \\ & & & 0 & 1 & \cdots\cdots & 0 \\ & 0 & & & & & \\ & & & & & & 1 \end{pmatrix} \tag{3.6}$$

We multiply this matrix to the matrix:

$$G_m = I + \begin{pmatrix} 0 & \cdots\cdots & \cdots & 0 & & 0 \\ & \cdots\cdots & & & & \\ 0, & \cdots & (e_m - 1) & t_{m,1}, & \cdots\cdots & t_{m,d-m} \\ 0 & & \cdots\cdots\cdots\cdots & & & 0 \\ 0 & & \cdots\cdots & & & 0 \end{pmatrix} \tag{3.7}$$

from the left,noting that the m-th row of the first matrix has only zero elements except for the m-th, and that the second term of the second matrix has no non-zero elements in the last (d-m) rows. This completes the proof of Theorem 3.

Remark. The theorem has an important implication when all the diagonal elements $a_{1,1},\ldots\ldots a_{d,d}$ of the matrix A are rational and if our generators has a positive definite hermitian invariant. Because in such a case, we have a unitary group generated by reflections of finite periods. There is a classification of all finite subgroups (Shephard,Todd [18]) and we can prove that the solutions of our equation are algebraic functions, genralizing the famous results of H. Schwarz ([17], see Bannai-Takano [19]).

3.4. Computing Connections.

Lemma 1.

Let C_j be a d by d matrix whose elements are all zero except the j-th row:

$$C_j = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \\ c_{j,1} & c_{j,2} & \dots & c_{j,j} & \dots & c_{j,d} \\ & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix} \tag{4.1}$$

And let U and L be the upper and lower triangular matrices defined by:

$$U = \begin{pmatrix} 0 & c_{1,2} & c_{1,3} & \cdots & c_{1,d} \\ 0 & 0 & c_{2,3} & \cdots & c_{2,d} \\ & & \cdots\cdots & & \\ & & \cdots\cdots & & \\ 0 & 0 & \cdots & & c_{d-1,d} \\ 0 & 0 & & & 0 \end{pmatrix} \tag{4.2}$$

$$L = \begin{pmatrix} c_{1,1} & 0 & 0 & 0 & \cdots & 0 \\ c_{2,1} & c_{2,2} & 0 & 0 & \cdots & 0 \\ c_{3,1} & c_{3,2} & c_{3,3} & 0 & \cdots & 0 \\ & & \cdots\cdots & & & \\ & & \cdots\cdots & & & \\ c_{d,1} & c_{d,2} & c_{d,3} & \cdots & & c_{d,d} \end{pmatrix} \tag{4.3}$$

Then the following identity holds:

$$[\ I+C_1\ ][\ I+C_2\ ]\ \cdots\cdots\ [\ I+C_d\ ] = [\ I-U\ ]^{-1}[\ I+L\ ] \tag{4.4}$$

Proof.

We first show the following auxiliary identity for $k=0,1,\dots,(d-1)$.

$$\left[\sum_{j=1}^{k} C_j - U \right] \cdot C_{k+1} = 0 \tag{4.5}$$

Since the non-zero elements of C_{k+1} appears only on the (k+1)-th row, we are only concerned with the (k+1)-th column of the matrix $[\Sigma_{j=1}^{k} C_j - U]$. By definition, the first k-rows of this matrix is equal to the first k rows of the lower triangular matrix L, and the rest of (d-k) rows are those of -U.

$$\left[\sum_{j=1}^{k} C_j - U \right] = \begin{pmatrix} c_{1,1} & 0 & 0 & \cdots\cdots & 0 & 0 & & 0 \\ c_{2,1} & c_{2,2} & 0 & \cdots\cdots & 0 & 0 & & 0 \\ & \cdots\cdots\cdots\cdots & & & & & & \\ c_{k,1} & c_{k,2}, \cdots & & c_{k,k} & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & -c_{k+1,k+2} & \cdots & -c_{k+1,d} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -c_{k+2,d} \\ & & \cdots\cdots\cdots\cdots & & & & & \\ 0 & 0 & & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{4.6}$$

Hence the (k+1)-th column vector of the matrix is a null vector showing (4.5).

Now we prove by induction:

$$[I-U][I+C_1][I+C_2] \cdots [I+C_k] = I + \left[\sum_{j=1}^{k} C_j - U \right] \tag{4.7}$$

For k=1, we have

$$[I-U][I+C_1] = I - UC_1 + C_1 - U \tag{4.8}$$

where the matrix UC_1 is a null matrix because the upper triangular matrix U has only zeros on the first column, and the matrix C_1 has non-zero elements only on the first row. Hence the identity (4.7) is true for k=1.

To complete our induction, we multiply the matrix $[I+C_{k+1}]$ on the both sides of eq.(4.7), and write down the right hand side of it as follows:

$$[\ I + \sum_{j=1}^{k} C_j - U\ ]\ [\ I+C_{k+1}\ ] = I + \sum_{j=1}^{k+1} C_j - U + [\ \sum_{j=1}^{k} C_j - U\ ]\cdot C_{k+1} \qquad (4.9)$$

Again, the last term is a null matrix because of our identity (4.5). Finally, we have for k=d:

$$[\ I-U\ ][I+C_1][I+C_2] \quad [I+C_d] = I + \sum_{j=1}^{d} C_j - U = I + L \qquad (4.10)$$

which completes the proof of our lemma, because [I-U] is always invertible.

Now let us write

$$c_{j,k} = \begin{cases} (e_j - 1)t_{j,k} & (k \neq j) \\ (e_j - 1) & (k = j) \end{cases} \qquad (4.11)$$

and apply our decomposition theorem into generalized reflections to $M_1,\ldots,M_r$, reminding of the fact that

$$t_{j,k} = 0 \qquad \text{for} \quad \lambda_j = \lambda_k \qquad (4.12),$$

then we obtain the following explicit representation theorem for the generator M_∞ of eq.(2.7).

Theorem 4.

We have

$$M_\infty = M_1 \cdot M_2 \cdot \cdots\cdots M_r = [\ I-U\ ]^{-1}[\ I + L\ ] \qquad (4.13)$$

and

$$\det[\ M_\infty - fI\ ] = \det[\ [U+L] -(f-1)I\ ] \qquad (4.14).$$

Proof, We only have to show (4.14).

$$\det[\,[I-U]^{-1}[I+L]-fI\,] = \det[I-U]^{-1}\det[\,I+L-f[I-U]\,]$$

$$= \det[\,I+L+U-fI\,]. \qquad (4.15)$$

?

This completes the proof.

Remark. The number of unknown constants in the matrix I+U+L is

$$N' = d^2 - \sum_{j=1}^{r} m_j^2 \qquad (4.16)$$

by eq.(2.5). We call this matrix "the indicatrix" of our original system (1.1).

Theorem 5. If the number of accessory parameters is zero, then we can determine the generators with respect to a fundamental set $X(t)D$, where D is a some non-singular diagonal matrix.

Proof. The number of accessory parameters N is given by eq.(1.19) of Chap.II:

$$N = d^2-d+2 - \sum_{j=1}^{r} m_j^2 - \sum_{k=1}^{s} n_k^2 \qquad (4.17)$$

where $\operatorname{rank}(A-\rho I) = d-n_k$, or n_k is the multiplicity of the eigenvalue ρ of A without non-trivial subdiagonal elements in the Jordan canonical form of A. Then by the behavior of the local canonical set at $t = \infty$, $f_k=\exp(2\pi i\rho)$ has the same multiplicity in eq.(4.14). Then, by lemma 1 of Chap.II, we have $\sum_{k=1}^{s} n_k^2$ conditions. But by Riemann-Fuchs theorem, we have

$$e_1e_2\cdots\cdots e_d = \exp(2\pi i \sum_{j=1}^{d} a_{j,j}) = \exp(2\pi i \sum_{k=1}^{s} \rho_k) = f_1f_2\cdots f_d \qquad (4.18)$$

Consequently, we are left with

$$d^2 - \sum m_j^2 - (\,\sum n_k^2-1) = N+(d-1)$$

constants. A constant multiple of our singular solution $x_k(t)$ is a solution. And if we multiply some nonsingular diagonal matrix D from the right, we again have a fundamental set of solutions with prescribed singular behavior, since every solution is multiplied by a certain constant. This multiplication by D will give us (d-1) degree of freedom, and hence the number of undetermined constants becomes exactly the number N of accessory parameters.

In this way we have a monodromy representation unique up to an automorphism by a diagonal transformation. This completes the proof of our main assertion, Theorem 5.

3.5. Quadratic Invariant.

The conclusion of this section is not, yet, satisfactory. We believe that the very strong assumption we had to impose can be removed someday. We briefly summarise what we have done in sections 3.1.-3.4..

1. We found an intrinsically important fundamental set of solutions $X(t)$.
2. Local Canonical set $X_j(t)$ were found for every $t=\lambda_j$ leaving possibly the least number of connection coefficients in $X(t)=X_j(t)T_j$ (cf.(1.11)).
3. Generators M_j for the representation with respect to $X(t)$ were described in terms of T_j. (cf. (2.1)).
4. Generators M_j ($j=1,2,\ldots,r$) are decomposed into generalized reflections (cf.(3.5)).
5. From the topological property of $\mathbb{P}^1-S$, we showed the possibility of determining T_j within the equivalence by diagonal transformations.

We urge the reader to remind of the similarity of the step 5, with the step we determined the matrix A from given sets of singularities S and characteristic exponents $[a_{1,1},\ldots\ldots,a_{d,d}:\rho_1,\rho_2,\ldots,\rho_d]$, cf. Theorem 1.2., in Chap.II.. A diagonal transformation was introduced in the proof in the later step because it is the most general transformation which leaves given sets invariant. But in our step 5. of this chapter, there is no such intrinsic reason for the introduction of a diagonal transformation, we only know that a diagonal transformation of $X(t)$ and accompanying $X_j(t)$'s leaves the form of M_j's and T_j's invariant.

The heart of the analysis of H.Schwarz's study of hypergeometric differential equation is the classification of the universal covering surfaces of solutions into three types: conformal to an open disk, complex plane, or $\mathbb{P}^1$. An interpretation in our language, i.e., a representation in $GL(2,C)$, is given in our expository note (Okubo,K.[22]) in which the classification is done by constructing a quadratic hermitian invariant for the representation.

<u>Theorem 6</u>. Let H be the inverse of the matrix:

$$\begin{pmatrix} 1 & t_{1,2} & t_{1,3} & \cdots\cdots\cdots & & t_{1,d} \\ t_{2,1} & 1 & t_{2,3} & \cdots\cdots\cdots & & \\ & & & & & \\ t_{j,1} & t_{j,2}\cdots & & 1 & \cdots & t_{j,d} \\ & \cdots\cdots\cdots & & & & \\ & \cdots\cdots\cdots & & & & \\ t_{d,1} & t_{d,2} & \cdots & t_{d,j} & t_{d,d-1} & 1 \end{pmatrix} = H^{-1} \qquad (5.1)$$

(3,4)の $t_{ij}/(1-e_i)$ と思ってよい。
III-10.

where $t_{j,k}$ $(j,k=1,2,\ldots,d)$ are the connection coefficients determined by (1.3).

If the following conditions holds, then H defines a Hermitian invariant.

(i) H exists .

(ii) H is hermitian .

<u>Proof</u>. Let us write an element x of C^d as a horizontal vector:

$$x = (x^1, x^2, \ldots\ldots, x^d) \qquad (5.2)$$

We write the complex conjugate of the transpose of x by x*. Similarly, we express by H* the transposed complex conjugate of H, by e_j^* the complex conjugate of the number $e_j = \exp(2\pi i a_{j,j})$ and so on. We have, by definition:

$$H^* = H \qquad (5.3)$$

and

$$e_j e_j^* = e_j^* e_j = 1 \qquad (5.4).$$

H defines a quadratic form:

$$H(x) = xHx^* \qquad (5.5)$$

which is an invariant of our representation if for any element g of the group G generated by the generators $M_1, M_2, \ldots, M_r$, if the following identity holds:

$$H(x) = H(x\cdot g) = xgHg^*x = xHx^* \tag{5.6}$$

By our decomposition theorem 3., it is sufficient to show that for all generalized reflections:

$$G_j = I + (e_j - 1)\begin{pmatrix} 0 & 0 \ldots & 0 \ldots & 0 \\ \ldots\ldots & & & \\ t_{j,1} & t_{j,2} \ldots & 1 \ldots\ldots & t_{j,d} \\ 0 & 0 \ldots & 0 \ldots & 0 \\ & \ldots\ldots & & \\ 0 & 0 & \ldots\ldots & 0 \end{pmatrix} = I+(e_j - 1)Q_j \tag{5.7}$$

we have:

$$H = G_j H G_j^* \quad (j=1,2,\ldots,d) \tag{5.8}$$

Let us compute the right hand side of the identity (5.8).

$$H = [I+(e_j - 1)Q_j]H[I+(e_j^* - 1)Q_j^*]$$

$$= H + (e_j - 1)Q_jH + (e_j^* - 1)HQ_j^* + (e_j - 1)(e_j^* - 1)Q_jHQ_j^* \tag{5.9}$$

Since H is defined as the inverse of the matrix whose k-th row is identical with that of Q_j, and since $H=H^*$, we have

$$Q_jH = HQ_j^* = (Q_jH)^* = E_{j,j} \tag{5.10}$$

where $E_{j,j}$ is the matrix with all the elements zero but 1 on the j-th diagonal. Similarly, hence the sole non-zero elements on the j-th row of Q_j^* is 1, we have

$$H = H + [(e_j - 1) + (e_j^* - 1) + (e_j - 1)(e_j^* - 1)]E_{j,j} = H \tag{5.11}$$

In computing the number in bracket we used the relation (5.4) to eliminate the coefficient of the matrix $E_{j,j}$. This completes the proof of Theorem 6.

The first condition H^{-1} exists can be removed if we examine the relation with the matrix [U+L] with the definition (4.13). Namely, we have:

$$H^{-1} = [E-I]^{-1}[U+L] \qquad (E=\mathrm{diag}(e_1,e_2,\dots,e_d)) \tag{5.12}$$

On the other hand, we have from (4.14), we see that eigenvalues of the matrix [U+L] is given by: $f_j-1 = \exp(2\pi i\rho_j)-1$ ($j=1,2,\dots,d$). Consequently, we have

$$\det([E-I]^{-1}[U+L]) = \prod_{k=1}^{d} [f_k-1]/[e_k-1] \tag{5.13}$$

Thus the condition (i) of the theorem is removed under the assumptions made in the foregoing chapters. ($a_{k,k}$'s and ρ_j's are not integers).

If we can use the arbitrariness of (d-1) parameters by a diagonal transformations properly, we can construct H so that it is hermitian. This is true for the case of classical hypergeometric functions and genralized hypergeometric functions. But without assuming some symmetry for the matrix A, it might be impossible to remove the condition (ii) of the theorem.