Chapter 5 REDUCIBILITY

5.1. Definitions.

A system of linear ordinary differential equations of the first order with rational functions of t as coefficients:

$$dx_j/dt = \sum_{k=1}^{d} a_{j,k}(t)x_k \qquad (j=1,2,\ldots,d) \tag{1.1}$$

is said to be reducible iff there is a non-singular transformation:

$$x=T(t)y \quad : x_j(t) = \sum_{k=1}^{d} T_{j,k}(t)y_k \tag{1.2}$$

$$\det(T(t)) \neq 0 \tag{1.3}$$

such that the transformed system:

$$dy/dt = B(t)y = [T^{-1}(t)A(t)T(t)-T^{-1}(t)T'(t)]y \tag{1.4}$$

has the reducible coefficient matrix B(t), where,of course, we assumed that all the elements $T_{j,k}(t)$ $(j,k=1,2,\ldots,d)$ of the transformation matrix T(t) are rational functions of t.

A matrix B(t) of size d by d is a reducible matrix when it is considered as a linear transformation

$$B(t): C^d \to C^d \tag{1.5}$$

it has an invariant proper subspace V:

$$B(t)V \subset V \tag{1.6}$$

for all t.

For an example, if we can take this invariant subspace V as the linear subspace of the first p ($1\le p<d$) components of the vector y, then the matrix B(t) necessarily has the form:

$$B(t) = \left(\begin{array}{c|c} B_{1,1}(t) & 0 \\ \hline B_{2,1}(t) & B_{2,2}(t) \end{array} \right) \qquad (1.7)$$

where the respective sizes of the blocks $B_{1,1}(t)$, $B_{2,1}(t)$ and $B_{2,2}(t)$ are p by p, p by (d-p) and (d-p) by (d-p) respectively.

Similarly, if there is a non-singular constant matrix C such that the transformed matrix $C^{-1}B(t)C$ has a off diagonal block with all the elements zero:

$$C^{-1}B(t)C = \left(\begin{array}{c|c} B_{1,1}(t) & 0 \\ \hline B_{2,1}(t) & B_{2,2}(t) \end{array} \right) \qquad (1.8)$$

then B(t) is reducible.

Let $S=[\lambda_1,\lambda_2,\dots,\lambda_r,\infty]$ be the set of poles of the matrix A(t). Let X(t) be some fundamental set of solutions of the system (5.1). Let be a closed circuit in $\mathbb{P}^1-S$, and let $X(t)M(\gamma)$ be the results of analytic continuation of X(t) along γ . We call the representation of the fundamental group $\pi_1(\mathbb{P}^1-S)$ in GL(d,C) defined by:

$$\rho:\gamma \to M(\gamma), \quad (\rho: G=\pi_1(\mathbb{P}^1-S) \to GL(d,C)) \qquad (1.9)$$

the monodromy representation of (5.1) with respect to the fundamental set X(t), as in the preceding chapter.

Alinear representation of a group G is reducible if there is a proper non-trivial invariant subspace for all the elements.

Theorem. 1.

If the system (5.1) is reducible, then every monodromy representation is reducible.

Corollary.

If a monodromy representation is irreducible for (5.1), then (5.1) is irreducible (= not reducible).

Proof of the theorem. By a non-singular rational transformation (5.2), we get a new system of the form:

$$\begin{aligned} y_1' &= B_{1,1}(t) \\ y_2' &= B_{2,1}(t)y_1 + B_{2,2}(t) \end{aligned} \qquad (1.10).$$

We have a non-trivial set of solutions for which $y_1=0$ identically, that is , we have a fundamental set of solutions of the form:

$$Y(t) = \left(\begin{array}{c|c} Y_{1,1} & 0 \\ \hline Y_{2,1} & Y_{2,2} \end{array} \right) \qquad (1.11)$$

If $Y_{1,1}(t)$ be a matrix of the size p by p, $Y_{2,1}(t)$ be (d-p) by p, and $Y_{2,2}(t)$ be (d-p) by (d-p), then it is clear that the representation with respect to this set has the form:

$$\rho(\gamma) = \left(\begin{array}{c|c} C_{1,1} & 0 \\ \hline C_{2,1} & C_{2,2} \end{array} \right) \qquad (1.12)$$

Hence any proper non-trivial vector space contained in the subspace whose first p components are zero is invariant.

Remark. Although the converse to the theorem seems to be known as a pure existence theorem, my intension was to give a constructive proof for it. The following theorem is a very poor answer for my intension which can only be applied to the classical second order equations.

Theorem 2. If a monodromy representation of the system (5.1) has a (d-1)-dimensional invariant subspace, and if it is a fuchsian system, then the system it self is reducible.

Procf. We may assume that the invariant subspace V is the set of vectors whose d-th component is zero. Let the group be generated by $g_1, g_2, \ldots\ldots, g_p$ with respect to the definite fundamental set of solutions

$$X(t) = [\ x_1(t), x_2(t), \ldots\ldots, x_d(t)\ ] \qquad (\ x_j(t) \text{ is a vertical vector}) \quad (1.13).$$

By the particular choice of the invariant subspace V, the generators have the form:

$$g_j = \left(\begin{array}{c|c} G_{1,1} & \begin{matrix} 0 \\ 0 \\ \cdot \\ 0 \end{matrix} \\ \hline g_{d,1}, \ldots\ldots, g_{d,d-1}, & \exp(-2\pi i c_j) \end{array}\right) \qquad (1.14)$$

where $G_{1,1}$ is a matrix of the size (d-1) by (d-1), and the constant c_j is the negative value of a characteristic root at the singular point $t=\lambda_j$, which can be uniquely determined up to an integral difference.

By a simple observation, we see the vector solution $x_d(t)$ is transformed into $x_d(t)\cdot\exp(-2\pi i c_j)$ by the circuit around the singular point $t=\lambda_j$. Hence the column vector

$$y(t) = (t-\lambda_1)^{c_1}\cdots\cdots(t-\lambda_r)^{c_r}x_d(t) = r(t)x_d(t) \qquad (5.15)$$

is single-valued in the entire complex plane. On the other hand we assumed the system to be fuchsian, that is, $y(t)$ is a rational function with at most poles on $S=[\lambda_1,\ldots\ldots,\lambda_r,\infty]$. We can easily compute the system of first order equations satisfied by the rational vector $y(t)$ as follows:

$$dy/dt = [d/dt][r(t)x_d(t)] = r'(t)x_d(t) + r(t)A(t)x_d(t)$$

$$= [\ r'(t)/r(t)\cdot I + A(t)\ ]y(t) \qquad (1.16).$$

Let $R(t)$ be a d by d non-singular rational matrix whose d-th column vector is $y(t)$. Then the d-th column vector of the matrix:

$$dR/dt - A(t)R(t) - r'(t)/r(t)\cdot R(t)$$

is identically zero. Multiply $R^{-1}(t)$ from the left, and we see that the matrix:

$$T(t) = R(t)^{-1}[dR(t)/dt - A(t)R(t) - [r'(t)/r(t)]R(t)] \qquad (1.17)$$

has zero d-th column vector. Consequently, the matrix

$$B(t) = R(t)^{-1}(dR(t)/dt - A(t)R(t)\ ) \qquad (1.18)$$

has the d-th column vector

$$b_d(t) = {}^t(0,0,\ldots\ldots,r'(t)/r(t)) \qquad (1.19)$$

That is the matrix $B(t)$ has an invariant non-trivial $(d-1)$-dimensional subspace V' whose d-th component is zero. The actual transformation matrix which carries the system (5.1) to (5.4) in this case is given by $R(t)$. We only specified the d-th column vector of $R(t)$, and the rest of $(d-1)$ columns can be chosen arbitrarily so that $R(t)$ is non-singular: e.g.,

$$R(t) = \left(\begin{array}{cccc|c} 1 & & & 0 & \\ & 1 & & & \\ & & 1 & & \\ 0 & & & & y(t) \\ & & & 1 & \\ \hline 0 & 0 & \dots & 0 & \end{array}\right) \qquad (1.20)$$

is a possible choice. This completes the proof of theorem 2..

We can not give no more general results of the above type, but for some specific examples, we can give certain criterions for the reducibility or the irreducibility of the given system.

5.2. An Irreducibility Theorem.

A directed finite graph is a pair (V,F) where V is a finite set of points called vertices:

$$V = [\ v_1, \dots, v_r\ ] \qquad (2.1)$$

and F is a set of multivalued functions defined on V into V:

$$F: V \to V.$$

A directed arc u is defined by a pair of vertices (v,v') such that v' is in $F(v)$:

$$u = [\ (v,v'): v' \varepsilon F(v)\ ] \qquad (2.2)$$

A path P is a string of arcs $u_j=(v_j,v_j')$ such that $v_{j+1}=v_j'$:

$$P = [\ (v_j,v_j'):\ v_{j+1}=v_j'\varepsilon F(v_j)\quad j=1,\dots,p\ ] \qquad (2.3).$$

A graph is strongly connected if there is atleast one path P connecting any two given vertices.

An adjacency matrix of a graph (V,F) is a matrix whose (j,k) element is either zero or one according to

$$v_k \not\varepsilon F(v_j) \qquad \text{or} \qquad v_k \ \varepsilon\ F(v_j).$$

A finite directed graph determines its matrix of adjacency uniquely: and conversely, a square matrix determines an equivalent class of directed graphs if all of its elements are either zero or one. The equivalence is defined by certain permutations of vertices and identifying arcs connecting the same pair of points.

The following theorem gives a practical method of determining a given group generated finitely by generalized reflections is irreducible or not.

Theorem 3. Let G be a subgroup G of GL(d,C) generated by d generalized reflections:

$$[I+C_j:\ j=1,2,\dots,d]$$

where C_j is a matrix of the size d by d whose row vectors are all null vectors except for the j-th row $(c_{j,1},\dots\dots,c_{j,d})$.

And let (V,F) be one graph defined by the matrix Q whose (j,k) the element is zero or one according to

$$c_{j,k} = 0 \qquad \text{or} \qquad c_{j,k} \neq 0 \qquad (2,4).$$

Then when $\det(U+L)=\det(\sum_{j=1}^{d} C_j) \neq 0$, the group G is irreducible if the graph (V,F) is strongly connected.

Proof. Suppose V is an invariant linear subspace of C^d under the transformations of G. This is equivalent that

$$V(I+C_j) \subset V \qquad \text{for } j=1,2,\ldots,d \tag{2.5}$$

We will show that all the row vectors:

$$v_j = (c_{j,1}, c_{j,2}, \ldots\ldots, c_{j,d}) \quad j=1,2,\ldots\ldots,d \tag{2.6}$$

are contained in V, and by the assumption, we can show that are linearly independent and span the whole space C^d

If V is non-trivial, then there must be a vector v which has at least one of the component non-zero. Let this component be the j-th component and let μ be its value. Then since both v and $v(I+C_j)$ are contained in V, we have

$$v_j = (1/\mu)[\ v(I+C_j) - v\ ] \tag{2.7}$$

contained in V. Then by the strong connectivity of the graph (V,F), there is a sequence of numbers $p_1, p_2, \ldots, p_s$ such that there is a path

$$[(v_{p_r}, v_{p_r}') : r=1,2,\ldots,s\ ] \tag{2.8}$$

such that for any number k ($k \neq j$, $k=1,2,\ldots.,d$) we have

$$v'_{p_s} = v_k \tag{2.9}$$

The sequence of paths (2.8) is equivalent to the sequence of transformations of the type (2.7), and we may interprete vertices as vectors (2.6).

That is, any other vector v_k is obtained as a linear combinations of the transformed

$$\begin{cases} v_j \\ v_j(I+C_{p_1}) \\ \cdots\cdots \\ v_j(I+C_{p_1})(I+C_{p_2})\cdots\cdots(I+C_{p_s}) \end{cases} \tag{2.10}$$

Consequently, we have all of the vectors $v_1,\ldots,v_d$ contained in the vector space V if VG is contained in V. This completes the proof.

The irreducibility criterion can be applied directly for d=2 without any modifications. We have two generators:

$$M_o = I + (e_o-1)\begin{pmatrix} 1 & p \\ 0 & 0 \end{pmatrix} \tag{2.11}$$

$$M_1 = I + (e_1-1)\begin{pmatrix} 0 & 0 \\ q & 1 \end{pmatrix} \tag{2.12}$$

where we have

$$e_o = \exp(2\pi i a_{2,2}) \tag{2.13}$$

$$e_1 = \exp(2\pi i a_{1,1}) \tag{2.14}$$

$$pq = [e_o+e_1-f_1-f_2]/(e_o-1)(e_1-1) = \sin(\rho_1-a_{1,1})\pi\cdot\sin(\rho_2-a_{1,1})\pi/K$$

$$(K = \sin(a_{1,1}\pi)\sin(a_{2,2}\pi)\sin(a_{1,1}-a_{2,2})\pi\ )$$

and

$$\det(M_oM_1) = (e_o-1)(e_1-1)(1-pq) = (e_o-1)(e_1-1)\prod_{j=1}^{2}\sin(\pi\rho_j)/\sin(\pi a_{j,j})$$

$$\ldots\ldots \quad (2.15)$$

For the computations (2.11)-(2.15) we refer pp.17-19 of Chap.IV.

<u>Proposition.</u> The system

$$(t-\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix})\,dx/dt = \begin{pmatrix}a_{1,1} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{pmatrix} x$$

is irreducible if

(i) $a_{1,1}, a_{2,2}, \rho_1, \rho_2$ are not integers.

(ii) $\rho_j - a_{k,k}$ for $j,k=1,2$ are not integers.

The above conditions for more general systems can be stated without any difficulty by modifying the arguments used in the proof of Theorem 3. There are two well known partition pairs:

$d=1+1+ \ldots\ldots +1 =(d-1)+1$ (Genralized Hypergeometric equation or Jordan Pochhammer equation)

$2d = d+d = d+(d-1)+1$ (Goursat-Sasai equation for $d=2$).

We refer the articles T.Kimura,[7] ,K.Okubo [10] and T.Sasai [15] for further informations.

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