Chapter I Reduction to the Canonical Form

1. The Set of Finite Singular Points.

We prepare some elementary propositions about the set of finite singular points of the differential equations, i.e., the set of the roots of the leading coefficients.

Let S be a set of finite complex numbers:

$$S = [\ \lambda_1, \lambda_2, \ldots\ldots, \lambda_d] \tag{1.1}$$

The number d of the elements in the set S is fixed throughout this section. The members λ_j's are not necessarily distinct. Hence we partition the set S into the equivalent classes of identical members:

$$S = P_1 \cup P_2 \cup \ \ldots\ldots \cup P_r \tag{1.2}$$

where for each j, P_j consists of m_j identical complex number a_j:

$$P_j = [\ \underbrace{a_j, a_j, \ldots\ldots\ldots\ldots, a_j}_{m_j}] \tag{1.3}$$

Naturally, we have the identity:

$$\sum_{j=1}^{r} m_j = d \qquad (\ 0 < m_j \leq d\ ) \tag{1.4}$$

To give a definite description of the properties of the family of subsets of S, we assume that the partitioned classes $P_1, P_2, \ldots\ldots, P_r$ are so ordered that the sequence of numbers $m_1, m_2, \ldots, m_j$ makes a non-decreasing sequence:

$$0 < m_1 \leq m_2 \leq \ldots\ldots \leq m_r \tag{1.5}$$

And, similarly, we assume that the members $\lambda_1,\lambda_2,\ldots,\lambda_d$ of the set S is ordered induced from the ordering of the partition. Thus for an exmample, if

$$S = P_1 \cup P_2 \cup P_3 \tag{1.6}$$

$$P_1 = [\ a,a\ ],\ P_2 = [\ b,b,b\ ],\ P_3 = [\ c,c,c,c\ ]$$

then we understand that

$$\lambda_1=\lambda 2=a,\ \lambda_3=\lambda_4=\lambda_5=b,\ \lambda_6=\lambda_7=\lambda_8=\lambda_9=c$$

In the following discussions, we call two subsets A and B of S are equal to each other if they are the same sets as sets of finite complex numbers. Thus in the above example the set $A=[\lambda_1,\lambda_5,\lambda_8]$ is equal to the set $B=[\lambda_2,\lambda_3,\lambda_6]$. We find it convenient to attach a label to a subset of S so that two same sets to have a same label. If A consists of n_1-elements from the set P_1, n_2-elements from $P_2,\ldots\ldots$, and n_r-elements from the set P_r, then we define the label L(A) to be the string of r numbers:

$$L(A) = n_1 n_2 \ldots . n_r \tag{1.7}$$

$$A = [\ a_1,\ldots,a_1\ ] \cup [\ a_2,\ldots,a_2] \cup \ldots\ldots [\ a_r,\ldots,a_r\ ]. \tag{1.8}$$

Since A is a subset of S, we cnly admit of numbers $n_1,n_2,\ldots\ldots,n_r$ such that

$$0 \leq n_j \leq m_j \qquad (\ j = 1,2,\ldots,r), \tag{1.9}$$

consequently, if we define $p=m_r+1$, and consider the label L(A) as an expression of a natural number with the base p:

$$L(A)= n_1 \cdot p^{r-1} + n_2 \cdot p^{r-2} + \ \ldots\ldots +n_r \tag{1.10}$$

we have a one to one correspondence between the equivalent classes of subsets of S and some subsets S* of natural numbers. The important point is that we introduced a linear order into the family of subsets of the set S.

For a given positive number k less than d (=the number of elements in S), we generates a sequence of subsets $S_o, S_1, \ldots, S_k$ as follows:

$$
\begin{aligned}
S_o &= [\ \lambda_1, \lambda_2, \ldots\ldots, \lambda_j, \lambda_{j+1}, \ldots\ldots, \lambda_k] \\
S_1 &= [\ \lambda_1, \lambda_2, \ldots\ \qquad\qquad , \lambda_{k-1}, \lambda_d\ ] \\
S_2 &= [\ \lambda_1, \ldots\ldots \qquad\qquad , \lambda_{k-2}, \lambda_{d-1}, \lambda_d\ ] \\
&\ldots\ldots\ldots\ldots\ldots \\
S_j &= [\ \lambda_1, \ldots, \lambda_{k-j}, \lambda_{d-j+1}, \ldots\ldots \quad \lambda_d\ ] \\
S_{j+1} &= [\ \lambda_1, \ldots, \lambda_{k-j-1}, \lambda_{d-j}, \ \ldots\ldots\ , \lambda_d\ ] \\
&\ldots\ldots\ldots\ldots \\
S_k &= [\ \lambda_{d-k+1}, \ldots\ldots \qquad \ldots\ldots \quad \lambda_d\ ]
\end{aligned}
\tag{1.11}
$$

The Rule for the generation of this sequence of subsets is that the set S_{j+1} is obtained from the set S_j by replacing the elements λ_{k-j} with the element λ_{d-j} ($k < d$, $1 \leq j \leq k$) and the starting set is the set consists of the first k elements.

<u>Proposition 1.1</u>. For the sequence of subsets $S_o, S_1, \ldots, S_k$, we have

$$L(S_o) \geq L(S_1) \geq \ldots\ldots\ldots \geq L(S_k) \tag{1.12}$$

where the total number of equalities q in the above sequence is given by the sum

$$q = \sum_j [m_j-(d-k)] \tag{1.13}$$

where the summation is taken over those j for which the summands are positive.

Corollary. q is the number of elements in the intersection:

$$Q = \bigcap_{j=o}^{k} S_j \tag{1.14}$$

This intersection is taken as the intersection of finite complex numbers, not as the ordered set of λ_j's. ない

Proof. Two successive numbers $L(S_j)$ and $L(S_{j+1})$ are the same if and only if the two elements λ_{d-j} and λ_{k-j} are the same because S_{j+1} is obtained from S_j by interchanging these two elements. This happens when these two elements belong to the same equivalence class, say, P_i. Since P_i contains m_i elements, the possible number of equalities in this class is given by $[\ m_i-(d-k)+1\ ]$. See the illustration of the situation below.

$$\begin{array}{l} \overbrace{a_i,a_i,\ldots\ldots\ldots,a_i,\ldots\ldots,a_i}^{m_i} \\ \| \qquad\qquad\qquad\quad \| \\ \underbrace{\lambda_{k-j} \longleftrightarrow \lambda_{d-j}}_{(d-k)} \end{array} \tag{1.15}$$

Naturally, if m_i+1 is not greater than $(d-k)$ then there is no possible chance for an equality,hence the proposition follows.

To prove the corollary, we observe that every set S_j contains exactly k elements from S. That is, we obtain S_j from S by excluding $(d-k)$ elements. Similarly, we can not exclude more than $(d-k)$ elements from one specific class P_i. Hence we always include in S_j at least $[m_i-(d-k)]$ elements of the class P_i if this number is positive. Thus $q^* = \sum_i [m_i-(d-k)]$ is the least number of elements in the intersection of any family of subsets of S with exactly k elements. Let Q^* be the subset of S with the label $L(Q^*)=n_1n_2\ldots n_r$ where n_i is defined by

$$n_i = \begin{cases} [m_i-(d-k)] & (m_i > (d-k)) \\ 0 & (m_i \leq (d-k)). \end{cases} \tag{1.16}$$

Since Q^* is the least possible common intersection of all the subsets of S with exactly k elements, Q^* is appearently a subset of $Q=\bigcap_{j=o}^{k} S_j$.

To show Q is actually equal to Q^*, we assume that some a_i is contained in every S_j more than $[m_i-(d-k)]$ times. But this is impossible because every S_j is the union of the first (k-j)-elements of S and the last j-elements of S, and in between, there are only (d-k)-elements. See the situation below.

$$\underbrace{a_1,\ldots,a_1;\ a_2,\ldots\ldots;a_i,\ldots,a_i}_{k-j}\,\boxed{\overset{d-k}{a_i,\ldots,a_i}}\,\underbrace{a_i,\ldots;a_{i+1},\ldots\ldots;a_r,\ldots\ldots,a_r}_{j} \tag{1.17}$$

It is clear that this set S_j can not contain more than $[m_i-(d-k)]$-elements. Consequently, i-th digit of the label $L(Q) \leq n_i$ for every i, which shows that Q is a subset of Q^*. This completes the proof of the corollary.

<u>Proposition 1.2.</u> There are (k-q+1) distinct numbers among

$$L(S_o-Q), L(S_1-Q), \ldots\ldots, L(S_k-Q) \tag{1.18}$$

<u>Proof</u>. It is evident from the definition of the label that

$$L(A-B)=L(A)-L(B) \tag{1.19}$$

when B is a proper subset of A. By subtracting L(Q) from the inequalities of the prop.1.1., we see there are only q equalities among the inequalities:

$$L(S_o-Q) \geq L(S_1-Q) \geq \ldots\ldots\ldots \geq L(S_k-Q) \tag{1.20}$$

which shows there are exactly (k-q+1) distinct numbers among the sequence.

Example. Consider the set $S=[\lambda_1,\lambda_2,\ldots\ldots,\lambda_9]$ with

$$\lambda_1=\lambda_2=a, \quad \lambda_3=\lambda_4=\lambda_5=b, \quad \lambda_6=\lambda_7=\lambda_8=\lambda_9=c$$

The set S, with d=9, is now partitioned into:

$$P_1=[a,a], P_2=[b,b,b], P_3=[c,c,c,c]$$

with $m_1=2, m_2=3, m_3=4$. We illustrate our propositions for the cases k=8 and 7.

(i) The case k=8. By definition, we write down the sets $S_o,\ldots\ldots,S_8$:

$$S_o=S_1=S2=S_3 = [\ a,a,b,b,b,c,c,c\ ]$$
$$S_4=S_5=S_6 = [\ a,a,b,b,c,c,c,c\ ]$$
$$S_7=S_8= [\ a,b,b,b,c,c,c,c\ ]$$

Now we compute $n_1=m_1-(d-k)=1$, $n_2=m_2-(d-k)=2$, $n_3=m_3-(d-k)=3$. This shows L(Q)=123 and Q=[a,b,b,c,c,c] which is exactly the intersection of the sets $S_o,\ldots\ldots,S_9$. Since q=1+2+3, we have k-q+1=3 distinct members among $S_o-Q,\ldots\ldots,S_8-Q$:

$$S_o-Q=S_1-Q=S_2-Q=S_3-Q = [a,b]$$
$$S_4-Q=S_5-Q=S_6= [a,c]$$
$$S_7-Q=S_8-Q= [b,c]$$

(ii) The case k=7. We have $(d-k)=2, n_1=m_1-2=0, n_2=m_2-2=1, n_3=m_3-2=2, q=1+2=3, k-q+1=5$. We list the sets by the labels:

$L(S_o)=L(S_1)=L(S_2)=232$, $L(S_3)=L(S_4)=223$, $L(S_5)=214$, $L(S_6)=124$, $L(S_7)=034$

$L(Q)=012$

There are five distinct labels 220,211,202,112,022 for $L(S_j-Q)$.

Let us compute the labels L(S -Q):

$$L(S - Q) = L(S) - L(Q) = m_1 m_2 \ldots m_r - n_1 n_2 \ldots n_r \qquad \text{(base p)}$$

where

$$m_i - n_i = \begin{cases} m_i - 0 & (m_i \leq (d-k)) \\ m_i - [m_i - (d-k)] & (m_i > (d-k)) \end{cases}$$

Since the numbers m_i are non-decreasing sequence of numbers, there is a natural number h such that:

$$L(S - Q) = m_1 \ldots m_h \overbrace{(d-k) \ldots (d-k)}^{r-h} \qquad \text{(base p).} \qquad (1.21).$$

Proposition 1.3. Let $S_o^*, S_1^*, \ldots\ldots, S^*_{k-q}$ be defined from the set $S^* = S - Q$ (dコ, qコ)
by the same rule as (1.11), then there is a number j ($0 \leq j \leq k$) such that

S*より、k-qコ d-qコ.
で、S^*_0,… をつくる。

$$S_j - Q = S^*_i \qquad (1.22)$$

for each $i = 0, 1, 2, \ldots, k-q$.

Proof. It is clear that each set S^*_i contains exactly k-q elements, so that the union of S^*_i and Q consists of exactly k elements.

$$\begin{array}{|ccccccc|l}
\hline
a_1 & a_2 & \cdots\cdots & a_h & a_{h+1} & \cdots & a_r & \\
\cdots & \cdots & \cdots & \cdots & & & & \\
a_1 & a_2 & \cdots\cdots & a_h & a_{h+1} & & a_r & \\
 & \cdots & \cdots & & & & & (d-k) \\
 & a_2 & & a_h & a_{h+1} \cdots & & a_r & \\
 & & & \cdots & \cdots & \cdots & & \\
 & & & a_h & a_{h+1} \cdots & & a_r & \\
 & & & & \cdots & \cdots & & \\
\hline
 & & & & a_{h+1} \cdots & \cdots & a_r & \\
 & & & & & \cdots & & Q \\
 & & & & & & a_r & \\
\hline
\end{array}$$

$(m_1 \; m_2 \; \ldots\ldots \; m_h \; m_{h+1} \; \ldots\ldots \; m_r$ .. ...number of elements in $P_1, \ldots, P_r$).

We tabulated the elements of S. Those elements above the horizontal line are the the elements of $S^* = S - Q$, and those below are the elements of the intersection Q. All these elements are numbered λ_j ($j = 1, 2, \ldots, d$) vertically from above starting from the left most column first and then moving to the right.

Since the set S^*_i consists of the first k-q-i elements of S^* and the last i elements of S^*, we can obtain the set S^*_i by deleting the successive

$$(d-k)-(k-q-i)-i = (d-k)$$

elements from S^*. Now there are only three possibilities.

(i) All the deleted elements are the elements of the classes $P_1,\ldots,P_h$.

(ii) All the deleted elements are the elements of the classes $P_1,\ldots,P_h$ and some elements of P_{h+1}. Since the class P_{h+1} contains more than (d-k) elements it is impossible that any further classes contribute some elements to S^*_i.

(iii) All the deleted elements belong to some class $P_{h+\mu}$, or $P_{h+\mu+1}$.

In the first case it is evident that S^*_i Q consists of the first (k-i) elements and the last i-elements of S. Hence $S^*_i=S_i-Q$. In the second case, suppose there are σ elements from the class P_{h+1} deleted, then we have deleted $(d-k-\sigma)$-successive elements from the classes from $P_1,\ldots,P_h$. But even if we adjoin the elements of Q to S^*, these elements are successive (d-k) elements, and consequently, there is at least one (possibly n_i) j such that $S^*_i=S_j-Q$.

In the third case, after adjoining the elements of Q, we renumber some of the elements λ_j's in the class $P_{h+\mu}$ so that these (d-k) elements have successive numbers as their index. Then we claim that these (d-k) elements are successive elements of S, so that there is some number j such that $S^*_i=S_j-Q$ as a set of complex numbers. This completes the proof of the proposition 1.3..

<u>Definition</u>. Let A be a subset of the set S, then we define a polynomial $\varphi_A(t)$ by:

$$\varphi_A(t) = \prod_{\lambda\varepsilon A} (t-\lambda) \tag{1.23}$$

when A is the subset $S_j(j=0,1,2,\ldots,k)$ specified by the rule (1.11), we abbreviate

$$\varphi_{S_j}(t) = \varphi_j(t) \qquad (j=0,1,\ldots,k<d\ ) \tag{1.24}$$

Lemma. If the set of polynomials $[\varphi_o(t),\varphi_1(t),\ldots,\varphi_k(t)]$ is free of a common factor, then the set is a linearly independent set. (1.25)에 의하여.

Proof. We prove this lemma by induction on k (k<d).

For k=1, we have $\varphi_o(t)=(t-\lambda_1)$, and $\varphi_1(t)=(t-\lambda_d)$. They are naturally linearly independent if $\lambda_1 \neq \lambda_d$.

Suppose the statement is true for all 1,2,....,(k-1). Now we remind of the fact that the last one $\varphi_k(t)$ is obtained from $\varphi_{k-1}(t)$ by:

$$\varphi_k(t)=[(t-\lambda_{d-k+1})/(t-\lambda_1)]\,\varphi_{k-1}(t) \tag{1.25}$$

The factor in the bracket can not be 1, since in that case $(t-\lambda_1)$ is a common factor. Suppose there is a non-trivial linear relation

$$\sum_{j=o}^{k} c_j\varphi_j(t) = 0 \tag{1.26}$$

Since every polynomial $\varphi_o(t)$, $\varphi_1(t),\ldots,\varphi_{k-1}(t)$ contains a factor $(t-\lambda_1)$, by setting $t=\lambda_1$, we have $c_k=0$. Now we claim $[\varphi_o(t)/(t-\lambda_1),\ldots,\varphi_{k-1}(t)/(t-\lambda_1)]$ can not have a common factor. If they have one, then from eq.(1.25), then it is a factor of $\varphi_k(t)$. Hence this is a linearly independent set and we have:

$$c_o=c_1=\ldots\ldots=c_{k-1}=0$$

which completes the proof of the lemma.

Remark. To be more precise, we apply the induction to the set $S-[\lambda_1]$.

The following theorem is the main result of this section.

Theorem 1. For Q defined by eq.(1.14), $\varphi_Q(t)$ is the greatest common factor of the polynomials $[\varphi_j(t): j=0,1,2,\ldots,k]$. And the sequence $[\varphi_j(t)/\varphi_Q(t): j=0,1,2,\ldots,k]$ contains $(k-q+1)$ distinct polynomials and they are linearly independent.

Proof. From Prop.1.3., we can select $(k-q+1)$ distinct polynomials from the set $[\varphi_j(t)/\varphi_Q(t): j=0,1,2,\ldots,k]$. They are linearly independent by the preceding lemma.

Example. Let us consider the set

$$S= [a,a,a,a,b,b,b,b,c,c,c,c,c,c,c,c] \quad , \quad d=16 \ ,$$
$$\varphi_S(t)=(t-a)^4(t-b)^4(t-c)^8 \quad , \quad m_1=m_2=4, m_3=8$$

We take up the special case k=11, then since d-k=5, we have $n_1=4, n_2=4$, and $n_3=5$; q=3. From the set $S^*=S-Q = [a,a,a,a,b,b,b,b,c,c,c,c,c]$, we construct a sequence of polynomials of order $(k-q+1)=9$ by deleting $[(n_1+n_2+n_3)-(k-q)]=(d-k)=5$ successive elements. We list the corresponding polynomials $\varphi_j(t)/\varphi_Q(t)$ $(j=0,\ldots,9)$

$(t-a)^4(t-b)^4, (t-a)^4(t-b)^3(t-c), (t-a)^4(t-b)^2(t-c)^2, (t-a)^4(t-b)(t-c)^3, (t-a)^4(t-c)^4, (t-a)^3(t-c)^5, (t-a)^2(t-b)(t-c)^5, (t-a)(t-b)^2(t-c)^5$, and $(t-b)^3(t-c)^5$.

Their labels form a non-increasing sequence of natural numbers base 6;
440,431,422,413,404,305,215,125,035.

To show the linear independence of these 9 polynomials, we substitute t=a in the linear relation:

$$\sum_{j=0}^{9} c_j[\varphi_j(t)/\varphi_Q(t)] = 0$$

and we have $c_9=0$. Since the rest of the polynomials have (t-a) as a common factor, we list the labels of the quotient $[\varphi_j(t)/\varphi_Q(t)]/(t-a)$:

340,331,322,304,205,115,025

It is clear, that these are the labels of the sequence of polynomials obtained from the set:

$$S' = [a,a,a,b,b,b,b,c,c,c,c,c]$$

by deleting 5 successive elements. Note that the label of the set S'

$$L(S') = 345$$

satisfy the condition (1.5). We apply the same argument inductively, each time showing that the coefficient of the last term is zero in the linear combinations of the polynomials which are expressed in labels as follows:

440,431,422,413,404,305,215,125,035
340,331,322,313,304,205,115,025
240,231,222,213,204,105,015
140,131,122,113,104,005
40, 31, 22, 13, 4
30, 21, 12, 3
20, 11, 2
10, 1

Naturally, the last pair corresponds to [(t-a),(t-b)] and they are linearly independent.

Finally, just for the sake of completeness, we list the labels of the original subsets $S_o,\ldots\ldots,S_{11}$.

443,443,443,443,434,425,416,407,308,218,128,038

There are 3 equalities among the 4 labels in the left most position.

2. Reduction Process.

Proposition 2.1. If the following d-th order linear ordinary differential equation is fuchsian, namely every singular point is a regular singular point, with one at infinity:

$$L[x] = \sum_{k=o}^{d} C_k(t)[d/dt]^{n-k} x = 0 \tag{2.1}$$

where all the coefficients $C_o(t), C_1(t), \ldots\ldots, C_d(t)$ are polynomials in t, then we have:

$$\deg(C_k(t)) \leq \deg(C_{k-1}(t)) - 1 \tag{2.2}$$

$$m(C_k, \lambda) \leq m(C_{k-1}, \lambda) - 1 \tag{2.3}$$

where by $m(f,\lambda)$ we mean the multiplicity of the root λ of the given polynomial $f(t)$.

Proof. The condition (2.2) is the well-known condition that the singular point at infinity is regular. Similarly, the inequality (2.3) is the condition that the finite point λ is a regular singular point of eq. (2.1).

Remark. Naturally, we use the convention $m(f,\lambda)=0$ if $f(\lambda)\neq 0$. We are only concerned with the case where $m(f,\lambda)$ is a natural number.

Given a single d-th order differential equation (2.1) with

$$\deg(C_o(t)) = d, \tag{2.4}$$

we will transform the equation into a system of d first order equations of the form:

$$\begin{cases} (t-\lambda_j)[dy_j/dt] = \sum_{k=1}^{j} a_{j,k}\, y_k + y_{j+1} & (j=1,2,\ldots,d-1) \\ (t-\lambda_d)[dy_d/dt] = \sum_{k=1}^{d} a_{d,k}\, y_k & \end{cases} \tag{2.5}$$

where, of course, we define the dependent variables $y_1, y_2, \ldots, y_d$ recursively as:

$$\begin{cases} y_1 = x & \\ y_{k+1} = (t-\lambda_k)[dy_k/dt] - \sum_{p=1}^{k} a_{k,p}\, y_p & (k=1,2,\ldots,d) \end{cases} \tag{2.6}$$

and where $a_{j,k}$ ($1 \leq k \leq j$) are constants to be determined so that

$$y_{d+1} = 0 \tag{2.7}$$

holds. The set of numbers $[\lambda_1, \lambda_2, \ldots, \lambda_d] = \hat{S}$ is the set of roots of the polynomial $C_o(t)$ ($\deg(C_o(t)) = d$) arranged in the order described as in the preceding section.

<u>Lemma 2</u>. If we write $y_1, y_2, \ldots, y_{d+1}$ in the form:

$$y_k = \sum_{j=1}^{k} b_{k,j}(t)\, x^{(j-1)} \tag{2.8}$$

then the coefficients $b_{k,j}(t)$ ($k \geq j$, $k=1,2,\ldots,d$) satisfy:

$$b_{k+1,j}(t) = (t-\lambda_k) b_{k,j-1}(t) + (t-\lambda_k)[d/dt] b_{k,j}(t) - \sum_{p=j}^{k} a_{k,p} b_{p,j}(t) \tag{2.9}$$

$$b_{k+1,k+1}(t) = \varphi_S(t) \qquad (S=[\lambda_1, \lambda_2, \ldots, \lambda_k]) \tag{2.10}$$

and

$$b_{k+1,1}(t) = \det\begin{pmatrix} a_{1,1} & 1 & 0 & \cdots\cdots & \\ a_{2,1} & a_{2,2} & 1 & & \\ & \cdots\cdots\cdots & & & \\ & \cdots\cdots\cdots & & & 1 \\ a_{k,1} & a_{k,2} & & \cdots\cdots & a_{k,k} \end{pmatrix} \cdot (-1)^k \qquad (2.11)$$

<u>Proof</u>. By induction on k. For k=1, we have $y_2=(t-\lambda_1)x' - a_{1,1}x$. That is,

$$b_{2,2}(t) = (t-\lambda_1), \quad b_{2,1}(t) = -a_{1,1} \quad (b_{1,1}(t)=1)$$

The equalities (2.10) and (2.11) are trivially satisfied. The right hand side of eq.(2.9) has only the first term well-defined meaning for j=2, and it has only the third term well-defined for j=1. Thus eq.(2.9) is true for k=1.

Then we substitute eq.(2.8) into the right hand side of eq.(2.6). We have

$$y_{k+1} = (t-\lambda_k)[\sum_{j=2}^{k+1} b_{k,j-1}x^{(j-1)} + \sum_{j=1}^{k} b'_{k,j}\, x^{(j-1)}]$$
$$- \sum_{p=1}^{k} a_{k,p} [\sum_{j=1}^{p} b_{p,j}\, x^{(j-1)}]$$

where $b'_{k,j}$ denotes $(d/dt)b_{k,j}(t)$.

Since the derivative of order k appears only in the first summation with j=k+1, we have:

$$b_{k+1,k+1}(t) = (t-\lambda_k)\; b_{k,k}(t) = (t-\lambda_k)(t-\lambda_{k-1}) \;\cdots\cdots (t-\lambda_1) = \varphi_S(t)$$

which shows (2.10).

Now for j=1 to k, we pick up the coefficients of $x^{(j-1)}$, and interchanging the order of summation as:

$$\sum_{p=1}^{k} \sum_{j=1}^{p} = \sum_{j=1}^{k} \sum_{p=j}^{k}$$

we obtain (2.9) for $j \neq 1$. For $j=1$, we have $b_{1,1}=1$ and $b_{2,1} = -a_{1,1}$. Suppose $b_{k,1}$ is independent of t for k=1 up to k, then we have:

$$b_{k+1,1}(t) = -\sum_{p=j}^{k} a_{k,p} b_{p,1} \tag{2.12}$$

independent of t.

Now, if we write down (2.12) for k=1 to k+1, we have

$$\left\{\begin{array}{l} b_{2,1} + a_{1,1} = 0 \\ b_{3,1} + a_{2,2} b_{2,1} + a_{2,1} = 0 \\ \quad \cdots\cdots\cdots\cdots\cdots \\ b_{k,1} + a_{k-1,k-1} b_{k-1,1} + \cdots\cdots \qquad + a_{k-1,1} = 0 \\ b_{k+1} + a_{k,k} b_{k,1} + \cdots\cdots \qquad + a_{k,2} b_{2,1} + a_{k,1} = 0. \end{array}\right. \tag{2.13}$$

If we use the vector matrix notation , then we can write (2.13) in the following form:

$$\begin{bmatrix} a_{1,1} & 1 & & & \\ a_{2,1} & a_{2,2} & 1 & & \\ & \cdots & \cdots & \cdots & \\ & & & & 1 \\ a_{k,1} & a_{k,2} & & & a_{k,k} \end{bmatrix} \begin{bmatrix} 1 \\ b_{2,1} \\ \\ b_{k-1,1} \\ b_{k,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \\ -b_{k+1,1} \end{bmatrix} \tag{2.14}$$

We solve this equation as if we knew the vector on the right hand side, and the vector $(1, b_{2,1}, \ldots\ldots, b_{k-1,1}, b_{k,1})$ as unknown. We are interested only in the first component, that is, by Cramer's rule,

$$1 = \det \begin{bmatrix} 0 & 1 & \cdots\cdots & & \\ 0 & a_{2,2} & 1 & & \\ a_{3,} & a_{3,2} & a_{3\,3} & 1 & \\ & \cdots & \cdots & \cdots & \\ 0 & a_{k-1,2} & a_{k-2,3} & & 1 \\ -b_{k+1,1} & a_{k,2} & & & a_{k,k} \end{bmatrix} \Big/ \det A$$

$$= (-1)^k b_{k+1,1} \,/\, \det A \tag{2.15}$$

where A is the matrix on the left hand side of eq.(2.14). This completes the proof of lemma 2.

<u>Theorem 2.1.</u> The coefficient $b_{j+1,k+1}(t)$ for $j \geq k$, $j=1,2,\ldots,d$, is a polynomial of degree k. If we define the set S and its family of subsets $S_o,\ldots,S_k$ by,

$$\begin{aligned}
S &= [\lambda_1,\lambda_2,\ldots\ldots,\lambda_k,\ldots,\lambda_j] \\
S_o &= [\lambda_1,\lambda_2,\ldots\ldots,\lambda_k] \\
S_1 &= [\lambda_1,\lambda_2,\ldots,\lambda_{k-1},\lambda_j] \\
&\ldots\ldots\ldots\ldots \\
S_k &= [\lambda_{j-k+1},\ldots,\lambda_j]
\end{aligned} \tag{2.16}$$

and the associated polynomials of order k, $\varphi_o(t),\ldots,\varphi_k(t)$ by,

$$\varphi_p(t) = \prod_{\lambda \varepsilon S_p} (t-\lambda) \qquad (p=0,1,\ldots,k) \tag{2.17}$$

then we have:

$$b_{j+1,k+1}(t) = \psi(t) - \sum_{p=o}^{k} a_{j-p,k+1-p} \cdot \varphi_p(t) \tag{2.18}$$

where $\psi(t)$ is a polynomial of order k whose coefficients dependent on those $a_{r,s}$ for which

$$(r-s) \geq (j-k) \tag{2.18}$$

hold.

<u>Proof.</u> For the sake of convenience, we call (j-k) the index of the coefficient

$a_{j,k}$. Then the diagonal elements $a_{k,k}$ of the matrix A all have the index zero. For our specific matrix A in (2.11), all the elements with index (-1) has numerical value equal to one. We prove eq.(2.18) by induction on the index (j-k).

For (j-k)=0, we have $S=S_o=S_j$. By eq.(2.10), we have:

$$b_{k+1,k+1}(t) = \varphi_S(t)$$

the right hand side of this equation does not depend on any $a_{1,1}, \ldots, a_{k,k}$.

Suppose now the induction hypothesis is satisfied for (j-k)=0,1,...,(q-1), and we try to show eq.(2.18) for (j-k)=q. This we do by induction on k. For k=0, we have

$$b_{j+1,1}(t)=b_{q+1,1}(t) = \det A_q = \begin{vmatrix} a_{1,1} & 1 & & \\ \cdots & \cdots & \cdots & \\ & \cdots & \cdots & 1 \\ a_{q,1}, & \cdots & \cdots & a_{q,q} \end{vmatrix}$$

Every element appearing in this matrix A_q except $a_{q,1}$ has index less than (q-2). Hence, we have

$$b_{q+1,1} = (-1)^{q-1} a_{q,1} + \psi_o \qquad (2.20)$$

where ψ_o is a constant (a polynomial of degree 0) depending only on those $a_{r,s}$ with index less than (q-1).

By lemma 2, we have

$$b_{j+1,k+1}(t) = (t-\lambda_j)b_{j,k}(t) + (t-\lambda_j)b'_{j,k+1}(t) - \sum_{p=k+1}^{j} a_{j,p}b_{p,k+1}(t) \qquad (2.21)$$

This is nothing but eq.(2.9) with the index pair (k+1,j) replaced by (j+1,k+1). In the last summation, we have $(j-p) \leq (j-k-1)=q-1$, and $(p-k-1) \leq (j-k-1)=q-1$.

We pick up terms with coefficients having indices less than (q-1) and group them together under the name $\psi(t)$. The rest of terms on the right, namely, those with coefficients of index (q-1) are:

$$(t-\lambda_j)[\ -\sum_{p=o}^{k-1} a_{j-1-p,k-p}\cdot\varphi^*_p(t)\ ]\ -a_{j,k+1}\cdot b_{k+1,k+1}(t)$$

$$= -\sum_{p=o}^{k} a_{j-p,k+1-p}\ \varphi_p(t) \qquad (2.22)$$

because $b_{k+1,k+1}(t) = \varphi_S(t)$ by eq.(2.10), and

$$(t-\lambda_j)\varphi^*_p(t) = (t-\lambda_j)\prod_{\lambda\varepsilon S^*_p}(t-\lambda)$$

where, by the induction hypothesis, we have:

$S^* = [\lambda_1,\lambda_2,\ldots\ldots,\lambda_k,\ldots.\lambda_{j-1}]$

$S^*_o = [\lambda_1,\lambda_2,\ldots\ldots,\lambda_{k-1}]$

$S^*_1 = [\lambda_1,\ldots.,\lambda_{k-2},\lambda_{j-1}]$

....

$S^*_{k-1} = [\lambda_{j-k+1},\ldots\ldots,\lambda_{j-1}]$

There is no contribution of index (q-1) from the second term $(t-\lambda_j)b'_{j,k+1}(t)$ because $b_{j,k+1}(t)$ contains terms $a_{j-1-p,k+1-p}$ $(p=0,1,2,..,k)$ as terms of the highest index but the index is (q-2). This completes the proof of Theorem 2.

Example. Consider $S=[\lambda_1,\lambda_2,\lambda_3]$, $d=3$. Then, we have by lemma 2. and Theorem 2.1.,

$$y_1=x,\ y_2=(t-\lambda_1)x'-a_{1,1}x,\ y_3=(t-\lambda_1)(t-\lambda_2)x''+[\psi_1(t)-a_{2,2}(t-\lambda_1)-a_{1,1}(t-\lambda_2)]x' +(a_{1,1}a_{2,2}-a_{2,1})x$$

$$y_4 = (t-\lambda_1)(t-\lambda_2)(t-\lambda_3)x^{(3)} + [\psi_2(t) -a_{3,3}(t-\lambda_1)(t-\lambda_2)-a_{2,2}(t-\lambda_1)(t-\lambda_3) -a_{1,1}(t-\lambda_2)(t-\lambda_3)]x'' + [\psi_3(t)-a_{3,2}(t-\lambda_1)-a_{2,1}(t-\lambda_3)]x' - \det(A_3)x$$

3. Structure of the Coefficients of Fuchsian Equations.

Proposition 3.1. For a factored monic polynomial:

$$\varphi(t) = \prod_{\lambda \varepsilon S} (t-\lambda) \qquad (|S| = d) \tag{3.1}$$

the k-th order derivative $\varphi^{(k)}(t)$ is the sum,

$$\varphi^{(k)}(t) = \Sigma \prod_{\lambda \varepsilon T} (t-\lambda) \tag{3.2}$$

where T runs for all possible subset of the set S with (d-k) elements, possibly with some repetitions.

Proof. For k=1, we have

$$\varphi'(t) = \sum_{k=1}^{d} \prod_{\lambda \varepsilon [S-[\lambda_k]]} (t-\lambda) \tag{3.3}$$

We can repeat the same computation to the terms in summation any number of times. Hence the proposition follows.

Proposition 3.2. Every polynomial $\psi(t)$ appeared in Theorem 2.1., is a linear combination of polynomials $\varphi_A(t)$ where A's are some finite subsets of $S=[\lambda_1,\lambda_2,\ldots\ldots,\lambda_d]$.

Proof. Since the proof of Theorem 2.1. depends on the recurrence formulae (2.9) and (2.10), we examine the operations involved therein. The first term in (2,9) is a multiplication by $(t-\lambda_k)$. That is:

$$\varphi_A(t) \quad \to \quad \varphi_B(t) \qquad (B= A \cup [\lambda_k]).$$

where A is a subset of $[\lambda_1,\lambda_2,\ldots\ldots,\lambda_{k-1}]$. The second term contains the multiplication by $(t-\lambda_k)$ and a differentiation. By the preceding proposition, we have

$$[d/dt]\varphi_A(t) = \sum_B \varphi_B(t) \qquad (B\subset A) .$$

The third term is a linear combination of polynomials.

Now consider the set of all linear combinations of the form:

$$L_k = [\varphi\colon \varphi = \sum_p c_p \varphi_{A_p}(t) \quad A_p \subset [\lambda_1,\lambda_2,\ldots\ldots,\lambda_k] \subset S]$$

Then this set is invariant under the operations

(i) multiplication by $(t-\lambda_k)$ when all the A_p are subsets of $[\lambda_1,\ldots\ldots,\lambda_{k-1}]$.

(ii) differentiation

(iii) by taking a linear combination .

Now the proof is complete with the formula (2.10) when we observe that all the coefficients $b_{j,k}(t)$ $(k\geq j,k=1,2,\ldots,d)$ are the elements of respective set L_k.

<u>Proposition 3.3.</u> If $t=\lambda$ is a finite regular singular point of the d-th order linear differential equation:

$$L[x] = C_o(t)x^{(d)} + C_1(t)x^{(d-1)}+ \ldots\ldots\ldots +C_{d-1}(t)x' + C_d(t)x = 0 \qquad (3.4)$$

where the leading coefficient $C_o(t)$ is given by:

$$C_o(t) = \varphi_S(t) \qquad (3.5)$$

$$S = [\lambda_1,\lambda_2,\ldots\ldots,\lambda_d] \qquad (3.6)$$

then we have:

$$m(C_k,\lambda) \geq m(C_o,\lambda) - k \qquad (3.7)$$

Proof. This follows immediately from the inequality (2.3) of lemma 2.

<u>Corollary</u>. Let the sets $S_o, S_1, \ldots\ldots, S_k$ $(k<d)$ be the subsets of the zeros of $C_o(t)$ defined by (1.11). And let $Q = \bigcup_{j=o}^{k} S_j$, then $C_{d-k}(t)$ is divisible by $\varphi_Q(t)$.

Proof. The inequality (3.7) implies:

$$m(C_{d-k},\lambda) \geq m(C_o,\lambda)-(d-k)$$

Consequently, $C_{d-k}(t)$ is divisible by $(t-\lambda)^{m-(d-k)}$ ($m=m(C_o,\lambda)$), and by eq.(1.16) $C_{d-k}(t)$ is divisible by $Q(t)$.

<u>Theorem 3.1.</u> Every fuchsian equation of order d of the type (3.4) can be transformed into a system of first order equations of the form:

$$(tI-B)[dy/dt] = Ay \tag{3.8}$$

where A is a matrix of the type:

$$\begin{bmatrix} a_{1,1} & 1 & 0 & \cdots\cdots & & 0 \\ a_{2,1} & a_{2,2} & 1 & \cdots\cdots & & 0 \\ & \cdots\cdots\cdots & & & & \\ a_{d-1,1} & \cdots\cdots & & & a_{d-1,d-1} & 1 \\ a_{d,1} & a_{d,2} & \cdots\cdots & & a_{d,d-1} & a_{d,d} \end{bmatrix}$$

and B is a diagonal matrix with the roots of $C_o(t)$ on the diagonal, and where y is a d-dimensional vector defined recursively by (2.6).

Proof. From the definition (2.6), all we have to show is the existence of $1+2+\ldots.+d = d(d+1)/2$ elements of the matrix A such that:

$$y_{d+1} = L[x] \tag{3.9}$$

By eq.(2.8), we write:

$$y_{d+1}(t) = \sum_{j=o}^{d} b_{d+1,j}(t)x^{(j)} \tag{3.10}$$

where $b_{d+1,j}(t)$ ($j=0,1,\ldots,d$) are given by (2.18) as:

$$b_{d+1,j}(t) = \psi(t) - \sum_{p=o}^{j-1} a_{d-p,j-p}\, \varphi_p(t) \tag{3.11}$$

and where by eq.(2.10):

$$b_{d+1,d+1}(t) = \varphi_S(t) = C_o(t) \tag{3.12}$$

We determine $a_{j,k}$ inductively with respect to the index j-k. Let us put j=d in (3.11). We try to determine $a_{1,1},\ldots,a_{d,d}$ from the equation:

$$C_1(t) = \psi(t) - \sum_{p=o}^{d-1} a_{d-p,d-p}\varphi_p(t) \tag{3.14}$$

where $C_1(t)$ is divisible by $\varphi_Q(t)$, the maximal common factor of $\varphi_o,\ldots,\varphi_{d-1}$, by Corollary to the proposition 3.3., while $\psi(t)$ is a linear combination of polynomials $\varphi_{S*}(t)$ of order (d-1) where S* is a subset of S with (d-1) elements independent of $a_{j,k}$. Hence, $\psi(t)$ is, also, divisible by $\varphi_Q(t)$. Let the order of φ_Q be q. Then the order of $[C_1(t)-\psi(t)]/\varphi_Q(t)$ is equal to (d-1-q) and it contains (d-q) preassigned constants. But by theorem 1, there are (d-q) linearly independent polynomials among the set [$\varphi_p(t)/\varphi_Q(t)$: $p=0,1,\ldots,(d-1)$]. Hence we can determine $a_{1,1},\ldots,a_{d,d}$ so that eq.(3.14) is satisfied.

To determine the constants [$a_{d-p,j-p}$: $p=0,1,2,\ldots,(d-j)$] from

$$C_{d-j}(t) = b_{d+1,j}(t) = \psi(t) - \sum_{p=o}^{j-1} a_{d-p,j-p}\, \varphi_p(t) \tag{3.15}$$

where $\varphi_o(t),\ldots,\varphi_{j-1}(t)$ corresponding to the sets

$$S_o = [\lambda_1,\lambda_2,\ldots\ldots,\lambda_{j-1}]$$
$$S_1 = [\lambda_1,\ldots\ldots,\lambda_{j-2},\lambda_d\ ]$$
$$\ldots\ldots\ldots$$
$$S_{j-1} = [\lambda_{d-j+1},\ldots\ldots,\lambda_d]$$

and where $\psi(t)$ is a linear combination of polynomials of the type $\varphi_{S*}(t)$ for some $S^*\ \ S$ with $(j-1)$ elements whose coefficients depend on those $a_{r,s}$ with indices $(r-s)$ less than $(d-j)$. Similarly, we have both $\psi(t)$ and $C_{d-j}(t)$ divisible by $\varphi_Q(t)$ for $Q= \ _p S_p$. The degree of the polynomial $(C_{d-j}(t)-\psi(t))/\varphi_Q(t)$ is $(j-1-q)$ and hence contains $j-q$ constants, but this is exactly the number of linearly independent polynomials of degree $(j-1)$: $\varphi_o(t),\varphi_1(t),\ldots,\varphi_{j-1}(t)$. We choose $(j-q)$ polynomials which are linearly independent and let

$$a_{d-p,\,j-p} = 0 \tag{3.16}$$

if $\varphi_p(t)$ is not chosen, and determine other constants $a_{d-p,\,j-p}$'s so that the following equation is satisfied;

$$[\ C_{d-j}(t)-\psi(t)\ ]/\varphi_Q(t) = \sum_{(a_{d-p,\,j-p}\neq 0)} a_{d-p,\,j-p}\varphi_p(t) \tag{3.17}$$

Of course, it is not necessary to have the right hand side of eq.(3.16) always equal to be zero, we may assign any convenient value if there is any. In any case, we have a unique set of constants from eq.(3.17) for a given numerical value on the right hand side of eq.(3.16). This completes the proof.

<u>Corollary.</u> Every single linear ordinary differential equation of fuchsian type is equivalent to a system of first order equations of the type (3.8).

Proof. By a fractional linear transformation of the independent variable t, we may assume that the given equation has a regular singular point at infinity so that the condition (2.2) is satisfied when certain polynomial is multiplied.

1. $C_o(t)x^{(d)}+C_1(t)x^{(d-1)} + \ldots\ldots+C_d(t)x = 0 \qquad \deg(C_d) > 0,\ \deg(C_c)=d+k.$

2. $C_o(t)x^{(d)}+ \quad \ldots\ldots \quad +C_d(t)x=0 \qquad \deg(C_d)=0, \deg(C_o)=d.$

3. $C_o(t)x^{(d)}+ \quad \ldots\ldots +C_k(t)x^{(d-k)} = 0 \qquad \deg(C_k)=0, \deg(C_o)=k.$

Then above three cases exhaust the possibility. The preceding theorem deals with the case 2 only. For the first case we change the dependent variable x by

$$x= z^{(k)}$$

similarly, for the third case, we use

$$x^{(d-k)} = z$$

to reduce the equation to the second case. This completes the proof of the corollary.

Remark. The classical problem of "one accessary parameter" occurs for the type 1, with d=2,k=1. Our reduction process brings such an equation to a system of three first order equations. This process is an introduction of appearent excess order, in contrast to the introduction of appearent singular points by classical analysists.