# Chapter II Study of the Canonical System.

## 1. Number of Parameters.

Let d be a fixed integer, and we consider a system of first order equations for $x=(x_1,x_2,\ldots\ldots,x_d)$ of the form:

$$(tI-B)x'=Ax \tag{1.1}$$

where both A and B are d by matrix with constants as their elements, and where I denotes d by d identity matrix. We write the system (1.1) in the form:

$$(t-B)x' = Ax$$

when the existence of the identity matrix is clear from the context, we will use this convention throughout this paper.

We assume the following conditions.

(i) B is a diagonal matrix with eigenvalues ( diagonal elements )

$$S = [\lambda_1,\lambda_2,\ldots\ldots,\lambda_d] \tag{1.2}$$

For the sake of definiteness we assume that these eigenvalues are ordered as in the section 1 of the preceding chapter. Namely, identical eigenvalues are grouped together, and these groups are ordered in the non-decreasing order of the number of elements in respective groups. We, also, use labels $L(S) = m_1m_2,\ldots m_r$.

(ii) The eigenvalues of the matrix A is denoted by:

$$R = [\rho_1,\rho_2,\ldots\ldots,\rho_d] \tag{1.3}$$

and we use the same ordering and label to R. We assume none of the elements in R is a negative integer:

$$\rho_j \notin [-1,-2,\ldots,-n,\ldots \quad ] \tag{1.4}$$

and if for a pair of distinct integers $(j,k : j\neq k\ j,k=1,2,\ldots\ldots,d)$

$$\rho_j \neq \rho_k \qquad \text{implies} \qquad (\rho_j-\rho_k) \notin [\ \pm 1,\pm 2,\ldots\ldots,\pm n,\ldots\ldots\ ] \tag{1.5}$$

Before we state the third condition for our system, we simplify the system under the assumptions (i).

Proposition 1.1. The most general constant linear transformation T which leaves the matrix B invariaant has the form:

$$T = \left[\begin{array}{c|c|c|c} T_1 & 0 & \ldots & 0 \\ \hline 0 & T_2 & \ldots & 0 \\ \hline & \cdots & \cdot & \\ \hline 0 & 0 & \ldots & T_r \end{array}\right] \tag{1.6}$$

where $T_i$ $(i=1,2,\ldots,r)$ denotes a non-singular $m_i$ by $m_i$ matrix, and where O's are null matrices of appropriate sizes.

Proof. Trivial.

(iii) When we write the matrix A blockwise as:

$$A= \left[\begin{array}{c|c|c|c} A_{1,1} & A_{1,2}, & \ldots\ldots & A_{1,r} \\ \hline A_{2,1} & A_{2,2}, & \ldots & A_{2,r} \\ \hline & \begin{matrix}\ldots\ldots \\ \ldots\ldots\end{matrix} & \begin{matrix}\ldots\ldots \\ \ldots\ldots\end{matrix} & \\ \hline A_{r,1} & A_{r,2}, & \ldots\ldots & A_{r,r} \end{array}\right] \tag{1.7}$$

where $A_{j,k}$ is a matrix of the size $m_j$ by $m_k$ for the label of S: $L(S)=m_1 m_2,\ldots m_r$, then each diagonal block $A_{1,1},\ldots,A_{r,r}$ are diagonalizable, hence we assume they are already diagonalized. Further, we assume that none of the diagonal elements $a_{1,1}, a_{2,2},\ldots\ldots, a_{d,d}$ is an integer:

$$a_{j,j} \notin [\ 0,\pm 1,\pm 2,\ldots\ldots,\pm n,\ldots\ldots\ ] \tag{1.8}$$

Proposition 1.2. The most general constant linear transformation which leaves the matrices $B, A_{1,1}, A_{2,2},\ldots\ldots,A_{r,r}$ invariant is a diagonal transformation if we are to fix the ordering of the diagonal elements of each block $A_{k,k}$'s.

Proof. Trivial. Let $D=\mathrm{diag}[d_1,d_2,\ldots\ldots,d_d]$, then the $(j,k)$ element of the matrix $D^{-1}AD$ becomes $a_{j,k}\cdot d_j^{-1}\cdot d_k$. Hence the effect of a diagonal transformation results in choosing $(d-1)$ elements not on the diagonal line arbitrary by defining $(d-1)$ ratios $d_1/d_d, d_2/d_d,\ldots\ldots, d_{d-1}/d_d$.

Proposition 1.3. If $\lambda$ is an eigenvalue of the matrix B of multiplicity m, then $t=\lambda$ is a regular singular point of the system (1.1) with (d-m) holomorphic solutions and m singular solutions with characteristic exponents

$$a_{j,j},\ldots\ldots,a_{j+r,j+r} \tag{1.9}$$

Proof. Since none of the exponents takes an integral value, there is no logarithmic solution with exponent zero..

For the sake of simplicity, we assume :

(iv) There is no pair of diagonal elements $a_{j,j}$ and $a_{k,k}$ with an integral difference if they belong to the same block $A_{p,p}$ for some p.

Proposition 1.4. The point at infinity is an regular singular point of the system (1.1). There is no logarithmic solution. If $\rho$ is an exponent of multiplicity n, then there is a set of n independent solutions belonging to the same exponent $\rho$.

Proof. The first statement is trivial. The second and the third statement follow from the assumption (ii).

Lemma 1. If $n=\mathrm{rank}(A-\rho I)$, then $(d-n)^2$ elements of the matrix A are determined from the rest of elements.

Proof. We may suppose without loss of generality, that the first n by n principal diagonal block of the matric $(A-\rho I)$ has non-zero determinant. We write the matrix $(A-\rho I)$ in the form:

$$(A-\rho I) = \left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right] \tag{1.10}$$

where P is n by n, Q is n by (d-n), R is (d-n) by n, and where S is (d-n) by (d-n). By the assumption we can find a matrix X of the size n by (d-n) such that

$$PX = Q \tag{1.11}$$

Similarly, we have to have

$$RX = S \tag{1.12}$$

since the rank of the matrix is n. Hence we have:

$$S = RP^{-1}Q$$

which completes the proof.

Theorem 1.1. (Riemann-Fuchs Relation) We have

$$\sum_{j=1}^{d} a_{j,j} = \sum_{k=1}^{d} \rho_k \qquad (1.13)$$

Proof. This is nothing but the invariance of the trace of the matrix A.

Now we try to determine a system of the form (1.1) by giving 3 sets

$$S = [\lambda_1, \lambda_2, \ldots\ldots, \lambda_d] \quad \text{with label } L(S)=m_1 m_2 \ldots m_r \qquad (1.14)$$

$$E = [a_{1,1}, a_{2,2}, \ldots\ldots, a_{d,d}] \qquad (1.15)$$

$$R = [\rho_1, \rho_2, \ldots\ldots, \rho_d] \quad \text{with label } L(R)=n_1 n_2 \ldots n_s \qquad (1.16)$$

by counting how many elements of the matrix A is determined from them. Evidently, A has $d^2$ elements. By Prop.1.2., (d-1) elements can be chosen arbitrary by a diagonal transformation. Each block of the size $m_j$ determine $(m_j)^2$ elements ( $m_j$ diagonal elements from the set E, and $m_j{}^2-m_j$ zeros on off diagonal position). If $\rho$ is an eigenvalue of multiplicity $n_k$ of A, then we have

$$\operatorname{rank}(A-\rho I) = d-n_k \qquad (1.17),$$

hence, by Lemma 1, determine $(n_k)^2$ elements. And finally, there is one condition (1.13) known by the name Riemann-Fuchs relation.

Definition 1.1. Two partitions $m_1+m_2+ \cdots\cdots m_r = d$, and $n_1+n_2+ \cdots\cdots+n_s= d$ of a positive integer d are paired partitioned if

$$d^2 -d + 2 = \sum_{j=1}^{r} (m_j)^2 + \sum_{k=1}^{s} (n_k)^2 \qquad (1.18)$$

holds.

Definition 1.2. Given a system of first order equations of the form (1.1) with assumptions (i),(ii),(iii), and (iv), the number of accessary parameters is defined to be the difference

$$N = d^2-d+2 - \sum_{j=1}^{r} m_j{}^2 - \sum_{k=1}^{s} n_k{}^2 \tag{1.19}$$

Theorem 1.2. A triple of sets [S,E,R] determine a system of equations of the form (1.1) under the conditions (i),(ii),(iii) and (iv), if N=0.

Proof. Trivial from the definition of the number of accessary parameters.

Example 1.1. For $d=2$, $L(S)=11$, $L(R)=11$ $E=[a_{1,1},a_{2,2}]$ and $R=[\rho_1,\rho_2]$.

$N = 4-2+2-(1+1)-(1+1)=0$

In this case we specify the off-diagonal element $a_{2,1}=1$ ( or $a_{1,2}=1$) and determine $a_{1,2}$ (or $a_{2,1}$) so that

$$\det(A-\rho I) = (a_{1,1} - \rho)(a_{2,2}-\rho)-a_{1,2} = [a_{1,1}a_{2,2}-a_{1,2}]-[a_{1,1}+a_{2,2}]\rho + \rho^2$$

$$=(\rho_1-\rho)(\rho_2-\rho) .$$

The invariance of the trace relation appears as the coefficient of the $\rho$. We have

$$a_{1,2} = a_{1,1}a_{2,2} - \rho_1\rho_2$$

As is seen from this example, since the diagonal transformation leaves the exponents of the system, we should rather say that the product of the form

$$a_{j,k}\cdot a_{k,j}$$

is determined if we let the (d-1) degree of freedom left for a diagonal transformation. This remark has an important meaning when we deal with an hermitian invariant of the group.

2. Extended Gauss' Formula.

Proposition 2.1. Under the conditions (i) - (iv) of the preceding section, the system of equations (1.1) has d singular solutions of the form:

$$x_k(t) = (t-\lambda_k)^{a_{kk}} \sum_{m=0}^{\infty} g_k(m)(t-\lambda_k)^m \qquad (k=1,2,\ldots,d) \tag{2.1}$$

convergent in a disk of radius R with center at $t=\lambda_k$, where R is the minimum distance from $t=\lambda_k$ to the nearest singular point $\lambda_j$ $(\lambda_j \neq \lambda_k)$.

Proof. Since each finite singular point $\lambda_k$ of (1.1) is a regular singular point, the convergence is trivial. There is no logarithmic solution because we assumed by (iii) and (iv) that there is no integral difference among the characteristic exponents:

$$0,0,\ldots,0,a_{p,p},\ldots,a_{p+m_j,p+m_j},0,\ldots,0$$

where $m_j$ is the size of the block to which the eigenvalue $\lambda_k$ of the matrix B belongs.

For the sake of simplicity, we assume the following condition:

(v) if $\lambda_j$ is not equal to $\lambda_k$ then we have

$$|\lambda_j-\lambda_k| > |\lambda_k| > 0 \tag{2.2}$$

The condition is refered as "the pentagonal condition" because when the inequalities are replaced by equalities then the set of singularities S are at the six vertices of a regular hexagon with center at the origin. This condition can be relaxed when there is no triple of ditinct singularities $[\lambda_i,\lambda_j,\lambda_k]$ lies on a straight line. (See M.Hukuhara [ 4],R.Schafke [16] or K.Okubo [23] ).

Proposition 2.2. Under the condition (v) there is a simply connected compact domain D in which all the solutions (2.1) are convergent.

Proof. Each solution converges in an open disc, and the origin is in the intersection of these discs. Hence the set of t for which all the solutions converge is non-empty and open.

We introduce a parameter $\mu$ into (1.1) by:

$$(t-B)x' = (A+\mu)x \tag{2.3}$$

The motivation for introducing such a parameter is manyfold. Gauss called the series:

$$F(a,b,c+\mu:x) = \sum_{n=o}^{\infty} \frac{(a)_n(b)_n}{(c+\mu)_n(1)_n} x^n \tag{2.4}$$

contiguous to the series:

$$F(a,b,c:x) = \sum_{n=o}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} x^n \tag{2.5}$$

when $\mu$ is an integer.

The so-called Jordan-Pochhammer equation:

$$L[w] = Q(t)w^{(n)}-\mu Q'(t)w^{(n-1)}+(\mu+1)\mu/2.1Q''(t)w^{(n-1)}- \ldots\ldots$$

$$-R(t)w^{(n-1)}+(\mu+1)R'(t)w^{(n-2)}- \ldots\ldots = 0 \tag{2.6}$$

where $Q(t)$ and $R(t)$ are polynomials such that one of $Q(t)$ and $tR(t)$ is of degree n, while the other has degree (n-1), is obtained from a simple first order equation:

$$L[z] = Q(t)z' - R(t)z = 0 \tag{2.7}$$

by applying the Euler transformation ( Riemann-Liouville generalized differentiation):

$$D^{-\mu}L[z] = \int_{o}^{t} (t-s)^{-\mu-1}[Q(t)z'-R(t)]ds \,/\Gamma(-\mu) \tag{2.8}$$

since we may apply Leibnitz's Rule to a product of two functions when one of them is a polynomial.

The following two propositions and a corollary appears in K.Okubo [ 9 ].

Proposition 2.3. (Truesdell's F-equation) Let $X(t,\mu)$ be a solution matrix of the system (2.3), then for some solution matrix $X(t,\mu-1)$ of

$$(t-B)x' = (A+\mu-1)x \tag{2.9},$$

we have

$$[d/dt]X(t,\mu) = X(t,\mu-1) \tag{2.10}$$

Proposition 2.4. (Invariance under Euler's transformation)

We have

$$X(t,\mu) = \int_{C} (s-t)^{\mu-1}X(s,0)ds \tag{2.11}$$

for some path of integration C such that

$$[\ (s-t)^{\mu-1}(s-B)X(s,o)\ ]_{C} = 0 \tag{2.12}.$$

Corollary. If for some $\mu$, system (2.9) is integrable, then we have integral representation of the solution matrix.

In a series of papers T.Osler investigated a family of functions which can be obtained from the known special functions by applying Leibnitz Rule to infinite series. The above propositions together with the main theorem of the first chapter guarantee that the functions defined as a set of solutions of fuchsian equations are invariant under generalized derivatives and that the product of two such functions satisfies a fuchsian equation since the product has finitely many regular singular points in the entire complex plane. Although we do not treat products of solutions of fuchsian equations, we can say, that the study of the class of functions invariant under generalized derivative has a fruitful realization in the class of functions defined as solutions of fuchsian equations. (cf. [ 9],[10],[11],[15],[20] ).

Theorem 2.1.(Fundamental Theorem)

Under the assumations (i)-(v), the set of solutions defined by (2.1) is a fundamental set of solutions in a simply connected compact domain D mentioned in Prop.2.2., and we have the Wronskian:

$$W(t) = \det [\ x_1(t), x_2(t), \ldots\ldots, x_d(t)\ ] = \prod_{k=1}^{d} (t-\lambda_k)^{a_{k,k}} \frac{\Gamma(a_{k,k}+1)}{\Gamma(\rho_k+1)} \qquad \cdots\cdots\cdots (2.13)$$

Proof.

The following proof is an exact reproduction of the proof given in K.Okubo ([ 9]) and T.Sasai ([15]) except for the fact that we assumed arbitrary multiplicity to the eigenvalue of the matrix B.

Let $\mu$ be an integer and we define solutions:

$$X(t,\mu) = [x_1(t,\mu), x_2(t,\mu), \ldots\ldots, x_d(t,\mu)] \tag{2.14}$$

where $x_k(t,\mu)$ $(k=1,2,\ldots,d)$ are defined by

$$x_k(t,\mu) = (t-\lambda_k)^{a_{k,k}+\mu} \cdot \sum_{s=o}^{\infty} g_k(s,\mu)(t-\lambda_k)^s \tag{2.15}$$

with

$$g_k(0,\mu) = (\ 0,0,\ldots,1,0,\ldots.,0\ ) = e_k \tag{2.16}$$

The restriction (2.16) about the constant multiplicity of a singular solution is important in the derivation of the formula (2.13). Now if we transform the coefficient vector $g_k(s,\mu)$ by

$$g_k(s,\mu) = h_k(s,\mu) \frac{\Gamma(a_{k,k}+s+1)}{\Gamma(a_{k,k}+\mu+s+1)} \tag{2.17}$$

then by direct computation we know that the vectors $h_k(s,\mu)$ satisfy the reccurrence relation:

$$(B-\lambda_k)h_k(s+1,\mu) = (a_{k,k}-A)h_k(s,\mu) + sh_k(s,\mu) \tag{2.18}$$

with

$$h_k(0,\mu) = e_k \tag{2.19}$$

We observe that the relation (2.18) has coefficents independent of $\mu$ and the starting value $h_k(0,\mu)$ is, also, independent of $\mu$. That is, the series solution (2.15) is actually an inverse factorial series in the parameter $\mu$:

$$x_k(t,\mu) = (t-\lambda_k)^{a_{kk}+\mu} \sum_{s=o}^{\infty} [h_k(s)\Gamma(a_{k,k}+s+1)(t-\lambda_k)^s\ ] \,/\, \Gamma(a_{k,k}+s+\mu+1)$$

$$= (t-\lambda_k)^{a_{k,k}+\mu} \sum_{s=o}^{\infty} \varphi(s,t)/\Gamma(a_{k,k}+\mu+s+1) \qquad (2.20).$$

By the well known theorem for inverse factorial series ([ ],[ ]) if t is fixed or in a compact set D and if the series is convergent for some $\mu_o$, then the series (2.20) is uniformly convergent in the domain $Re(\mu) \geq \mu_o+1$ of the complex $\mu$-plane. Thus the series is not only convergent but is an asymptotic expansion in the complex parameter $\mu$, and we have for large values of $\mu$:

$$x_k(t,\mu) \cong (t-\lambda_k)^{a_{k,k}+\mu}\, e_k \qquad (\mu \to \infty) \qquad (2.21)$$

where $e_k$ is the unit vector whose components are all zero but the k-th.

On the other hand the solution $x_k(t,\mu)$ has a derivative starting with

$$[d/dt]x_k(t,\mu) = (a_{k,k}+\mu)(t-\lambda_k)^{a_{kk}+\mu-1}\, e_k + \ldots.. \qquad (2.22)$$

Our assumptions (i)-(iv) guarantee that this is the particular solution:

$$x_k(t,\mu-1) = (t-\lambda_k)^{a_{kk}+\mu-1} \sum_{s=o}^{\infty} g_k(\mu-1,s)(t-\lambda_k)^s \qquad (2.23)$$

multiplied by the factor $(a_{k,k}+\mu)$. In this way, we have:

$$[d/dt]\, X(t,\mu) = X(t,\mu-1)(D+\mu) \qquad (2.24)$$

where D is the diagonal matrix whose k-th diagonal element is $a_{k,k}$.

If we write the wronskian of the solution matrix (2.14) as $w(t,\mu)$, then we have

$$\det[\ (t-B)X'(t,\mu)\ ] = \det[\ (t-B)X(t,\mu-1)\ ] = \det[\ (A+\mu)X(t,\mu)\ ] \qquad (2.25)$$

from which it easily follows that:

$$w(t,\mu)=w(t,o)\prod_{k=1}^{d}(t-\lambda_k)^{\mu}\frac{\Gamma(\mu+a_{k,k}+1)\Gamma(\rho_k+1)}{\Gamma(\mu+\rho_k+1)\Gamma(a_{k,k}+1)} \tag{2.26}$$

Now from the asymptotic expansion (2.21) for large positive integer $\mu$, we have the left hand side of the above identity is,

$$w(t,\mu) = \prod_{k=1}^{d}(t-\lambda_k)^{\mu+a_{kk}} \det[\, e_1,e_2,e_3,\dots,e_d] \,(1 + O(1/\mu)\,) \tag{2.27}$$

Now we combine (2.26) and (2.27) together and let $\mu$ tends to positive infinity reminding of the fact:

$$\frac{\Gamma(a+n)}{\Gamma(b+n)} = n^{\mathrm{Re}(a-b)}\,(\,1 + O(1/n)\,) \qquad (\text{as } n \to \infty\,) \tag{2.28}$$

which yields fundamental identity (2.13). This completes the proof of Th.2.1..

Corollary. The identity holds throughout the domain obtained from the entire complex plane by removing simple arcs joining the infinity and each singular points.

Proof. $w(t,o)$ is a solution of the following differential equation:

$$w'/w = \mathrm{Trace}[\,(t-B)^{-1}A\,] = \sum_{k=1}^{d} a_{k,k}/(t-\lambda_k) \tag{2.29}$$

Since the domain defined in the corollary is a simply connected domain in $\mathbb{P}^1$-S we can continue $w(t,o)$ analytically by monodromy theorem.

<u>Example(Classical Gauss Formula)</u>.

Consider the system (2.18) with $(\lambda_1,\lambda_2) = (0,1)$ at dimension $d=2$.

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} h_1(s+1) = \begin{pmatrix} a_{1,1}+s-a_{1,1} & -a_{1,2} \\ -a_{2,1} & a_{1,1}+s-a_{2,2} \end{pmatrix} h_1(s) \tag{2.30}$$

with the initial condition

$$h_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.31}$$

If we write the vector $h_1(s)$ componentwise as

$$h_1(s) = \begin{pmatrix} h^1(s) \\ h^2(s) \end{pmatrix} \tag{2.32}$$

then we have a recurrence relation for $h^1(s)$:

$$\begin{cases} (s+1)h^1(s+1) = [(s+a_{11}-a_{11})(s+a_{11}-a_{22})-a_{12}a_{21}]h^1(s) \\ \qquad h^1(0) = 1 \end{cases} \tag{2.33}$$

Since the coefficient on the right hand side of (2.33) is

$$\det[(s+a_{11})-A] = (s+a_{11}-\rho_1)(s+a_{11}-\rho_2) \tag{2.34}$$

we can easily derive the expression:

$$h^1(s) = (a_{11}-\rho_1)_s(a_{11}-\rho_2)_s/(1)_s$$

where $(p)_s = p(p+1)\cdots\cdots(p+s-1) = \Gamma(p+s)/\Gamma(p)$.

If we set:

$$\begin{cases} a_{11}-\rho_1 = a \\ a_{11}-\rho_2 = b \\ a_{11}+1 \ = c \end{cases} \tag{2.35}$$

then clearly, the first component of the solution $x_1(t)$ at $t=0$ becomes

$$t^{a_{11}} \sum_{s=0}^{\infty} h^1(s)/(a_{11}+1)_s \, t^s = t^{1-c} \sum_{s=0}^{\infty} (a)_s(b)_s \,/(1)_s(c)_s \, t^s$$

$$= t^{c-1}F(a,b,c:t) \tag{2.36}$$

The second component of the solution $x_1(t)$ vanishes in the order $O(t^c)$ and we will denote it by $\theta(t)$.

For the second solution $x^2(t)$ defined locally near $t=1$, we first compute the exponent $a_{22}$ from (2.35):

$$a_{22} = \rho_1 + \rho_2 - a_{11} = c-1-a-b. \tag{2.37}$$

Now we write the second solution componentwise as

$$x_2(t) = (t-1)^{c-1-a-b} \begin{pmatrix} \varphi(t)(t-1) \\ \psi(t) \end{pmatrix} \tag{2.38}$$

where both $\varphi(t)$ and $\psi(t)$ are holomorphic at $t=1$, and we specifically have

$$\psi(1) = 1$$

by definition (2.16).

Let us compute the wronskian $w(t)=\det[x_1(t),x_2(t)]$ near the singular point $t=1$:

$$w(t) = t^{c-1}(t-1)^{c-1-a-b}[\Psi(t)F(a,b,c:t)-(t-1)\varphi(t)\theta(t)] \tag{2.39}$$

The solution $x_1(t)$ may have the representation:

$$x_1(t) = c_1x_2(t) +c_2H(t)$$

where $H(t)$ is an holomorphic solution at $t=1$. But when we assume

$$\mathrm{Re}(c-1-a-b) > 0 \tag{2.40}$$

we have $H(t)$ as the significan part, and we have $\det[x_2,c_1x_2+H] = \det[x_1,H]$ . Consequently, by letting $t$ tend to 1, and by comparing (2.39) with the fundamental identity (2.13) we have

$$F(a,b,c:1) = \Gamma(c)\Gamma(c-a-b) \,/\, \Gamma(c-a)\Gamma(c-b) \tag{2.41}.$$

Other summation formulae will appear in the last chapter.