Modified $\mathcal{A}$-hypergeometric Systems

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Abstract We will introduce a modified system of $\mathcal{A}$-hypergeometric system (GKZ system) by applying a change of variables for Gröbner deformations and study its Gröbner basis and the indicial polynomial along the "exceptional hypersurface".

1 Introduction

Since the work of Gel'fand, Zelevinsky, and Kapranov [3], studies of $\mathcal{A}$-hypergeometric system (GKZ system) have attracted a lot of mathematicians, who want to understand hypergeometric differential equations in a general way. We refer to the book [8] on the status of the art in 2000, and the recent papers [4] and [9] and their reference trees on recent advances. We also note that these studies have had fruitful interactions with frontiers of computational commutative algebra and computational D-modules.

In this short paper, we will introduce a modified version of this $\mathcal{A}$-hypergeometric system and provide the first step to study it (see also [6]). The original system is defined on the $y = (y_1, \ldots, y_n)$ space and the modified system is defined on the $(t, , x_1, \ldots, x_n)$ space with one more variable t. We consider the direct sum of the $\mathcal{A}$-hypergeometric system on the y space and the D-module $D/D \cdot s\partial_s$ on the s-space. For a weight vector $w \in \mathbf{Z}^n$, the original system restricted on the complex torus is transformed into the modified system on (t, x) space by the map

$$\mathbf{C}^n \times \mathbf{C}^* \ni (y_1, \ldots, y_n, s) \mapsto (t^{w_1} x_1, \ldots, t^{w_n} x_n, t) \in \mathbf{C}^n \times \mathbf{C}^*$$

The transformed system can be naturally extended on $\mathbf{C}^{n+1}$. Intuitively speaking, the variety $t = 0$ is analogous to the exceptional hypersurface of a blowing-up operation. We will study the indicial polynomial along $t = 0$ as the first step to make a local and global analysis of the modified system.

2 Definition and Holonomic Rank of Modified $\mathcal{A}$-hypergeometric systems

Let $A = (a_{ij})_{ij}$ be a $d \times n$-matrix whose elements are integers and $w = (w_1, \ldots, w_n)$ a vector of integers. We suppose that the set of the column vectors of A spans $\mathbf{Z}^d$. Define

$$\tilde{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 \\ & \cdots & & 0 \\ a_{dn} & \cdots & a_{dn} & 0 \\ w_1 & \cdots & w_n & 1 \end{pmatrix}.$$

Definition 1 We call the following system of differential equations $H_{A,w}(\beta)$ a *modified $\mathcal{A}$-hypergeometric differential system*:

$$\begin{aligned} \left(\sum_{j=1}^{n} a_{ij}x_j\partial_j - \beta_i\right) \bullet f &= 0, \qquad (i = 1, \ldots, d) \\ \left(\sum_{j=1}^{n} w_j x_j \partial_j - t\partial_t\right) \bullet f &= 0, \\ \left(\prod_{i=1}^{n} \partial_i^{u_i} t^{u_{n+1}} - \prod_{j=1}^{n} \partial_j^{v_j} t^{v_{n+1}}\right) \bullet f &= 0. \qquad (u, v \in \mathbf{N}^{n+1} \text{ and } \tilde{A}u = \tilde{A}v) \end{aligned}$$

Let $I_{\tilde{A}}^{sat}$ be the saturation of the toric ideal $I_{\tilde{A}}$ generated by

$$\prod_{i=1}^{n} \partial_i^{u_i} t^{u_{n+1}} - \prod_{j=1}^{n} \partial_j^{v_j} t^{v_{n+1}} \qquad (u, v \in \mathbf{N}^{n+1} \text{ and } \tilde{A}u = \tilde{A}v) \tag{1}$$

in $\mathbf{C}[\partial_1, \ldots, \partial_n, t]$ with respect to t. In other words,

$$I_{\tilde{A}}^{sat} = (I_{\tilde{A}} : t^\infty) = \{\ell \,|\, t^m \ell \in I_{\tilde{A}} \text{ for a non-negative integer } m\} \tag{2}$$

Note that $I_{\tilde{A}}^{sat}$ is an ideal generated by binomials.

Definition 2 We call the following system of differential equations $H_{A,w}^{sat}(\beta)$ a *saturated modified $\mathcal{A}$-hypergeometric differential system*:

$$\left(\sum_{j=1}^{n} a_{ij}x_j\partial_j - \beta_i\right) \bullet f = 0, \qquad (i = 1, \ldots, d) \tag{3}$$

$$\left(\sum_{j=1}^{n} w_j x_j \partial_j - t\partial_t\right) \bullet f = 0, \tag{4}$$

$$\ell \bullet f = 0, \quad \ell \in I_{\tilde{A}}^{sat} \tag{5}$$

Throughout this paper, we will use notations and facts shown in [8]. In particular, we do not cite original papers for text level well-known facts in the theory of D-modules. Refer references of [8] as to these original papers.

Let a_i be the i-th column vector of the matrix A and $F(\beta, x, t)$ the integral

$$F(\beta, x, t) = \int_C \exp\left(\sum_{i=1}^{n} x_i t^{w_i} s^{a_i}\right) s^{-\beta-1} ds, \qquad s = (s_1, \ldots, s_d),\ \beta = (\beta_1, \ldots, \beta_d).$$

The integral $F(\beta, x, t)$ satisfies the modified $\mathcal{A}$-hypergeometric differential system "formally". We use the word "formally" because, there is no general and rigorous description about the cycle C. However, the integral representation gives an intuitive figure of what are solutions of modified $\mathcal{A}$-hypergeometric systems. The proof is analogous to [8, 221–222]. We note that if $a_{di} = 1$ for all i, we also have the following "formal" integral representation

$$\begin{aligned} F(\beta, x, t) &= \int_C \left(\sum_{i=1}^{n} x_i t^{w_i} \tilde{s}^{\tilde{a}_i}\right)^{-\beta_d} \tilde{s}^{-\tilde{\beta}-1} d\tilde{s}, \\ & \quad \tilde{a}_i = (a_{1i}, \ldots, a_{d-1,i})^T, \tilde{s} = (s_1, \ldots, s_{d-1}),\ \beta = (\beta_1, \ldots, \beta_d). \end{aligned}$$

We denote by D the ring of differential operators $\mathbf{C}\langle x_1, \ldots, x_n, t, \partial_1, \ldots, \partial_n, \partial_t\rangle$. We will regard modified $\mathcal{A}$-hypergeometric system and saturated one as left ideals in D. We will denote by $H_{A,w}(\beta)$ and $H^{sat}_{A,w}(\beta)$ these left ideals respectively as long as no confusion arises.

Theorem 1 *1. The left D-module $D/H_{A,w}(\beta)$ is holonomic.*

2. The rank of $H_{A,w}(\beta)$ agrees with the holonomic rank of $H_A(\beta)$ for any w.

3. The left D-module $D/H^{sat}_{A,w}(\beta)$ is a submodule of $D/H_{A,w}(\beta)$

4. The rank of $H^{sat}_{A,w}(\beta)$ agrees with the holonomic rank of $H_A(\beta)$ for any w.

Proof. (1) We apply the *Laplace transformation* with respect to the variable t ($t \mapsto -\partial_{t'}$, $\partial_t \mapsto t'$) for the modified $\mathcal{A}$-hypergeometric system $H_{A,w}(\beta)$. Then, the transformed system is nothing but $\mathcal{A}$-hypergeometric system for the matrix $\tilde{A}$ and the parameter vector $(\beta_1, \ldots, \beta_n, -1)$. It is known that the transformed system is holonomic, then the original system is also holonomic by showing the Hilbert polynomials with respect to the Bernstein filtration of each system agree.

(2) We consider the biholomorphic map φ on $\mathbf{C}^n \times \mathbf{C}^*$

$$\mathbf{C}^n \times \mathbf{C}^* \ni (y_1, \ldots, y_n, s) \mapsto (t^{w_1} x_1, \ldots, t^{w_n} x_n, t) \in \mathbf{C}^n \times \mathbf{C}^* \tag{6}$$

The map φ induces a correspondence of differential operators on $\mathbf{C}^n \times \mathbf{C}^*$

$$\begin{aligned} \frac{\partial}{\partial y_i} &= t^{-w_i} \frac{\partial}{\partial x_i} \\ -s\frac{\partial}{\partial s} &= -t\frac{\partial}{\partial t} + \sum_{j=1}^{n} w_n x_n \frac{\partial}{\partial x_i} \end{aligned}$$

Consider a left ideal H_Y in $D_Y = \mathbf{C}\langle y_1, \ldots, y_n, s, \partial_{y_1}, \ldots, \partial_{y_n}, \partial_s\rangle$ generated by $H_A(\beta)$ and $s\partial_s$. The holonomic rank of D_Y/H_Y is that of $H_A(\beta)$. We can see that the image of $\mathcal{D}_Y/\mathcal{D}_Y H_Y$ by the biholomorphic map φ on $\mathbf{C}^n \times \mathbf{C}^*$ is $\mathcal{D}_X/\mathcal{D}_X H_{A,w}(\beta)$ by utilizing the correspondence of differential operators. Here, $\mathcal{D}_X$ and $\mathcal{D}_Y$ denote the sheaves of differential operators on $\mathbf{C}^n \times \mathbf{C}^*$ of (y, s)-space and (x, t)-space respectively. Since the holonomic rank agrees with the multiplicity of the zero section of the characteristic variety at generic points, the holonomic ranks of the both systems agree [8, pp 28–40].

(3) Since $I_A^{sat} \subseteq I_A$, it holds.

(4) The holonomic rank is characterized as the dimension of classical solutions as a $\mathbf{C}$-vector space [8, p.36]. It follows from the definition of the saturation, the both systems has the same classical solutions. Q.E.D.

Example 1 We take $A = (1, 3)$, $\beta = (-1)$, and $w = (-1, 0)$ (*Airy type integral*) [8, p.223]. Define a sequence d_m by

$$d_0 = 1, d_{m+1} = \frac{-(3m+1)(3m+2)(3m+3)}{m} d_m$$

The divergent series

$$\begin{aligned} f(x;t) &= \sum_{m=0}^{\infty} \left(d_m x_1^{-3m-1} x_2^m\right) t^{3m+1} \\ &= \sum_{m=0}^{\infty} \left(\frac{\Gamma(3m+1)}{\Gamma(m+1)} x_1^{-3m-1} x_2^m\right) t^{3m+1} \end{aligned} \qquad (7)$$

is a formal solution of the modified system. Fix a point $(x_1, x_2) = (a_1, a_2)$ such that $a_1, a_2 \neq 0$. Then this is a Gevrey formal power series solution at $(a_1, a_2, 0)$ along $t = 0$ in the class $s = 1 + 2/3$ from the definition of Gevrey series. The slope of this system can be computed by our program [7, command `sm1.slope`, `slope`], [1], [2] and the set of the slopes is $\{-3/2\}$. Since $1/(1-s)$ is the slope, we have constructed a formal power series standing for the slope.

3 A Gröbner Basis of Saturated Modified $\mathcal{A}$-Hypergeometric Systems

We will call $t = 0$ the *exceptional hypersurface* and we are interested in local analysis near $t = 0$. We denote by $\tau = (\mathbf{0}, -1; \mathbf{0}, 1)$ the weight vector such that t has the weight -1 and ∂_t has the weight 1. We also denote by $\tilde{A}_{\theta,w,\beta}$ the first $(d+1)$ Euler operators (3) and (4).

It is easy to see that, for generic w, $\mathrm{in}_\tau(D \cdot I_{\tilde{A}}^{sat})$ is generated by monomials in $\mathbf{C}[\partial_1, \ldots, \partial_n]$ and we will regard it as a monomial ideal in this commutative ring.

Theorem 2 *For generic β and w, we have*

$$\mathrm{in}_{(\mathbf{0},-1;\mathbf{0},1)}(H^{sat}_{A,w}(\beta)) = D \cdot \mathrm{in}_\tau(D \cdot I^{sat}_{\tilde{A}}) + D \cdot \tilde{A}_{\theta,w,\beta} \tag{8}$$

Proof. The proof is analogous to [8, Theorem 3.1.3]. Let $s = (s_1, \ldots, s_d)$ be a vector of new indeterminates. Consider the algebra

$$D[s] = \mathbf{C}\langle x_1, \ldots, x_n, t, \partial_1, \ldots, \partial_n, \partial_t, s_1, \ldots, s_d\rangle$$

and its homogenized Weyl algebra by h $D[s]^h$. Let H be the left ideal in $D[s]^h$ generated by $\tilde{A}_{\theta,w,s^2}$ and the homogenization of $I^{sat}_{\tilde{A}}$. We define a partial order $>_\tau$ on monomials in $D[s]$ by

$$s^a x^b \partial^c t^d \partial_t^e >_\tau s^{a'} x^{b'} \partial^{c'} t^{d'} \partial_t^{e'} \quad \Leftrightarrow \quad \begin{array}{l} -d+e > -d'+e', \text{ or} \\ -d+e = -d'+e' \text{ and } (a,e,d) >_{lex} (a',e',d') \end{array}$$

We refine this partial order by any monomial order and define orders $<$ in $D[s]$. (This order on $D[s]$ is extended to the order in the homogenized Weyl algebra and $D[s]^h$ as in [8, Chapter 1].)

Let $\mathcal{G}$ be the reduced Gröbner basis of the homogenized binomial ideal $I^{sat}_{\tilde{A}}$ in $D[s]^h$ with respect to the order $<$. Note that the reduced Gröbner basis consists of elements of the form $\underline{\partial^u h^p} - \partial^v t^{v_{n+1}} h^{p'}$, $v_{n+1} > 0$ since w is generic and $I^{sat}_{\tilde{A}}$ is saturated. Note that either $p = 0$ or $p' = 0$ holds.

We will show that $\mathcal{G}$ and $\tilde{A}^h_{\theta,w,s^2}$ is a Grobner basis $\mathcal{G}'$ with respect to $<$ in $D[s]^h$. This fact can be shown by checking the S-pair criterion in $D[s]^h$. It is easy to see that

$$sp(\underline{\theta_t} - \sum w_j\theta_j, \underline{s_i^2} - \sum a_{ij}\theta_j) \to_{\mathcal{G}'} 0$$
$$sp(\underline{s_k^2} - \sum a_{kj}\theta_j, \underline{s_i^2} - \sum a_{ij}\theta_j) \to_{\mathcal{G}'} 0$$

We assume $p > 0$ and $p' = 0$.

$$\begin{aligned}
& sp(\underline{\partial^u h^p} - \partial^v t^{v_{n+1}}, \underline{s_i^2} - \sum a_{ij}\theta_j) \\
= \quad & s_i^2(\underline{\partial^u h^p} - \partial^v t^{v_{n+1}}) - \partial^u h^p(\underline{s_i^2} - \sum a_{ij}\theta_j)) \\
= \quad & -s_i^2 \partial^v t^{v_{n+1}} + \partial^u h^p \sum a_{ij}\theta_j \\
= \quad & -s_i^2 \partial^v t^{v_{n+1}} + \left(\sum a_{ij}\theta_j\right) \partial^u h^p + \left(\sum a_{ij}u_j\right) \partial^u h^p \\
& \text{since } \partial^u h^p > \partial^v t^{v_{n+1}} \text{ we may rewrite it as} \\
= \quad & -s_i^2 \partial^v t^{v_{n+1}} + \left(\sum a_{ij}\theta_j\right)(\partial^u h^p - \partial^v t^{v_{n+1}}) + \left(\sum a_{ij}\theta_j\right) \partial^v t^{v_{n+1}} + \left(\sum a_{ij}u_j\right) \partial^u h^p \\
= \quad & \left(\sum a_{ij}\theta_j\right)(\partial^u h^p - \partial^v t^{v_{n+1}}) + \partial^v t^{v_{n+1}} \left(\sum a_{ij}\theta_j - \sum a_{ij}v_j - s_i^2\right) + \left(\sum a_{ij}u_j\right) \partial^u h^p \\
& \text{since } \sum a_{ij}u_j = \sum a_{ij}v_j \\
= \quad & \left(\sum a_{ij}\theta_j\right)(\underline{\partial^u h^p} - \partial^v t^{v_{n+1}}) + \partial^v t^{v_{n+1}} \left(\sum a_{ij}\theta_j - s_i^2\right) + \left(\sum a_{ij}u_j\right)(\partial^u h^p - \partial^v t^{v_{n+1}}) \\
\to_{\mathcal{G}'} \quad & 0
\end{aligned}$$

The case $p = 0$, and $p' > 0$ can be shown analogously.

The final case we have to check is that

$$
\begin{aligned}
& sp(\underline{\partial^u h^p} - \partial^v t^{v_{n+1}}, \underline{\theta_t} - \sum a_{ij}\theta_j) \\
= \quad & -\theta_t \partial^v t^{v_{n+1}} + h^p \partial^u \sum w_j \theta_j \\
= \quad & -\theta_t \partial^v t^{v_{n+1}} + \left(\sum w_j\theta_j + \sum w_j u_j\right) h^p \partial^u \\
= \quad & -\theta_t \partial^v t^{v_{n+1}} + \left(\sum w_j\theta_j + \sum w_j u_j\right) (h^p \partial^u - \partial^v t^{v_{n+1}}) \\
& \qquad + \left(\sum w_j\theta_j + \sum w_j u_j\right) \partial^v t^{v_{n+1}} \\
= \quad & -\theta_t \partial^v t^{v_{n+1}} + \left(\sum w_j\theta_j + \sum w_j u_j\right) (h^p \partial^u - \partial^v t^{v_{n+1}}) \\
& \quad + \partial^v t^{v_{n+1}} \left(\sum w_j\theta_j + \sum w_j u_j - \sum w_j v_j\right) \\
= \quad & \left(\sum w_j\theta_j + \sum w_j u_j\right) (h^p \partial^u - \partial^v t^{v_{n+1}}) \\
& + \partial^v t^{v_{n+1}} \left(\sum w_j\theta_j + \sum w_j u_j - \sum w_j v_j - \theta_t - v_{n+1}\right) \\
= \quad & \left(\sum w_j\theta_j + \sum w_j u_j\right) \left(\underline{h^p \partial^u} - \partial^v t^{v_{n+1}}\right) + \partial^v t^{v_{n+1}} \left(\sum w_j\theta_j - \underline{\theta_t}\right) \\
\to_{\mathcal{G}'} \quad & 0
\end{aligned}
$$

The rests of the proof are analogous to [8, p.106]. Q.E.D.

4 Indicial Polynomial along $t = 0$

We fix generic w. Let M be the monomial ideal $\mathrm{in}_\tau(I_{\tilde{A}}^{sat})$ in $\mathbf{C}[\partial_1, \ldots, \partial_n]$. The top dimensional standard pairs are denoted by $\mathcal{T}(M)$ [8, p.112] and $\beta^{(\partial^\beta, \sigma)}$ is the zero point in $\mathbf{C}^n$ of the distraction of M and $A\theta - \beta$ associated to the standard pair (∂^β, σ).

Theorem 3 *Let β and w both be generic. Then, the indicial polynomial (b-function) of $H_{A,w}^{sat}(\beta)$ along $t = 0$ is*

$$\sum_{(\partial^\beta, \sigma) \in \mathcal{T}(M)} (s - w \cdot \beta^{(\partial^\beta, \sigma)}) \tag{9}$$

If $\mathcal{T}(M)$ is the empty set, the indicial polynomial is 0.

Proof. Under Theorem 2, the proof is analogous to [8, p.198, Proposition 5.1.9].

If the indicial polynomial is not zero and the difference of roots are not integral, we can construct formal series solution of the form

$$t^e \sum_{k=0}^{\infty} c_k(x) t^k, \quad c_k \in \mathbf{C}[1/x_1, \ldots, 1/x_n, x_1, \ldots, x_n]$$

where e is a root of the indicial polynomial and $t^e c_0(x)$ is a solution of the initial system $\mathrm{in}_{(\mathbf{0},-1;\mathbf{0},1)}(H_{A,w}(\beta))$. If the indicial polynomial is zero, there is no formal series solution of the form above. It is an interesting problem to study ordinary differential equations in $H^{sat}_{A,w}(\beta)$ with respect to t when x is fixed.

Example 2 (Continuation of Example 1). Note that $\mathrm{in}_\tau(I^{sat}_{\tilde{A}}) = \langle \partial_2 \rangle$. The distraction [8, p.68] of $\mathrm{in}_\tau(H^{sat}_{A,w}(\beta))$ is generated by $\theta_2, \theta_1 + 3\theta_2 + 1, -\theta_1 - \theta_t$. Therefore, the set of zero points are $\{(-1, 0, 1)\}$. Then, the indicial polynomial is $s - 1$. The formal solution (7) stands for the root $s = 1$. Incidentally, the equality (9) holds in this case.

Example 3 Consider the saturated modified hypergeometric system for $A = (-1, 1, 2)$, $\beta = (1/2)$, $w = (-2, -1, 0)$. This is the *Bessel function in two variables* called by Kimura and Okamoto [5]. A 3-D Graph of a solution of this system can be seen at
`http://www.math.kobe-u.ac.jp/HOME/taka/test-bess2m.html`. You will be able to see waves in two directions. The indicial polynomial is 0, because $I^{sat}_{\tilde{A}} \ni \underline{1} - \partial_1^2\partial_3$. Then, there exists no series solution of the form above. Incidentally, the set of the slopes along $t = 0$ at $x = (2, 2, 1)$ is equal to $\{-2, -3/2\}$. The values are obtained by our program [7].

Let us change w into $w = (3, 2, 1)$. The set of zero points of the distraction are
$\{(-1/2, 0, 0), (0, 0, 0), (0, 1, -1/4)\}$ (which are obtained by computing the primary decomposition of the ideal generated by the distraction of $\mathrm{in}_\tau(I^{sat}_{\tilde{A}})$ and $\tilde{A}_{\theta,w,\beta}$) and the indicial polynomial is $(s - 3/2)(s + 1/4)(s - 7/4)$ (we use the `Risa/Asir` command `generic_bfct`). In this case, the generic condition for the Theorem satisfied and the equality (9) holds. Incidentally, the local monodromy group of the local solutions around $t = 0$ is generated by $\mathrm{diag}(-1, \exp(\pi\sqrt{-1}/2), -\exp(\pi\sqrt{-1}/2))$. The set of the slopes along $t = 0$ at $x = (2, 2, 1)$ is empty.

It is an interesting open problem to construct rank many series solutions in terms of formal puiseux series and exponential functions along $t = 0$.

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