T. Kimura Hypergeometric Functions of Two Variables, 1973

CHAPTER II

The Hypergeometric Functions of Two Variables

7. Definition of the hypergeometric functions of two variables. In the preceding chapter, we stated several properties of the hypergeometric function $F(\alpha, \beta, \gamma, x)$ and the hypergeometric differential equation. We shall now proceed to the hypergeometric functions of two variables. We start with two hypergeometric series

$$F(\alpha, \beta, \gamma, x) = \sum_{m=0}^{\infty} \frac{(\alpha, m)(\beta, m)}{(\gamma, m)(1, m)} x^m$$

and

$$F(\alpha', \beta', \gamma', y) = \sum_{m=0}^{\infty} \frac{(\alpha', n)(\beta', n)}{(\gamma', n)(1, n)} y^n .$$

Consider the product of these two functions:

$$(7.1) \quad F(\alpha, \beta, \gamma, x)F(\alpha', \beta', \gamma', y) = \sum_{m,n=0}^{\infty} \frac{(\alpha,m)(\alpha',n)(\beta,m)(\beta',n)}{(\gamma,m)(\gamma',n)(1,m)(1,n)} x^m y^n$$

This is a double power series in x, y. This function of x and y, however, can not be regarded really as a function of two variables, since this is a product of a function of x and a function of y. To obtain functions which are really regarded as functions of two variables, we shall modify the coefficients of the general terms of this double power series.

To do this, let us consider the three products

$$(\alpha, m)(\alpha', n), \quad (\beta, m)(\beta', n), \quad (\gamma, m)(\gamma', n)$$

and assign to these products the three quantities

$$(\alpha, m+n), \quad (\beta, m+n), \quad (\gamma, m+n)$$

respectively, i.e.

$$(\alpha, m)(\alpha', n) \longleftrightarrow (\alpha, m+n),$$

$$(\beta, m)(\beta', n) \longleftrightarrow (\beta, m+n),$$

$$(\gamma, m)(\gamma', n) \longleftrightarrow (\gamma, m+n).$$

If we replace just one of the three products by the corresponding quantity, we obtain from the double series (7.1) the following three double power series:

$$(7.2\text{-}1) \quad \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m)(\gamma', n)(1, m)(1, n)} x^m y^n ,$$

$$(7.2\text{-}2) \quad \sum_{m,n=0}^{\infty} \frac{(\alpha, m)(\alpha', n)(\beta, m+n)}{(\gamma, m)(\gamma', n)(1, m)(1, n)} x^m y^n ,$$

$$(7.2\text{-}3) \quad \sum_{m,n=0}^{\infty} \frac{(\alpha, m)(\alpha', n)(\beta, m)(\beta', n)}{(\gamma, m+n)(1, m)(1, n)} x^m y^n .$$

Replacing just two of the three products by the corresponding quantities, we obtain the following three double power series:

$$(7.3\text{-}1) \quad \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m+n)}{(\gamma, m)(\gamma', n)(1,m)(1,n)} x^m y^n ,$$

$$(7.3\text{-}2) \quad \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)(1,m)(1,n)} x^m y^n ,$$

$$(7.3\text{-}3) \quad \sum_{m,n=0}^{\infty} \frac{(\alpha, m)(\alpha', n)(\beta, m+n)}{(\gamma, m+n)(1, m)(1, n)} x^m y^n .$$

The double power series

(7.4) $$\sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m+n)}{(\gamma, m+n)(1, m)(1, n)} x^m y^n$$

is obtained from (7.1) by replacing all three products by the corresponding quantities.

Among the seven series, (7.2-1) and (7.2-2) are essentially the same, while (7.3-2) and (7.3-3) are also essentially the same, by the symmetry on the parameters α and β. On the other hand, rearranging the double series (7.4) into the series of homogeneous polynomials, we obtain $F(\alpha, \beta, \gamma, x+y)$. In fact,

$$\sum_{m,n=0}^{\infty} \frac{(\alpha,m+n)(\beta,m+n)}{(\gamma,m+n)(1,m)(1,n)} x^m y^n = \sum_{k=0}^{\infty} \sum_{m+n=k} \frac{(\alpha,m+n)(\beta,m+n)}{(\gamma,m+n)(1,m)(1,n)} x^m y^n$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha,k)(\beta,k)}{(\gamma,k)(1,k)} \sum_{m+n=k} \frac{(1,k)}{(1,m)(1,n)} x^m y^n$$

Observing

$$\frac{(1,k)}{(1,m)(1,n)} = \binom{k}{m} \qquad \text{if } m+n=k,$$

we get

$$\sum_{m+n=k} \frac{(1,k)}{(1,m)(1,n)} x^m y^n = (x+y)^k .$$

Therefore

$$\sum_{m,n=0}^{\infty} \frac{(\alpha,m+n)(\beta,m+n)}{(\gamma,m+n)(1,m)(1,n)} x^m y^n = \sum_{k=0}^{\infty} \frac{(\alpha,k)(\beta,k)}{(\gamma,k)(1,k)} (x+y)^k$$

$$= F(\alpha, \beta, \gamma, x+y) .$$

Thus we arrived at four power seires to which Appel gave the following notations:

DEFINITION 7.1: The power series

$$F_1(\alpha, \beta, \beta', \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma, m+n)(1, m)(1, n)} x^m y^n ,$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m)(\gamma',n)(1,m)(1,n)} x^m y^n ,$$

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha,m)(\alpha',n)(\beta,m)(\beta',n)}{(\gamma,m+n)(1,m)(1,n)} x^m y^n ,$$

$$F_4(\alpha, \beta, \gamma, \gamma', x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha,m+n)(\beta, m+n)}{(\gamma,m)(\gamma',n)(1,m)(1,n)} x^m y^n$$

are called the hypergeometric series of two variables and the functions defined by these series are called the hypergeometric functions of two variables. We assume hereafter that $\gamma, \gamma' \neq 0, -1, -2, \cdots$.

Remark 1. F_1 is a polynomial if α is zero or a negative integer. F_1 is also a polynomial if β and β' are zero or negative integers. F_2 is a polynomial if α or (β and β') $= 0, -1, -2, \cdots$. F_3 is a polynomial if (α and α') or (β and β') $= 0, -1, -2, \cdots$. F_4 is a polynomial if α or $\beta = 0, -1, -2, \cdots$.

Remark 2. F_3 is symmetric in α, β and in α', β':

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) = F_3(\beta, \alpha', \alpha, \beta', \gamma, x, y)$$
$$= F_3(\alpha, \beta', \beta, \alpha', \gamma, x, y)$$
$$= F_3(\beta, \beta', \alpha, \alpha', \gamma, x, y) .$$

F_4 is symmetric in α, β:

$$F_4(\alpha, \beta, \gamma, \gamma', x, y) = F_4(\beta, \alpha, \gamma, \gamma', x, y) .$$

Remark 3. Using the Γ-function, we can write F_j (j = 1, 2, 3 and 4) respectively as

$$F_1 = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta')} \sum_{m,n=0}^{\infty} \frac{\Gamma(\alpha+m+n)\Gamma(\beta+m)\Gamma(\beta'+n)}{\Gamma(\gamma+m+n)\Gamma(1+m)\Gamma(1+n)} x^m y^n ,$$

$$F_2 = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta')} \sum_{m,n=0}^{\infty} \frac{\Gamma(\alpha+m+n)\Gamma(\beta+m)\Gamma(\beta'+n)}{\Gamma(\gamma+m)\Gamma(\gamma'+n)\Gamma(1+m)\Gamma(1+n)} x^m y^n$$

$$F_3 = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(\beta)\Gamma(\beta')} \sum_{m,n=0}^{\infty} \frac{\Gamma(\alpha+m)\Gamma(\alpha'+n)\Gamma(\beta+m)\Gamma(\beta'+n)}{\Gamma(\gamma+m+n)\Gamma(1+m)\Gamma(1+n)} x^m y^n$$

$$F_4 = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)} \sum_{m,n=0}^{\infty} \frac{\Gamma(\alpha+m+n)\Gamma(\beta+m+n)}{\Gamma(\gamma+m)\Gamma(\gamma'+n)\Gamma(1+m)\Gamma(1+n)} x^m y^n .$$

Remark 4. Rearranging the hypergeometric series into iterated series, we obtain, for example,

$$F_1(\alpha, \beta, \beta', \gamma, x, y) = \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)} F(\alpha+m, \beta', \gamma+m, y)\, x^m$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha,n)(\beta',n)}{(\gamma,n)(1,n)} F(\alpha+n, \beta, \gamma+n, x)\, y^n .$$

In fact, observing

$$(a, m+n) = a(a+1)\cdots(a+m-1)(a+m)\cdots(a+m+n-1) = (a, m)(a+m, n),$$

we get

$$\sum_{m,n=0}^{\infty} \frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m+n)(1,m)(1,n)} x^m y^n = \sum_{m,n=0}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)} \frac{(\alpha+m,n)(\beta',n)}{(\gamma+m,n)(1,n)} x^m y^n$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)} x^m \sum_{n=0}^{\infty} \frac{(\alpha+m,n)(\beta',n)}{(\gamma+m,n)(1,n)} y^n$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)} F(\alpha+m, \beta', \gamma+m, y) x^m$$

and

Similarly, we get the second identity. For other F_j we have

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) = \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)} F(\alpha+m, \beta', \gamma', y) x^m$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha,n)(\beta',n)}{(\gamma',n)(1,n)} F(\alpha+n, \beta, \gamma, x) y^n ,$$

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) = \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)} F(\alpha', \beta', \gamma+m, y) x^m$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha',n)(\beta',n)}{(\gamma,n)(1,n)} F(\alpha, \beta, \gamma+n, x) y^n ,$$

$$F_4(\alpha, \beta, \gamma, \gamma', x, y) = \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)} F(\alpha+m, \beta+m, \gamma', y) x^m$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha,n)(\beta,n)}{(\gamma',n)(1,n)} F(\alpha+n, \beta+n, \gamma, x) y^n .$$

Remark 5. If we give one of x and y the value zero, or if certain parameters take the value zero, then the hypergeometric series F_j are reduced to hypergeometric series of one variable. For example:

$$F_1(\alpha, \beta, \beta', \gamma, x, 0) = F_1(\alpha, \beta, 0, \gamma, x, y)$$
$$= F(\alpha, \beta, \gamma, x)$$

$$F_1(\alpha, \beta, \beta', \gamma, 0, y) = F_1(\alpha, 0, \beta', \gamma, x, y)$$
$$= F(\alpha, \beta', \gamma, y) .$$

Remark 6. If certain parameters take special values, the hypergeometric series F_j are reduced to known functions. For example:

$$F_1(\alpha, \beta, \beta', \alpha, x, y) = (1-x)^{-\beta}(1-y)^{-\beta'},$$

$$F_2(\alpha, \beta, \beta', \beta, \beta', x, y) = (1-x-y)^{-\alpha}.$$

Exercise 1. Prove that

$$F_2(\alpha, \beta, \beta', \beta, \beta', x, y) = (1-x-y)^{-\alpha}$$

and

$$x F_2(1, 1, \beta', 2, \beta', x, y) + y F_2(1, \beta, 1, \beta, 2, x, y)$$
$$= \log\{(1-x-y)^{-1}\}.$$

8. Domains of convergence of the hypergeometric series of two variables.

In this section, we shall study domains of convergence of the four double series F_1, F_2, F_3 and F_4. To do this, we begin with several remarks concerning double power series.

First of all, consider a series

$$\sum_{m=0}^{\infty} a_m . \tag{8.1}$$

As it is well known, the series (8.1) converges to a certain value a (or the series (8.1) has a sum a : $a = \sum_{m=0}^{\infty} a_m$) if and only if for $\varepsilon > 0$ there exists a positive integer M such that we have

$$\left| a - \sum_{m=0}^{p} a_m \right| < \varepsilon \qquad \text{for every} \quad p > M.$$

This definition implies that if $\sum_{m=0}^{\infty} a_m$ is convergent, then the sequence $\{a_m\}_{m=0}^{\infty}$ is bounded.

Let us consider next a double series:

$$\sum_{m,n=0}^{\infty} a_{m,n} . \tag{8.2}$$

The following definition might seem to be very natural:

"The series (8.2) converges to a (or (8.2) has a sum a), if and only if for any $\varepsilon > 0$ there exist two positive integers M and N such that

$$\left| a - \sum_{0 \leq m \leq p,\, 0 \leq n \leq q} a_{m,n} \right| < \varepsilon \qquad \text{for } p > M \text{ and } q > N."$$

This definition, however, does not necessarily implies that the double sequence $\{a_{m,n}\}$ is bounded. For example, if we define

$$a_{m,n} = \begin{cases} m & \text{for } n = 0 , \\ -m & \text{for } n = 1 , \\ 0 & \text{for } n > 1 , \end{cases}$$

then $\sum_{m,n=0}^{\infty} a_{m,n} = 0$ in the sense of the definition given above. It is evident that $\{a_{m,n}\}$ is not bounded. Therefore, we shall use the concept of absolute convergence for double power series, instead of the definition given above.

Let

$$(8.3) \qquad \sum_{m,n=0}^{\infty} a_{m,n} x^m y^n ,$$

be a double power series, and denote by A the set of all points (x_0, y_0) where (8.3) is absolutely convergent. Let D be the interior of A. It is clear that if $(x_0, y_0) \in A$, then the double sequence $\{a_{m,n} x_0^m y_0^n\}$ is bounded. We shall prove the following proposition:

PROPOSITION 8.1: Suppose that $|x_0|$ and $|y_0|$ are positive and that $0 < \theta < 1$. Then, if $\{a_{m,n} x_0^m y_0^n\}$ is bounded, the double power series (8.3) is absolutely and uniformly convergent for

$$|x| \leq \theta|x_0| , \quad |y| \leq \theta|y_0| .$$

Proof: By the hypothesis, there exists a positive constant K such that

$$\left| a_{m,n} x_0^m y_0^n \right| \leq K \quad (m, n = 0, 1, 2, \cdots) .$$

If $|x| \leq \theta|x_0|$ and $|y| \leq \theta|y_0|$, then

$$\left|a_{m,n} x^m y^n\right| = \left|a_{m,n} x_0^m y_0^n\right| \left|x/x_0\right|^m \left|y/y_0\right|^n \leq K\, \theta^{m+n}$$

Since $\sum_{m,n=0}^{\infty} K\,\theta^{m+n}$ is convergent, $\sum_{m,n=0}^{\infty} a_{m,n} x^m y^n$ is absolutely and uniformly convergent for $|x| \leq \theta|x_0|$, $|y| \leq \theta|y_0|$.

Observe that for any point $(x_0, y_0) \in D$, there exists a neighborhood U of (x_0, y_0) which is contained in A. From this fact it follows that for any point $(x_0, y_0) \in D$ there exists a neighborhood V of (x_0, y_0) such that series (8.3) is absolutely and uniformly convergent in V. The domain D is called the domain of convergence of (8.3).

Let us go back to the hypergeometric series of two variables and prove the following theorem.

THEOREM 8.1: Suppose that $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \neq 0, -1, -2, \ldots$ Then F_j $(j = 1, 2, 3, 4)$ are not polynomials in x, y and

The domain of convergence of
$$\begin{cases} F_1 = \{(x,y);\ |x| < 1,\ |y| < 1\}, \\ F_2 = \{(x,y);\ |x| + |y| < 1\}, \\ F_3 = \{(x,y);\ |x| < 1,\ |y| < 1\}, \\ F_4 = \{(x,y);\ \sqrt{|x|} + \sqrt{|y|} < 1\}, \end{cases}$$

Proof: We shall use the well known formula:

$$(8.4) \qquad \lim_{k\to\infty} \frac{(a,k)}{(k-1)!\,k^a} = \frac{1}{\Gamma(a)}$$

or

$$(8.4') \qquad (a, k) \sim \frac{(k-1)!\,k^a}{\Gamma(a)} \quad \text{as } k \longrightarrow \infty .$$

Case F_1: Put

$$A_{m,n} = \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)(1, m)(1, n)}.$$

It follows from (8.4') that

$$A_{m,n} \sim \frac{(m+n-1)!(m+n)^{\alpha}}{\Gamma(\alpha)} \frac{(m-1)!m^{\beta}}{\Gamma(\beta)} \frac{(n-1)!n^{\beta'}}{\Gamma(\beta')} \frac{\Gamma(\gamma)}{(m+n-1)!(m+n)^{\gamma}} \frac{1}{m!n!}$$

as $m, n \to \infty$, and hence

$$A_{m,n} \sim \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta')} (m+n)^{\alpha-\gamma} m^{\beta-1} n^{\beta'-1}$$

as $m, n \to \infty$. Therefore

$$|A_{m,n}| \leq K(m+n)^{\alpha_1-\gamma_1} m^{\beta_1-1} n^{\beta_1'-1} \quad \text{for } m, n = 0, 1, 2, \cdots$$

where K is a positive constant and $\alpha_1 = \mathrm{Re}(\alpha)$, $\beta_1 = \mathrm{Re}(\beta)$, $\beta_1' = \mathrm{Re}(\beta')$ and $\gamma_1 = \mathrm{Re}(\gamma)$. This shows that

(8.5)
$$\sum_{m,n=0}^{\infty} A_{m,n} x^m y^n$$

is absolutely convergent for $|x| < 1$, $|y| < 1$. On the other hand, there exists a positive constant K' such that

$$K'(m+n)^{\alpha_1-\gamma_1} m^{\beta_1-1} n^{\beta_1'-1} \leq |A_{m,n}|$$

for $m, n = 0, 1, 2, \cdots$. Hence if $|x| > 1$ or $|y| > 1$, the double sequence $\{A_{m,n} x^m y^n\}$ is not bounded. Therefore, the series (8.5) is not absolutely convergent if $|x| > 1$ or $|y| > 1$. This means that the domain of convergence of F_1 is $\{(x,y);\ |x| < 1$ and $|y| < 1\}$.

Case F_2: Put

$$A_{m,n} = \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m)(\gamma', n)(1, m)(1, n)}.$$

It follows from (8.4')

$$A_{m,n} \sim \frac{(m+n-1)!(m+n)^{\alpha}}{\Gamma(\alpha)} \frac{(m-1)!m^{\beta}}{\Gamma(\beta)} \frac{(n-1)!n^{\beta'}}{\Gamma(\beta')} \frac{\Gamma(\gamma)}{(m-1)!m^{\gamma}} \frac{\Gamma(\gamma')}{(n-1)!n^{\gamma'} m!n!}$$

$$= \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta')} (m+n)^{\alpha-1} m^{\beta-\gamma} n^{\beta'-\gamma'} \frac{(m+n)!}{m!n!}$$

and hence

$$|A_{m,n}| \leq K(m+n)^{\alpha_1-1} m^{\beta_1-\gamma_1} n^{\beta_1'-\gamma_1'} \frac{(m+n)!}{m!n!}$$

for $m, n = 0, 1, 2, \cdots$, where K is a positive constant. Take a positive number σ greater than $\beta_1 - \gamma_1$ and $\beta_1' - \gamma_1'$. Then we have

$$m^{\beta_1-\gamma_1} n^{\beta_1'-\gamma_1'} \leq m^{\sigma} n^{\sigma} \leq (m+n)^{2\sigma}/4^{\sigma} \quad \text{for } m, n \geq 0.$$

Therefore

$$|A_{m,n} x^m y^n| \leq K 4^{-\sigma} (m+n)^{2\sigma+\alpha_1-1} \frac{(m+n)!}{m!n!} |x|^m |y|^n.$$

Observing

$$\sum_{m+n=k} |A_{m,n} x^m y^n| \leq K 4^{-\sigma} k^{2\sigma+\alpha_1-1} \sum_{m+n=k} \frac{k!}{m!n!} |x|^m |y|^n$$
$$= K 4^{-\sigma} k^{2\sigma+\alpha_1-1} (|x| + |y|)^k,$$

we get

$$\sum_{m,n=0}^{\infty} |A_{m,n} x^m y^n| \leq K 4^{-\sigma} \sum_{k=0}^{\infty} k^{2\sigma+\alpha_1-1} (|x| + |y|)^k.$$

It follows from this that the domain $\{(x,y);\ |x| + |y| < 1\}$ is contained in the domain of convergence of F_2. To show that this domain is exactly the domain of convergence of F_2, it is sufficient to prove that the double sequence $\{A_{m,n}x^m y^n\}$ is not bounded if $|x|+|y|>1$. This will be left to the readers as an exercise.

Case F_3: This case can be treated in the same way as the case of F_1. Hence the proof of the theorem in this case will be left to the readers as an exercise.

Case F_4: Set

$$A_{m,n} = \frac{(\alpha, m+n)(\beta, m+n)}{(\gamma, m)(\gamma', n)(1, m)(1, n)} .$$

In the same way as in previous cases, we have

$$A_{m,n} \sim \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)}(m+n)^{\alpha+\beta-2}m^{1-\gamma}n^{1-\gamma'}\left(\frac{(m+n)!}{m!n!}\right)^2 .$$

Hence

$$|A_{m,n}| \leqq K(m+n)^{\alpha_1+\beta_1-2}m^{1-\gamma_1}n^{1-\gamma_1'}\left(\frac{(m+n)!}{m!\,n!}\right)^2 ,$$

where K is a positive constant. Take a positive number σ so that

$$\sigma \geqq \max\{1-\gamma_1, 1-\gamma_1'\}.$$

Then

$$m^{1-\gamma_1}n^{1-\gamma_1'} \leqq m^{\sigma}n^{\sigma} \leqq (m+n)^{2\sigma}/4^{\sigma} .$$

Using these estimates, we obtain

$$|A_{m,n}x^m y^n| \leqq K\,4^{-\sigma}(m+n)^{\alpha_1+\beta_1+2\sigma-2}\left(\frac{(m+n)!}{m!\,n!}\right)^2|x|^m|y|^n .$$

Consider the estimates

$$\sum_{m,n=0}^{\infty}|A_{m,n}x^m y^n| = \sum_{k=0}^{\infty}\sum_{m+n=k}|A_{m,n}x^m y^n|$$

$$\leqq K\,4^{-\sigma}\sum_{k=0}^{\infty}k^{\alpha_1+\beta_1+2\sigma-2}\sum_{m+n=k}\left(\frac{(m+n)!}{m!\,n!}\right)^2|x|^m|y|^n ,$$

and

$$\sum_{m+n=k}\left(\frac{(m+n)!}{m!n!}\right)^2|x|^m|y|^n = \sum_{m+n=k}\left(\frac{(m+n)!}{m!n!}\right)^2\left(|x|^{\frac{1}{2}m}\right)^2\left(|y|^{\frac{1}{2}n}\right)^2$$

$$\leqq\left(\sum_{m+n=k}\frac{(m+n)!}{m!\,n!}|x|^{\frac{1}{2}m}|y|^{\frac{1}{2}n}\right)^2$$

$$= ((\sqrt{|x|}+\sqrt{|y|})^k)^2 .$$

Therefore we obtain

$$\sum_{m,n=0}^{\infty}|A_{m,n}x^m y^n| \leqq K\,4^{-\sigma}\sum_{k=0}^{\infty}k^{\alpha_1+\beta_1+2\sigma-2}(\sqrt{|x|}+\sqrt{|y|})^{2k} .$$

The series on the right-hand member converges if $\sqrt{|x|}+\sqrt{|y|}<1$. This means that the set $\{(x,y);\sqrt{|x|}+\sqrt{|y|}<1\}$ is contained in the domain of convergence of F_4. On the other hand it is easily seen that the double sequence $\{A_{m,n}x^m y^n\}$ is not bounded if $\sqrt{|x|}+\sqrt{|y|}>1$. Thus we have

the domain of convergence of $F_4 = \{(x,y);\sqrt{|x|}+\sqrt{|y|}<1\}$.

As in the case of one variable, the hypergeometric series of two variables are Newton series with respect to the parameters α, α', β and β', and factorial series with respect to γ and γ'. Thus we obtain the following theorem:

THEOREM 8.2: The hypergeometric functions of two variables are meromorphic with respect to the parameters and their poles are at

$$\gamma = 0, -1, -2, \cdots \quad\text{and}\quad \gamma' = 0, -1, -2, \cdots .$$

Exercise 1. Complete the proof of Theorem 8.1.

9. Contiguous functions. The contiguous functions of $F(\alpha, \beta, \gamma, x)$ are obtained by increasing or decreasing one of the parameters by one. In the same way, we can define the contiguous functions of the hypergeometric functions of two variables.

DEFINITION 9.1: The functions obtained from the hypergeometric functions F_j $(j = 1, 2, 3, 4)$ by increasing or decreasing one of the parameters involved in F_j by one are called contiguous to F_j.

The function F_1 has eight contiguous functions:

$$F_1(\alpha+1, \beta, \beta', \gamma, x, y), \quad F_1(\alpha-1, \beta, \beta', \gamma, x, y),$$
$$F_1(\alpha, \beta+1, \beta', \gamma, x, y), \quad F_1(\alpha, \beta-1, \beta', \gamma, x, y),$$
$$F_1(\alpha, \beta, \beta'+1, \gamma, x, y), \quad F_1(\alpha, \beta, \beta'-1, \gamma, x, y),$$
$$F_1(\alpha, \beta, \beta', \gamma+1, x, y), \quad F_1(\alpha, \beta, \beta', \gamma-1, x, y).$$

The function F_2 has ten contiguous functions. etc.

It was proved that any contiguous function of $F(\alpha, \beta, \gamma, x)$ is expressible as a linear combination of F and its first derivative. We shall prove an analogous theorem for the function $F_1(\alpha, \beta, \beta', \gamma, x, y)$.

THEOREM 9.1: We have

$$\alpha F_1(\alpha+1, \beta, \beta', \gamma, x, y) = \alpha F_1 + x\,\partial F_1/\partial x + y\,\partial F_1/\partial y,$$
$$\beta F_1(\alpha, \beta+1, \beta', \gamma, x, y) = \beta F_1 + x\,\partial F_1/\partial x,$$
$$\beta' F_1(\alpha, \beta, \beta'+1, \gamma, x, y) = \beta' F_1 + y\,\partial F_1/\partial y,$$
$$(\gamma-1)F_1(\alpha, \beta, \beta', \gamma-1, x, y) = (\gamma-1)F_1 + x\,\partial F_1/\partial x + y\,\partial F_1/\partial y,$$
$$(\gamma-\alpha)F_1(\alpha-1, \beta, \beta', \gamma, x, y) = (\gamma-\alpha-\beta x-\beta' y)F_1 + x(1-x)\partial F_1/\partial x + y(1-y)\partial F_1/\partial y,$$

$$(\gamma-\beta-\beta')F_1(\alpha, \beta-1, \beta', \gamma, x, y) = (\gamma-\beta-\beta'-\alpha x)F_1 + x(1-x)\partial F_1/\partial x + x(1-y)\partial F_1/\partial y,$$
$$(\gamma-\beta-\beta')F_1(\alpha, \beta, \beta'-1, \gamma, x, y) = (\gamma-\beta-\beta'-\alpha y)F_1 + y(1-x)\partial F_1/\partial x + y(1-y)\partial F_1/\partial y,$$
$$(\gamma-\alpha)(\gamma-\beta-\beta')F_1(\alpha, \beta, \beta', \gamma+1, x, y) = (\gamma-\alpha-\beta-\beta')\gamma F_1 + \gamma(1-x)\partial F_1/\partial x + \gamma(1-y)\partial F_1/\partial y.$$

Sketch of the Proof: Set

$$F_1(\alpha, \beta, \beta', \gamma, x, y) = \sum_{m,n=0}^{\infty} A_{m,n} x^m y^n,$$

where

$$A_{m,n} = \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)(1, m)(1, n)}.$$

Clearly we have

$$x\,\partial F_1/\partial x = \sum_{m,n=0}^{\infty} m A_{m,n} x^m y^n,$$

$$y\,\partial F_1/\partial y = \sum_{m,n=0}^{\infty} n A_{m,n} x^m y^n.$$

Observe that

$$(9.1)\quad \begin{cases} F_1(\alpha+1, \beta, \beta', \gamma, x, y) = \displaystyle\sum_{m,n=0}^{\infty} \frac{\alpha+m+n}{\alpha} A_{m,n} x^m y^n, \\ F_1(\alpha, \beta+1, \beta', \gamma, x, y) = \displaystyle\sum_{m,n=0}^{\infty} \frac{\beta+m}{\beta} A_{m,n} x^m y^n, \\ F_1(\alpha, \beta, \beta'+1, \gamma, x, y) = \displaystyle\sum_{m,n=0}^{\infty} \frac{\beta'+n}{\beta'} A_{m,n} x^m y^n, \\ F_1(\alpha, \beta, \beta', \gamma-1, x, y) = \displaystyle\sum_{m,n=0}^{\infty} \frac{\gamma+m+n-1}{\gamma-1} A_{m,n} x^m y^n. \end{cases}$$

Using these identities, we obtain the first four identities of Theorem 9.1.

We shall next prove the fifth identity. Observe that

$$F_1(\alpha-1, \beta, \beta', \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{\alpha-1}{\alpha+m+n-1} A_{m,n} x^m y^n$$

On the other hand,

$$xF_1(\alpha, \beta, \beta', \gamma, x, y) = \sum_{m,n=0}^{\infty} A_{m,n} x^{m+1} y^n$$

$$= \sum_{m,n=0}^{\infty} A_{m-1,n} x^m y^n ,$$

where

$$A_{m-1,n} = \frac{(\alpha, m+n-1)(\beta, m-1)(\beta', n)}{(\gamma, m+n-1)(1, m-1)(1, n)}$$

$$= \frac{(\gamma+m+n-1)m}{(\alpha+m+n-1)(\beta+m-1)} A_{m,n}$$

and

$$A_{-1,n} = 0 .$$

Hence

$$xF_1(\alpha, \beta, \beta', \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\gamma+m+n-1)m}{(\alpha+m+n-1)(\beta+m-1)} A_{m,n} x^m y^n ,$$

Similarly

$$yF_1(\alpha, \beta, \beta', \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\gamma+m+n-1)n}{(\alpha+m+n-1)(\beta'+n-1)} A_{m,n} x^m y^n ,$$

$$x^2 \partial F_1/\partial x = \sum_{m,n=0}^{\infty} \frac{(\gamma+m+n-1)(m-1)m}{(\alpha+m+n-1)(\beta+m-1)} A_{m,n} x^m y^n ,$$

$$y^2 \partial F_1/\partial y = \sum_{m,n=0}^{\infty} \frac{(\gamma+m+n-1)(n-1)n}{(\alpha+m+n-1)(\beta'+n-1)} A_{m,n} x^m y^n .$$

Inserting these formulas into the right-hand member of the fifth identity of Theorem 9.1, we can verify this identity by somewhat

lengthy computation which will be omitted. The other three identities can be proved in the same manner

Exercise 1. Complete the proof of Theorem 9.1.

The eight identities given by Theorem 9.1 mean that every contiguous function of F_1 can be expressed as a linear combination of F_1 and its first derivatives whose coefficients are rational in $x, y, \alpha, \beta, \beta'$ and γ. Eliminating the first derivatives from any three identities, we obtain a linear homogeneous relation between three contiguous functions and F_1. It should be noted that there are cases when the first derivatives can be eliminated from two identities. In such cases, we obtain linear homogeneous relations between two contiguous functions and F_1.

THEOREM 9.2: Three contiguous functions and F_1, or two contiguous functions and F_1 are connected by a linear homogeneous relation whose coefficients are polynomials in $\alpha, \beta, \beta', \gamma, x, y$.

Proof: Eliminate the first derivatives from three or two identities of Theorem 9.1. For example, we obtain

$$\alpha F_1(\alpha+1, \beta, \beta', \gamma, x, y) - \beta F_1(\alpha, \beta+1, \beta', \gamma, x, y) - \beta' F_1(\alpha, \beta, \beta'+1, \gamma, x, y) - (\alpha-\beta-\beta')F_1(\alpha, \beta, \beta', \gamma, x, y) = 0 ,$$

$$(\gamma-\alpha)F_1(\alpha, \beta, \beta', \gamma+1, x, y) - \gamma F_1(\alpha, \beta-1, \beta', \gamma, x, y) + \gamma(1-x)F_1(\alpha, \beta, \beta', \gamma, x, y) = 0 .$$

Exercise 2. Complete the proof of Theorem 9.2.

Observe that on the right-hand members of (9.1) the indices m and n are in the numerators of the coefficients of power series. On the other hand, on the right-hand members of the following formulas:

$$(9.2)\begin{cases} F_1(\alpha-1,\ \beta,\ \beta',\ \gamma,\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{\alpha-1}{\alpha+m+n-1} A_{m,n} x^m y^n , \\ F_1(\alpha,\ \beta-1,\ \beta',\ \gamma,\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{\beta-1}{\beta+m-1} A_{m,n} x^m y^n , \\ F_1(\alpha,\ \beta,\ \beta'-1,\ \gamma,\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{\beta'-1}{\beta'+n-1} A_{m,n} x^m y^n , \\ F_1(\alpha,\ \beta,\ \beta',\ \gamma+1,\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{\gamma}{\gamma+m+n} A_{m,n} x^m y^n \end{cases}$$

the indices m and n are in the denominators of the coefficients of power series. This fact was one of reasons why the last four formulas of Theorem 9.1 are more complicated than the first four. The same fact may be observed also for F_2, F_3 and F_4, and we can prove the following theorem:

THEOREM 9.3: The contiguous functions of F_2, F_3 and F_4 with $\alpha+1$, $\alpha'+1$, $\beta+1$, $\beta'+1$, $\gamma-1$ or $\gamma'-1$ are expressible as linear combinations of the original functions F_j and their derivatives $\partial F_j/\partial x$, $\partial F_j/\partial y$. For example, since

$$F_2(\alpha+1,\ \beta,\ \beta',\ \gamma,\ \gamma',\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{\alpha+m+n}{\alpha} A_{m,n} x^m y^n ,$$

we obtain

$$\alpha F_2(\alpha+1,\ \beta,\ \beta',\ \gamma,\ \gamma',\ x,\ y) = \alpha F_2 + x\,\partial F_2/\partial x + y\,\partial F_2/\partial y .$$

Eliminating the derivatives from the recurrence formulas of Theorem 9.3, we obtain relations between contiguous functions and the original functions. For example, we have

$$\alpha F_2(\alpha+1,\ \beta,\ \beta',\ \gamma,\ \gamma',x,y) - \beta F_2(\alpha,\beta+1,\ \beta',\ \gamma,\ \gamma',x,y)$$
$$-\beta' F_2(\alpha,\ \beta,\ \beta'+1,\ \gamma,\ \gamma',x,y) - (\alpha-\beta-\beta')F_1(\alpha,\beta,\ \beta',\gamma,\ \gamma',x,y) = 0$$
$$\alpha F_3(\alpha+1,\ \alpha',\ \beta,\ \beta',\ \gamma,\ x,\ y) - \beta F_3(\alpha,\ \alpha',\beta+1,\ \beta',\ \gamma,\ x,\ y)$$
$$-(\alpha-\beta)\,F_3(\alpha,\ \alpha',\ \beta,\ \beta',\ \gamma,\ x,\ y) = 0 ,$$
$$\alpha F_4(\alpha+1,\beta,\ \gamma,\ \gamma',\ x,\ y) - \beta F_4(\alpha,\beta+1,\ \gamma,\ \gamma',\ x,\ y)$$
$$-(\alpha-\beta)\,F_4(\alpha,\ \beta,\ \gamma,\ \gamma',\ x,\ y) = 0 .$$

For other contiguous functions of F_2, F_3 and F_4 which are obtained by decreasing α, α', β or β' by one, or by increasing γ or γ' by one:

$$(9.3)\begin{cases} F_2(\alpha-1,\ \beta,\ \beta',\ \gamma,\ \gamma',\ x,\ y) , \\ \vdots \\ F_4(\alpha,\ \beta,\ \gamma,\ \gamma'+1,\ x,\ y) , \end{cases}$$

the situation is not so simple. Appell and Kampé de Fériet claimed in their book that

"Les fonctions contiguës à une fonction F_j donnée s'expriment linéairement au moyen de cette fonction F_j et de ses dérivées partielles $\partial F_j/\partial x$ et $\partial F_j/\partial y$."

Their statement is certainly true for those contiguous functions in Theorem 9.3. However, their statement is not necessarily true for the contiguous functions (9.3). The author does not know whether the recurrence formulas for those contiguous functions were already discovered.

PROBLEM 1: Find the recurrence formulas for the contiguous functions (9.3).

This problem might not be very difficult. In fact, the author found the following three formulas for F_2:

$$(\alpha-\gamma)(\alpha-\gamma')(\alpha-\gamma-\gamma'+1)F_2(\alpha-1,\beta,\beta',\gamma,\gamma',x,y)$$
$$=(\alpha-\gamma-\gamma'+1)\{(\alpha-\gamma)(\alpha-\gamma')+(\alpha-\gamma')\beta x+(\alpha-\gamma)\beta' y\}F_2$$
$$-\{(\alpha-\gamma')(\alpha-\gamma-\gamma'+1)(1-x)+(2\alpha-\gamma-\gamma')\beta' y\}x\,\partial F_2/\partial x$$
$$-\{(\alpha-\gamma)(\alpha-\gamma-\gamma'+1)(1-y)+(2\alpha-\gamma-\gamma')\beta x\}y\,\partial F_2/\partial y$$
$$+(2\alpha-\gamma-\gamma')(1-x-y)xy\,\partial^2F_2/\partial x\,\partial y,$$

$$(\gamma-\beta)F_2(\alpha,\beta-1,\beta',\gamma,\gamma',x,y)=(\gamma-\beta-\alpha x)F_2+x(1-x)\,\partial F_2/\partial x - xy\,\partial F_2/\partial y,$$

$$(\gamma'-\beta')F_2(\alpha,\beta,\beta'-1,\gamma,\gamma',x,y)=(\gamma'-\beta'-\alpha y)F_2 - xy\,\partial F_2/\partial x+y(1-y)\,\partial F_2/\partial y.$$

It seems to the author that the other recurrence formulas as well as the above formulas will be derived by careful computations together with inspections and that each of the contiguous functions (9.3) is expressible as linear combinations of the original function F_j, its first derivatives $\partial F_j/\partial x$, $\partial F_j/\partial y$ and the mixed second derivative $\partial^2F_j/\partial x\,\partial y$.

PROBLEM 2: Find a systematic (or intrinsic) method of deriving the recurrence formulas.

10. Euler integral representations. The hypergeometric functions of two variables, except for F_4, have Euler integral representations which are double integrals but very similar to that of $F(\alpha,\beta,\gamma,x)$.

THEOREM 10.1: We have

(10.1) $F_1(\alpha,\beta,\beta',\gamma,x,y)$

$$=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}\iint_{u,v,1-u-v\geq 0} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-xu-yv)^{-\alpha}\,du\,dv$$

if $Re(\beta)>0$, $Re(\beta')>0$, $Re(\gamma-\beta-\beta')>0$;

(10.2) $F_2(\alpha,\beta,\beta',\gamma,\gamma',x,y)$

$$=\frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}\iint_{\substack{0\leq u\leq 1\\ 0\leq v\leq 1}} u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}(1-xu-yv)^{-\alpha}\,du\,dv$$

if $Re(\beta)>0$, $Re(\beta')>0$, $Re(\gamma-\beta)>0$, $Re(\gamma'-\beta')>0$;

(10.3) $F_3(\alpha,\alpha',\beta,\beta',\gamma,x,y)$

$$=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}\iint_{u,v,1-u-v\geq 0} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-xu)^{-\alpha}(1-yv)^{-\alpha'}\,du\,dv$$

if $Re(\beta)>0$, $Re(\beta')>0$, $Re(\gamma-\beta-\beta')>0$.

Proof: The main idea of the proof of this theorem is similar to that of the corresponding theorem (Theorem 3.1 of Section 3, Chapter I) in the case of one variable. We shall show only formal calculations.

First of all, remark that

$$(1-xu)^{-\alpha}=\sum_{m=0}^{\infty}\frac{(\alpha,m)}{m!}x^m u^m,$$

$$(1-yv)^{-\alpha'} = \sum_{n=0}^{\infty} \frac{(\alpha', n)}{n!} y^n v^n ,$$

$$(1-xu-yv)^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha, k)}{k!} (xu+yv)^k = \sum_{k=0}^{\infty} \frac{(\alpha, k)}{k!} \sum_{m+n=k} \frac{k!}{m!n!} x^m y^n u^m v^n$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)}{m!n!} x^m y^n u^m v^n .$$

Furthermore let us note that we have

$$\int_0^1 u^{p-1}(1-u)^{q-1} du = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \quad \text{if} \quad \mathrm{Re}(p) > 0, \quad \mathrm{Re}(q) > 0,$$

and

$$\iint_{u,v,1-u-v\geq 0} u^{p-1} v^{q-1} (1-u-v)^{r-1} du dv = \frac{\Gamma(p)\,\Gamma(q)\,\Gamma(r)}{\Gamma(p+q+r)} .$$

The second formula will be derived by a repeated application of the first formula.

Now let us return to the Euler integral representations. Consider

$$I_1 = \iint_{u,v,1-u-v\geq 0} u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-xu-yv)^{-\alpha} du dv .$$

Replacing the last factor $(1-xu-yv)^{-\alpha}$ by the corresponding power series representation, we obtain

$$I_1 = \iint_{u,v,1-u-v\geq 0} u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)}{(1,m)(1,n)} x^m y^n u^m v^n du dv$$

Interchanging the order of the integration and the summation, we get

$$I_1 = \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)}{(1,m)(1,n)} x^m y^n \iint_{u,v,1-u-v\geq 0} u^{\beta+m-1} v^{\beta'+n-1} (1-u-v)^{\gamma-\beta-\beta'-1} du dv$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)}{(1,m)(1,n)} \frac{\Gamma(\beta+m)\Gamma(\beta'+n)\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma+m+n)} x^m y^n$$

$$= \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma)} F_1(\alpha, \beta, \beta', \gamma, x, y) .$$

Notice that we used here the identity $\Gamma(a+k) = (a, k)\Gamma(a)$.

Similarly we have

$$I_2 = \iint_{\substack{0\leq u\leq 1\\ 0\leq v\leq 1}} u^{\beta-1} v^{\beta'-1} (1-u)^{\gamma-\beta-1} (1-v)^{\gamma'-\beta'-1} (1-xu-yv)^{-\alpha} du\, dv$$

$$= \int_0^1 \int_0^1 u^{\beta-1} v^{\beta'-1} (1-u)^{\gamma-\beta-1} (1-v)^{\gamma'-\beta'-1} \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)}{(1,m)(1,n)} x^m y^n u^m v^n du\, dv$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)}{(1,m)(1,n)} x^m y^n \int_0^1 \int_0^1 u^{\beta+m-1} (1-u)^{\gamma-\beta-1} v^{\beta'+n-1} (1-v)^{\gamma'-\beta'-1} du\, dv$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)}{(1,m)(1,n)} x^m y^n \int_0^1 u^{\beta+m-1} (1-u)^{\gamma-\beta-1} du \int_0^1 v^{\beta'+n-1} (1-v)^{\gamma'-\beta'-1} dv$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)}{(1,m)(1,n)} x^m y^n \frac{\Gamma(\beta+m)\Gamma(\gamma-\beta)}{\Gamma(\gamma+m)} \frac{\Gamma(\beta'+n)\Gamma(\gamma'-\beta')}{\Gamma(\gamma'+n)}$$

$$= \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}{\Gamma(\gamma)\Gamma(\gamma')} F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y)$$

and

$$I_3 = \iint_{u,v,1-u-v\geq 0} u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-xu)^{-\alpha} (1-yv)^{-\alpha'} du\, dv$$

$$= \iint_{u,v,1-u-v\geq 0} u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} \sum_{m,n=0}^{\infty} \frac{(\alpha,m)(\alpha',n)}{(1,m)(1,n)} x^m y^n u^m v^n du\, dv$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha,m)(\alpha',n)}{(1,m)(1,n)} x^m y^n \iint_{u,v,1-u-v\geq 0} u^{\beta+m-1} v^{\beta'+n-1} (1-u-v)^{\gamma-\beta-\beta'-1} du\, dv$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha,m)(\alpha',n)}{(1,m)(1,n)} x^m y^n \frac{\Gamma(\beta+m)\Gamma(\beta'+n)\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma+m+n)}$$

$$= \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma)} F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) \ .$$

This completes the proof of Theorem 10.1.

The function F_4 does not have any simple integral representation such as (10.1), (10.2) and (10.3). This, however, does not mean that F_4 does not have any double integral representation. For example, the following formula is known:

$$F_4(\alpha, \beta, \gamma, \gamma', x(1-y), y(1-x)) = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma'-\beta)} \times$$

$$\int_0^1\int_0^1 u^{\alpha-1} v^{\beta-1} (1-u)^{\gamma-\alpha-1} (1-v)^{\gamma'-\beta-1} (1-xu)^{\alpha-\gamma-\gamma'} (1-yv)^{\beta-\gamma-\gamma'} (1-xu-yv)^{\gamma+\gamma'-\alpha-\beta-1} du\, dv$$

where $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma-\alpha) > 0$, $\mathrm{Re}(\gamma'-\beta) > 0$.

Picard discovered that F_1 admits another integral representation which is not a double integral, but a simple integral.

THEOREM 10.2: We have

(10.4) $F_1(\alpha, \beta, \beta', \gamma, x, y)$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1} (1-u)^{\gamma-\alpha-1} (1-xu)^{-\beta} (1-yu)^{-\beta'} du$$

if $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\gamma-\alpha) > 0$.

The proof of this theorem is very similar to that of Theorem 10.1.

From (10.4) we can conclude that F_1 is analytically continuable and holomorphic for $[\mathbb{C} - [1, \infty)] \times [\mathbb{C} - [1, \infty)]$. Similarly (10.3) implies that F_3 is holomorphic for $[\mathbb{C} - [1, \infty)] \times [\mathbb{C} - (1, \infty)]$. On the other hand, we can prove the following theorem by using (10.4).

THEOREM 10.3: We have

(10.5) $F_1(\alpha, \beta, \beta', \gamma, x, y)$

$$= (1-x)^{-\beta}(1-y)^{-\beta'} F_1(\gamma-\alpha, \beta, \beta', \gamma, x/(x-1), y/(y-1))$$

$$= (1-x)^{-\alpha} F_1(\alpha, \gamma-\beta-\beta', \beta', \gamma, x/(x-1), (x-y)/(x-1))$$

$$= (1-y)^{-\alpha} F_1(\alpha, \beta, \gamma-\beta-\beta', \gamma, (y-x)/(y-1), y/(y-1))$$

$$= (1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'} F_1(\gamma-\alpha, \gamma-\beta-\beta', \beta', \gamma, x, (y-x)/(y-1))$$

$$= (1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'} F_1(\gamma-\alpha, \beta, \gamma-\beta-\beta', \gamma, (x-y)/(x-1), y) \ .$$

<u>Proof</u>: This theorem can be proved by changing variable u respectively by

$$u = 1-v, \quad u = v/[(1-x)+vx], \quad u = v/[(1-y)+vy],$$

$$u = (1-v)/(1-vx) \quad \text{and} \quad u = (1-v)/(1-vy) \ .$$

<u>Exercise</u> 1. Prove Theorems 10.2 and 10.3.

It is easily verified that the following six changes of variables:

$$u = 1-u', \qquad v = v',$$
$$u = u', \qquad v = 1-v',$$
$$u = 1-u', \qquad v = 1-v',$$
$$u = 1-v', \qquad v = u',$$
$$u = v', \qquad v = 1-u',$$
$$u = 1-v', \qquad v = 1-u'$$

map the square $0 \le u, v \le 1$ onto itself and do not change the form of the integral (10.2). By using these transformations we can prove the following theorem.

THEOREM 10.4: We have

$$
\begin{aligned}
(10.6)\quad F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) &= (1-x)^{-\alpha} F_2(\alpha, \gamma-\beta, \beta', \gamma, \gamma', x/(x-1), y/(1-x)) \\
&= (1-y)^{-\alpha} F_2(\alpha, \beta, \gamma'-\beta', \gamma, \gamma', x/(1-y), y/(y-1)) \\
&= (1-x-y)^{-\alpha} F_2(\alpha, \gamma-\beta, \gamma'-\beta', \gamma, \gamma', x/(x+y-1), y/(x+y-1)) \\
&= (1-y)^{-\alpha} F_2(\alpha, \gamma'-\beta', \beta, \gamma', \gamma, y/(y-1), x/(1-y)) \\
&= (1-x)^{-\alpha} F_2(\alpha, \beta', \gamma-\beta, \gamma', \gamma, y/(1-x), x/(x-1)) \\
&= (1-x-y)^{-\alpha} F_2(\alpha, \gamma'-\beta', \gamma-\beta, \gamma', \gamma, y/(x+y-1), x/(x+y-1)).
\end{aligned}
$$

It can be shown that there is no change of variables of the form:

$$u = \frac{A'u' + B'v' + C'}{Au' + Bv' + C}, \qquad v' = \frac{A''u' + B''v' + C''}{Au' + Bv' + C}$$

which maps the triangle $u, v, 1-u-v \ge 0$ into itself and which does not change the form of the integral (10.3). Therefore, it is difficult to find any transformation formulas for F_3 which are similar to (10.5) and (10.6).

Let us next find some relations between contiguous functions. To do this, consider the representation (10.4) of F_1. Raising α by one we obtain

$$
\begin{aligned}
\frac{\Gamma(\alpha+1)\Gamma(\gamma-\alpha-1)}{\Gamma(\gamma)} F_1(\alpha+1, \beta, \beta', \gamma, x, y) \\
= \int_0^1 u^{\alpha}(1-u)^{\gamma-\alpha-2}(1-xu)^{-\beta}(1-yv)^{-\beta'}\, du .
\end{aligned}
$$

On the other hand, lowering γ by one we get

$$
\begin{aligned}
\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha-1)}{\Gamma(\gamma-1)} F_1(\alpha, \beta, \beta', \gamma-1, x, y) \\
= \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-2}(1-xu)^{-\beta}(1-yu)^{-\beta'}\, du .
\end{aligned}
$$

If we put

$$U(u) = u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-xu)^{-\beta}(1-yu)^{-\beta'},$$

we have

$$\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} F_1(\alpha, \beta, \beta', \gamma, x, y) = \int_0^1 U(u)\, du ,$$

$$
\begin{aligned}
\frac{\Gamma(\alpha+1)\Gamma(\gamma-\alpha-1)}{\Gamma(\gamma)} F_1(\alpha+1, \beta, \beta', \gamma, x, y) &= \int_0^1 u(1-u)^{-1} U(u)\, du \\
&= -\int_0^1 U(u)du + \int_0^1 (1-u)^{-1} U(u) du
\end{aligned}
$$

$$\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha-1)}{\Gamma(\gamma-1)} F_1(\alpha, \beta, \beta', \gamma-1, x, y) = \int_0^1 (1-u)^{-1} U(u) du ,$$

and hence

$$
\begin{aligned}
\alpha F_1(\alpha+1, \beta, \beta', \gamma, x, y) + (\gamma-\alpha-1) F_1(\alpha, \beta, \beta', \gamma, x, y) \\
- (\gamma-1) F_1(\alpha, \beta, \beta', \gamma-1, x, y) = 0 .
\end{aligned}
$$

In this computation, we used the identity $\Gamma(x+1) = x\,\Gamma(x)$.

Let us differentiate $U(u)$ to obtain

$$(10.7)\qquad U'(u) = \left[\frac{\alpha-1}{u} + \frac{\alpha+1-\gamma}{1-u} + \frac{\beta x}{1-xu} + \frac{\beta' y}{1-yu}\right] U(u) .$$

Assume that $\mathrm{Re}(\alpha) > 1$, $\mathrm{Re}(\gamma-\alpha) > 1$. Then

$$\lim_{u\to+0} U(u) = 0 , \qquad \lim_{u\to 1-0} U(u) = 0 .$$

Therefore, integrating both hand members of (10.7) from 0 to 1, we get

$$(\alpha-1)\int_0^1 u^{-1}U(u)du+(\alpha+1-\gamma)\int_0^1 (1-u)^{-1}U(u)du$$
$$+\beta x\int_0^1 (1-xu)^{-1}U(u)du+\beta' y\int_0^1 (1-yu)^{-1}U(u)du = 0 .$$

This means that

$$(\gamma-\alpha)F_1(\alpha-1,\beta,\beta',\gamma,x,y)+(\alpha-1)F_1(\alpha,\beta,\beta',\gamma,x,y)$$
$$-(\gamma-1)F_1(\alpha,\beta,\beta',\gamma-1,x,y)+\beta x F_1(\alpha,\beta+1,\beta',\gamma,x,y)$$
$$+\beta' y F_1(\alpha,\beta,\beta'+1,\gamma,x,y) = 0 .$$

Finally we shall present the following application of Euler integral representations. In the proof of Theorem 10.1, we expanded the kernels of integrals $(1-xu-yv)^{-\alpha}$ and $(1-xu)^{-\alpha}(1-yv)^{-\alpha'}$ into the corresponding double power series in u and v. If we expand these kernels into series of different types, we shall obtain some other expressions of F_1, F_2, F_3 in forms of certain series. We shall present such an expression for F_1. In (10.1), change variables by

$$u = s(1-t), \qquad v = st .$$

This maps the square $0 \leqq t, s \leqq 1$ into the triangle $u, v, 1-u-v \geqq 0$. If $s = 0$, then $u = 0$ and $v = 0$ for $0 \leqq t \leqq 1$. Hence the side: $s = 0$, $0 \leqq t \leqq 1$ of the square is mapped onto a point: $u = 0$, $v = 0$. Otherwise, the mapping is one-to-one. (See Fig. 10.1.)

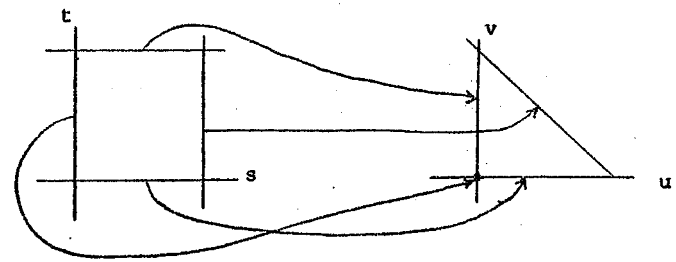


Fig. 10.1.

Then we have

$$\frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma)}F_1(\alpha,\beta,\beta',\gamma,x,y)$$
$$= \iint_{u,v,1-u-v\geq 0} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-xu-yv)^{-\alpha}dudv$$
$$= \int_0^1\int_0^1 s^{\beta-1}(1-t)^{\beta-1}s^{\beta'-1}t^{\beta'-1}(1-s)^{\gamma-\beta-\beta'-1}(1-xs+xst-yst)^{-\alpha}sdsdt$$
$$= \int_0^1\int_0^1 s^{\beta+\beta'-1}t^{\beta'-1}(1-s)^{\gamma-\beta-\beta'-1}(1-t)^{\beta-1}(1-xs+xst-yst)^{-\alpha}dsdt.$$

Consider the last factor

$$(1-xs+xst-yst)^{-\alpha} = (1-xs)^{-\alpha}(1-(y-x)st(1-sx)^{-1})^{-\alpha}$$
$$= \sum_{m=0}^{\infty}\frac{(\alpha,m)}{(1,m)}(y-x)^m s^m t^m (1-xs)^{-(m+\alpha)} .$$

Then

$$\frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma)} F_1 = \sum_{m=0}^{\infty} \frac{(\alpha,m)}{(1,m)} (y-x)^m \times$$

$$\int_0^1 s^{\beta+\beta'+m-1}(1-s)^{\gamma-\beta-\beta'-1}(1-xs)^{-(m+\alpha)}ds \int_0^1 t^{\beta'+m-1}(1-t)^{\beta-1}dt \ .$$

Observe that

$$\int_0^1 s^{\beta+\beta'+m-1}(1-s)^{\gamma+m-(\beta+\beta'+m)-1}(1-sx)^{-(m+\alpha)}ds$$

$$= \frac{\Gamma(\beta+\beta'+m)\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma+m)} F(\alpha+m, \ \beta+\beta'+m, \gamma+m, x)$$

and

$$\int_0^1 t^{\beta'+m-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\beta'+m)\Gamma(\beta)}{\Gamma(\beta+\beta'+m)} \ .$$

Hence

$$F_1(\alpha,\beta,\beta',\gamma,x,y) = \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta',m)}{(\gamma,m)(1,m)} F(\alpha+m,\beta+\beta'+m,\gamma+m,x)(y-x)^m \ .$$

In particular, putting $y = x$, we obtain

$$F_1(\alpha,\beta,\beta',\gamma,x,x) = F(\alpha,\beta+\beta',\gamma,x) \ .$$

<u>Exercise</u> 2. Derive the following formula:

$$F_1(\alpha,\beta,\beta',\gamma,x,y) = \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta+\beta',m)}{(\gamma,m)(1,m)} F(-m,\beta',\beta+\beta',1-y/x)x^m \ .$$

(It sould be noted that $F(-m,\beta',\beta+\beta',1-y/x)$ are polynomials.)

11. <u>Barnes integral representations</u>. In Section 4 (Chapter I) we proved that the function $F(\alpha, \beta, \gamma, x)$ admits an integral representation of the form

$$(11.1) \quad F(\alpha,\beta,\gamma,x) = \frac{1}{2\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\int_B \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-x)^s ds$$

if $\alpha, \beta, \gamma \neq 0, -1, -2, \cdots$, where B is the path given by Fig. 4.1. (See Theorem 4.1.) Notice that the path B is determined by α and β. Furthermore, if m is a nonnegative integer, the path B determined by $\alpha+m$ and β can be replaced by the path determined by α and β.

In Remark 4 (p.44) of Section 7 we proved the following formula:

$$F(\alpha,\beta,\beta',\gamma,x,y) = \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)} F(\alpha+m,\beta',\gamma+m,y)x^m \ .$$

If we utilize the Barnes integral representation (11.1) of $F(\alpha,\beta,\gamma,x)$, we obtain

$$\frac{\Gamma(\alpha)\Gamma(\beta')}{\Gamma(\gamma)} F_1(\alpha,\beta,\beta',\gamma,x,y) = \sum_{m=0}^{\infty} \frac{(\beta,m)}{(1,m)} \frac{\Gamma(\alpha,m)\Gamma(\beta')}{\Gamma(\gamma+m)} F(\alpha+m,\beta',\gamma+m,y)\, x^m$$

$$= \sum_{m=0}^{\infty} \frac{(\beta,m)}{(1,m)} x^m \frac{1}{2\pi i}\int_B \frac{\Gamma(\alpha+m+t)\Gamma(\beta'+t)}{\Gamma(\gamma+m+t)}\Gamma(-t)(-y)^t dt$$

$$= \frac{1}{2\pi i}\sum_{m=0}^{\infty}\int_B \left[\frac{(\alpha+t,m)(\beta,m)}{(\gamma+t,m)(1,m)}x^m\right]\frac{\Gamma(\alpha+t)\Gamma(\beta'+t)}{\Gamma(\gamma+t)}\Gamma(-t)(-y)^t dt$$

$$= \frac{1}{2\pi i}\int_B \sum \frac{(\alpha+t,m)(\beta,m)}{(\gamma+t,m)(1,m)}x^m \frac{\Gamma(\alpha+t)\Gamma(\beta'+t)}{\Gamma(\gamma+t)}\Gamma(-t)(-y)^t dt$$

$$= \frac{1}{2\pi i}\int_B F(\alpha+t,\beta,\gamma+t,x)\frac{\Gamma(\alpha+t)\Gamma(\beta'+t)}{\Gamma(\gamma+t)}\Gamma(-t)(-y)^t dt.$$

Thus we obtain the following results:

THEOREM 11.1: Suppose that $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \neq 0, -1, -2, \cdots$. Then the hypergeometric functions F_1, F_2, F_3 and F_4 admit integral representations

(11.2) $$\frac{\Gamma(\alpha)\Gamma(\beta')}{\Gamma(\gamma)}F_1(\alpha,\beta,\beta',\gamma,x,y) = \frac{1}{2\pi i}\int_B F(\alpha+t,\beta,\gamma+t,x)\frac{\Gamma(\alpha+t)\Gamma(\beta'+t)}{\Gamma(\gamma+t)}\Gamma(-t)(-y)^t dt,$$

(11.3) $$\frac{\Gamma(\alpha)\Gamma(\beta')}{\Gamma(\gamma')}F_2(\alpha,\beta,\beta',\gamma,\gamma',x,y) = \frac{1}{2\pi i}\int_B F(\alpha+t,\beta,\gamma,x)\frac{\Gamma(\alpha+t)\Gamma(\beta'+t)}{\Gamma(\gamma'+t)}\Gamma(-t)(-y)^t dt,$$

(11.4) $$\frac{\Gamma(\alpha')\Gamma(\beta')}{\Gamma(\gamma)}F_3(\alpha,\alpha',\beta,\beta',\gamma,x,y) = \frac{1}{2\pi i}\int_B F(\alpha,\beta,\gamma+t,x)\frac{\Gamma(\alpha'+t)\Gamma(\beta'+t)}{\Gamma(\gamma+t)}\Gamma(-t)(-y)^t dt,$$

(11.5) $$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma')}F_4(\alpha,\beta,\gamma,\gamma',x,y) = \frac{1}{2\pi i}\int_B F(\alpha+t,\beta+t,\gamma,x)\frac{\Gamma(\alpha+t)\Gamma(\beta+t)}{\Gamma(\gamma'+t)}\Gamma(-t)(-y)^t dt$$

respectively, where B is the path determined by α, β' in (11.2) and (11.3), by α', β' in (11.4), and by α, β in (11.5).

The function $F(\alpha+t,\beta,\gamma+t,x)$ in (11.2) admits the Barnes integral representation:

$$\frac{\Gamma(\alpha+t)\Gamma(\beta)}{\Gamma(\gamma+t)}F(\alpha+t,\beta,\gamma+t,x) = \frac{1}{2\pi i}\int_{B_t}\frac{\Gamma(\alpha+t+s)\Gamma(\beta+s)}{\Gamma(\gamma+t+s)}\Gamma(-s)(-x)^s ds,$$

where B_t is the path of integration determined by $\alpha+t$, β. Hence from (11.2), we derive

$$\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta')}{\Gamma(\gamma)}F_1(\alpha,\beta,\beta',\gamma,x,y) =$$

$$= \left(\frac{1}{2\pi i}\right)^2\iint_{\mathcal{B}}\frac{\Gamma(\alpha+t+s)\Gamma(\beta+s)\Gamma(\beta'+t)}{\Gamma(\gamma+s+t)}\Gamma(-s)\Gamma(-t)(-x)^s(-y)^t dt\,ds$$

where the domain $\mathcal{B}$ of integration is given by

$$\mathcal{B} = \{(s,t);\ s\in B_t,\ t\in B\}.$$

Thus we obtain the following results:

THEOREM 11.2: Suppose that $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \neq 0, -1, -2$, Then the hypergeometric functions F_j $(j = 1, 2, 3, 4)$ admit integral representations

(11.2') $$\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta')}{\Gamma(\gamma)}F_1(\alpha,\beta,\beta',\gamma,x,y) = \left(\frac{1}{2\pi i}\right)^2\iint_{\mathcal{B}}\frac{\Gamma(\alpha+t+s)\Gamma(\beta+s)\Gamma(\beta'+t)}{\Gamma(\gamma+s+t)}\Gamma(-s)\Gamma(-t)(-x)^s(-y)^t dt\,ds,$$

(11.3') $$\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta')}{\Gamma(\gamma)\Gamma(\gamma')}F_2(\alpha,\beta,\beta',\gamma,\gamma',x,y) = \left(\frac{1}{2\pi i}\right)^2\iint_{\mathcal{B}}\frac{\Gamma(\alpha+s+t)\Gamma(\beta+s)\Gamma(\beta'+t)}{\Gamma(\gamma+s)\Gamma(\gamma'+t)}\Gamma(-s)\Gamma(-t)(-x)^s(-y)^t dt\,ds,$$

(11.4') $$\frac{\Gamma(\alpha)\Gamma(\alpha')\Gamma(\beta)\Gamma(\beta')}{\Gamma(\gamma)}F_3(\alpha,\alpha',\beta,\beta',\gamma,x,y) = \left(\frac{1}{2\pi i}\right)^2\iint_{\mathcal{B}}\frac{\Gamma(\alpha+s)\Gamma(\alpha'+t)\Gamma(\beta+s)\Gamma(\beta'+t)}{\Gamma(\gamma+s+t)}\Gamma(-s)\Gamma(-t)(-x)^s(-y)^t dt\,ds,$$

(11.5') $$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\gamma')} F_4(\alpha,\beta,\gamma,\gamma',x,y)$$

$$= \left(\frac{1}{2\pi i}\right)^2 \iint_{\mathcal{B}} \frac{\Gamma(\alpha+s+t)\Gamma(\beta+s+t)}{\Gamma(\gamma+s)\Gamma(\gamma'+t)} \Gamma(-s)\Gamma(-t)(-x)^s(-y)^t \, dt\, ds ,$$

where the domain of integration in (11.k')

$$\mathcal{B} = \{(s, t);\ \ s \in B_t,\ t \in B\},$$

and B is the path of integration in (11.k), while B_t is the path of integration determined by $\alpha+t$, β for $k = 2, 3$, by α, β for $k = 4$, and by $\alpha+t$, $\beta+t$ for $k = 5$.

By utilizing (11.4') we can derive the following result:

THEOREM 11.3: The functions F_3 and F_2 are related by the formula:

(11.6) $F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y)$

$$= f(\alpha,\alpha',\beta,\beta')x^{-\alpha}y^{-\alpha'}F_2(\alpha+\alpha'+1-\gamma,\alpha,\alpha',\alpha+1-\beta,\alpha'+1-\beta',1/x,1/y)$$

$$+ f(\alpha,\beta',\beta,\alpha')x^{-\alpha}y^{-\beta'}F_2(\alpha+\beta'+1-\gamma,\alpha,\beta',\alpha+1-\beta,\beta'+1-\alpha',1/x,1/y)$$

$$+ f(\beta,\alpha',\alpha,\beta')x^{-\beta}y^{-\alpha'}F_2(\beta+\alpha'+1-\gamma,\beta,\alpha',\beta+1-\alpha,\alpha'+1-\beta',1/x,1/y)$$

$$+ f(\beta,\beta',\alpha,\alpha')x^{-\beta}y^{-\beta'}F_2(\beta+\beta'+1-\gamma,\beta,\beta',\beta+1-\alpha,\beta'+1-\alpha',1/x,1/y) ,$$

where

$$f(\lambda,\mu,\rho,\sigma) = (-1)^{\lambda}(-1)^{\mu} \frac{\Gamma(\gamma)\Gamma(\rho-\lambda)\Gamma(\sigma-\mu)}{\Gamma(\rho)\Gamma(\sigma)\Gamma(\gamma-\lambda-\mu)} .$$

Instead of giving a proof of this theorem, we shall prove a similar result in the case of one variable.

THEOREM 11.4: Suppose $\alpha, \beta, \gamma, \alpha-\beta, \gamma-\alpha, \gamma-\beta \neq$ integer. Then

(11.7) $$F(\alpha,\beta,\gamma,x) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)}(-x)^{-\alpha}F(\alpha,\alpha+1-\gamma,\alpha+1-\beta,1/x)$$

$$+ \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-x)^{-\beta}F(\beta+1-\gamma,\beta,\beta+1-\alpha,1/x) .$$

Proof: Let us consider the Barnes integral representation (11.1) of $F(\alpha, \beta, \gamma, x)$. This integral is uniformly convergent in x for

(11.8) $$|\arg(-x)| < \pi .$$

We shall restrict x to the domain (11.8). The multiple-valued functions on the right-hand member of (11.7) are also well defined in this domain. Take a sufficiently large positive number R, and let B_R be the arc of B between $-iR$ and iR. Let D_R be the semi-circle: $s = Re^{i\theta}$, $|\theta-\pi| \leq \frac{1}{2}\pi$. Put

$$\psi(s) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)} \Gamma(-s)(-x)^s .$$

Then the difference between two integrals of ψ along B_R and D_R:

$$\frac{1}{2\pi i}\int_{B_R} \psi(s)ds - \frac{1}{2\pi i}\int_{D_R} \psi(s)ds$$

is the sum of residues at the poles of $\psi(s)$ which are contained in the closed curve consisting of B_R and D_R. Those poles are located at $s = -\alpha-m$ and $s = -\beta-m$. Let us compute the residues of $\psi(s)$ at these poles. Observe that

$$\operatorname*{Res}_{s=-m} \Gamma(s) = (-1)^m/m! \quad \text{and} \quad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} .$$

Hence

$$\operatorname*{Res}_{s=-\alpha-m} \psi(s) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\beta-\alpha-m)\Gamma(\alpha+m)}{\Gamma(\gamma-\alpha-m)} \frac{(-1)^m}{m!} (-x)^{-\alpha-m}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)} \frac{(\alpha,m)}{(1,m)} \frac{\Gamma(\beta-\alpha-m)}{\Gamma(\gamma-\alpha-m)} (-x)^{-\alpha} x^{-m}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)} \frac{\Gamma(\alpha-\gamma+1)}{\Gamma(\alpha-\beta+1)} \frac{\sin(\pi(\gamma-\alpha))}{\sin(\pi(\beta-\alpha))} (-x)^{-\alpha} \frac{(\alpha,m)(\alpha-\gamma+1,m)}{(\alpha-\beta+1,m)} x^{-m}.$$

In this manner, we obtain

$$\operatorname*{Res}_{s=-\alpha-m} \psi(s) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} (-x)^{-\alpha} \frac{(\alpha,m)(\alpha+1-\gamma,m)}{(\alpha+1-\beta,m)(1,m)} x^{-m},$$

$$\operatorname*{Res}_{s=-\beta-m} \psi(s) = \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} (-x)^{-\beta} \frac{(\beta+1-\gamma,m)(\beta,m)}{(\beta+1-\alpha,m)(1,m)} x^{-m}$$

On the other hand, we have

$$\lim \int_{D_R} \psi(s)ds = 0 \quad \text{as} \quad R \quad \text{tends to} \quad +\infty$$

if $|x| > 1$ and $|\arg(-x)| < \pi$. To prove this, it is sufficient to estimate

$$\frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)} \Gamma(-s)(-x)^s$$

on D_R. We can not use Stirling's formula directly, because Stirling's formula is not valid in a sector $|\arg(s)-\pi| < \varepsilon$. To avoid this difficulty, let us use the formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

to obtain

$$\frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)} \Gamma(-s)(-x)^s = \frac{\Gamma(1-\gamma-s)\ (-s)}{\Gamma(1-\alpha-s)\Gamma(1-\beta-s)} \frac{\pi \sin(\pi(\gamma+s))}{\sin(\pi(\alpha+s))\sin(\pi(\beta+s))}(-x$$

Now by utilizing Stirling's formula, we can estimate the integral

of $\psi(s)$ on D_R.

Exercise 1. Complete the proof of Theorem 11.4.

At the end of Section 5, we mentioned that two functions

$$(-x)^{-\alpha} F(\alpha, \alpha+1-\gamma, \alpha+1-\beta, 1/x)$$

and

$$(-x)^{-\beta} F(\beta+1-\gamma, \beta, \beta+1-\alpha, 1/x)$$

are linearly independent solutions of the hypergeometric differential equation. The formula (11.7) was found by Kummer. Since these solutions are not single-valued functions, this formula will not hold, if branches of these functions are chosen in a suitable manner. This fact was not well understood at the time of Kummer.

In order to prove Theorem 11.3, it should be noticed that the domain of integration in the formula (11.4') is

$$\mathcal{B} = B_1 \times B_2 = \{(s, t); \quad s \in B_1, \ t \in B_2\},$$

where B_1 is the path determined by α, β, while B_2 is the path determined by α', β'. Therefore, the idea in the proof of Theorem 11.4 can be directly applied in the proof of Theorem 11.3. To prove Theorem 11.3, therefore, we must compute the residues of the integrand of the right-hand member of (11.4') at pairs of poles

$$\begin{cases} s = -\alpha-m, \\ t = -\alpha'-n, \end{cases} \quad \begin{cases} s = -\alpha-m, \\ t = -\beta'-n, \end{cases} \quad \begin{cases} s = -\beta-m, \\ t = -\alpha'-n, \end{cases} \quad \begin{cases} s = -\beta-m, \\ t = -\beta'-n. \end{cases}$$

For example,

$$\operatorname*{Res}_{\substack{s=-\alpha-m\\ t=-\alpha'-n}} \frac{\Gamma(\alpha+s)\Gamma(\alpha'+t)\Gamma(\beta+s)\Gamma(\beta'+t)}{\Gamma(\gamma+s+t)}\Gamma(-s)\Gamma(-t)(-x)^s(-y)^t$$

$$=\frac{\Gamma(\alpha)\Gamma(\alpha')\Gamma(\beta-\alpha)\Gamma(\beta'-\alpha')}{\Gamma(\gamma-\alpha-\alpha')}(-x)^{-\alpha}(-y)^{-\alpha'}\frac{(\alpha+\alpha'+1-\gamma,m+n)(\alpha,m)(\alpha',n)}{(\alpha+1-\beta,m)(\alpha'+1-\beta',n)(1,m)(1,n)}x^{-m}y^{-n}$$

12. <u>Systems of partial differential equations</u>. The function $F(\alpha, \beta, \gamma, x)$ satisfies the differential equation

$$x(1-x)y'' + [\gamma-(\alpha+\beta+1)x]y' - \alpha\beta y = 0 .$$

The hypergeometric functions F_j $(j = 1, 2, 3, 4)$ also satisfy systems of partial differential equations.

THEOREM 12.1: F_1, F_2, F_3 and F_4 satisfy respectively the following systems of partial differential equations:

$$(12.1)\quad \begin{cases} x(1-x)r+y(1-x)s+[\gamma-(\alpha+\beta+1)x]p-\beta yq-\alpha\beta z = 0 , \\ y(1-y)t+x(1-y)s+[\gamma-(\alpha+\beta'+1)y]q-\beta' xp-\alpha\beta' z = 0 , \end{cases}$$

$$(12.2)\quad \begin{cases} x(1-x)r-xys+[\gamma-(\alpha+\beta+1)x]p-\beta yq-\alpha\beta z = 0 , \\ y(1-y)t-xys+[\gamma'-(\alpha+\beta'+1)y]q-\beta xp-\alpha\beta' z = 0 , \end{cases}$$

$$(12.3)\quad \begin{cases} x(1-x)r+ys+[\gamma-(\alpha+\beta+1)x]p-\alpha\beta z = 0 , \\ y(1-y)t+xs+[\gamma-(\alpha'+\beta'+1)y]q-\alpha'\beta' z = 0 , \end{cases}$$

$$(12.4)\quad \begin{cases} x(1-x)r-y^2t-2xys+[\gamma-(\alpha+\beta+1)x]p-(\alpha+\beta+1)yq-\alpha\beta z = 0 , \\ y(1-y)t-x^2r-2xys+[\gamma'-(\alpha+\beta+1)y]q-(\alpha+\beta+1)xp-\alpha\beta z = 0 , \end{cases}$$

where

$$r = \partial^2 z/\partial x^2, \quad s = \partial^2 z/\partial x\partial y, \quad t = \partial^2 z/\partial y^2, \quad p = \partial z/\partial x, \quad q = \partial z/\partial y.$$

<u>Proof</u>: We shall show that F_1 satisfies (12.1). Others can be shown in a similar manner. Denote by θ and φ the operators

$$x\partial/\partial x \quad \text{and} \quad y\partial/\partial y$$

respectively. Then we have

$$\theta(\theta+\varphi+\gamma-1)F_1 = \sum_{m,n=0}^{\infty} m(m+n+\gamma-1)\frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m+n)(1,m)(1,n)}x^m y^n$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m+n-1)(1,m-1)(1,n)} x^m y^n$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha,m+n+1)(\beta,m+1)(\beta',n)}{(\gamma,m+n)(1,m)(1,n)} x^{m+1} y^n$$

$$= x \sum_{m,n=0}^{\infty} (\alpha+m+n)(\beta+m) \frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m+n)(1,m)(1,n)} x^m y^n$$

$$= x(\theta+\varphi+\alpha)(\theta+\beta)F_1 .$$

Hence

$$\theta(\theta+\varphi+\gamma-1)F_1 - x(\theta+\varphi+\alpha)(\theta+\beta)F_1 = 0 .$$

In the same way, we get

$$\varphi(\theta+\varphi+\gamma-1)F_1 - y(\theta+\varphi+\alpha)(\varphi+\beta')F_1 = 0 .$$

From these equations we can easily derive (12.1).

Similarly we see that F_2, F_3 and F_4 satisfy respectively

(12.2')
$$\begin{cases} \theta(\theta+\gamma-1)F_2 - x(\theta+\varphi+\alpha)(\theta+\beta)F_2 = 0 , \\ \varphi(\varphi+\gamma'-1)F_2 - y(\theta+\varphi+\alpha)(\varphi+\beta')F_2 = 0 , \end{cases}$$

(12.3')
$$\begin{cases} \theta(\theta+\varphi+\gamma-1)F_3 - x(\theta+\alpha)(\theta+\beta)F_3 = 0 , \\ \varphi(\theta+\varphi+\gamma-1)F_3 - y(\varphi+\alpha')(\varphi+\beta')F_3 = 0 , \end{cases}$$

(12.4')
$$\begin{cases} \theta(\theta+\gamma-1)F_4 - x(\theta+\varphi+\alpha)(\theta+\varphi+\beta)F_4 = 0 , \\ \varphi(\varphi+\gamma'-1)F_4 - y(\theta+\varphi+\alpha)(\theta+\varphi+\beta)F_4 = 0 . \end{cases}$$

From these the systems (12.2), (12.3) and (12.4) can be easily derived.

13. <u>Systems of total differential equations</u>. Before studying the systems of partial differential equations which F_j satisfy, we shall explain some aspects of total differential equations.

Let us consider a system of total differential equations of the form:

(13.1)
$$dz_j = f_j(x,y,z_1,\cdots,z_n)dx + g_j(x,y,z_1,\cdots,z_n)dy \qquad (j = 1,2,\cdots,n) ,$$

where $f_1, \cdots, f_n$, $g_1, \cdots, g_n$ are functions of $x, y, z_1, \cdots, z_n$ which are defined and continuously differentiable in a domain $\mathcal{D}$ contained in R^{n+2}. If we consider x, y as independent variables and $z_1, \cdots, z_n$ as dependent variables (i.e. unknown functions of x, y), then the system (13.1) is equivalent to the system

(13.2)
$$\begin{aligned} \partial z_j/\partial x &= f_j(x, y, z_1, \cdots, z_n) , \\ \partial z_j/\partial y &= g_j(x, y, z_1, \cdots, z_n) \end{aligned} \qquad (j = 1, \cdots, n) .$$

Suppose that there exists a solution of (13.1):

$$z_j = \varphi_j(x, y) \qquad (j = 1, \cdots, n)$$

which is defined and twice continuously differentiable in a domain $D \subset R^2$. The cross second derivatives of $\varphi_j(x, y)$ are independent of the order of differentiation with respect to x and y: i.e.

$$\frac{\partial^2 \varphi_j}{\partial x \partial y} = \frac{\partial^2 \varphi_j}{\partial y \partial x} \qquad (j = 1, \cdots, n) .$$

From (13.2) we derive

$$\frac{\partial}{\partial y} f_j(x,y,\varphi(x,y)) = \frac{\partial}{\partial x} g_j(x,y,\varphi(x,y)) \quad (j = 1,\cdots, n).$$

Hence

$$\partial f_j/\partial y + \sum_{k=1}^{n} [\partial f_j/\partial z_k][\partial \varphi_k/\partial y] = \partial g_j/\partial x + \sum_{k=1}^{n} [\partial g_j/\partial z_k][\partial \varphi_k/\partial x]$$

and hence

$$(13.3) \qquad \partial f_j/\partial y + \sum_{k=1}^{n} g_k \,\partial f_j/\partial z_k = \partial g_j/\partial x + \sum_{k=1}^{n} f_k \,\partial g_j/\partial z_k \qquad (j = 1, \cdots, n).$$

This condition holds for $(x, y, \varphi_1(x,y), \cdots, \varphi_n(x,y))$, $(x, y) \in D$. If for any point $(x_0, y_0, z_{10}, \cdots, z_{n0}) \in \mathcal{D}$ there exists a solution of (13.1) satisfying the initial condition $z_j(x_0, y_0) = z_{j0}$ $(j = 1, \cdots, n)$ and the smoothness condition, then (13.3) holds identically in $\mathcal{D}$. Therefore, (13.3) is a necessary condition for the existence of such solutions of (13.1). In general, in order that overdetermined systems such as (13.1) have solutions, we need some conditions such as (13.3). On the other hand, the theory of total differential equations guarantees that (13.3) is also a sufficient condition for the existence of solutions of (13.1) satisfying an arbitrary initial condition.

THEOREM 13.1: Suppose that $f_j(x, y, z_1, \cdots, z_n)$, $g_j(x, y, z_1, \cdots, z_n)$ are all continuously differentiable in $\mathcal{D} \subset R^{n+2}$ and satisfy (13.3). Then for any point $(x_0, y_0, z_{10}, \cdots, z_{n0}) \in \mathcal{D}$ there exists a unique solution of (13.1) which is defined and twice continuously differentiable in a neighborhood of (x_0, y_0) and satisfies the initial condition

$$(13.4) \qquad z_j(x_0, y_0) = z_{j0} \quad (j = 1, \cdots, n) .$$

DEFINITION 13.1: <u>A system of total differential equations of the form</u> (13.1) <u>is called completely integrable if</u> (13.3) <u>is satisfied</u>.

We shall now consider the case when f_j and g_j are all holomorphic in $x, y, z_1, \cdots, z_n$.

THEOREM 13.2: <u>Suppose that</u> f_j <u>and</u> g_j <u>are holomorphic at</u> $(x_0, y_0, z_{10}, \cdots, z_{n0})$ <u>and satisfy</u> (13.3). <u>Then there exists a unique solution of</u> (13.1) <u>which is holomorphic at</u> (x_0, y_0) <u>and satisfies the initial condition</u> (13.4).

Let us consider the case when f_j and g_j are all linear in $z_1, \cdots, z_n$: i.e.

$$(13.5) \qquad dz_j = \sum_{k=1}^{n} f_{jk}(x, y) z_k \, dx + \sum_{k=1}^{n} g_{jk}(x, y) z_k \, dy \qquad (j = 1, \cdots, n).$$

If we denote by

$$\omega_{jk} = f_{jk}(x, y)dx + g_{jk}(x, y)dy , \quad (j, k = 1, \cdots, n) ,$$

the system (13.5) can be written in the form

$$dz_j = \sum_{k=1}^{n} \omega_{jk} z_k \quad (j = 1, \cdots, n) .$$

Moreover if we put

$$Z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \quad \Omega = \begin{bmatrix} \omega_{11}, & \cdots, & \omega_{1n} \\ & \cdots\cdots & \\ \omega_{n1}, & \cdots, & \omega_{nn} \end{bmatrix},$$

we can write the system (13.5) as

$$dZ = \Omega Z .$$

The condition of integrability becomes

$$\sum_{k=1}^{n} z_k \Big[\partial f_{jk}/\partial y + \sum_{h=1}^{n} f_{jk} g_{hk}\Big] = \sum_{k=1}^{n} z_k \Big[\partial g_{jk}/\partial x + \sum_{h=1}^{n} g_{jh} f_{hk}\Big]$$

$$j = 1, \cdots, n .$$

Since these are identities, we obtain

$$(13.6) \qquad \partial f_{jk}/\partial y + \sum_{h=1}^{n} f_{jh} g_{hk} = \partial g_{jk}/\partial x + \sum_{h=1}^{n} g_{jh} f_{hk}$$

$$j, k = 1, \cdots, n .$$

As it is well known, solutions of linear ordinary differential equations exist in an interval or a domain where coefficients of the equations are continuous or holomorphic. The same fact holds for a system of linear total differential equations. We shall state such a result for the holomorphic case.

THEOREM 13.3: *Suppose that* f_{jk} *and* g_{jk} *are all holomorphic in* D *and satisfy the condition* (13.6) *there*. *Let*

$$z_j = \varphi_j(x, y) \qquad (j = 1, \cdots, n)$$

be a solution of (13.5) *which is holomorphic at* (x_0, y_0). *Then*

$\varphi_1, \cdots, \varphi_n$ *are analytically continuable along any path in* D *starting at* (x_0, y_0).

It is clear that the set of all solutions of a system of linear total differential equations is a vector space. Since solutions are uniquely determined by their initial values, the dimension of this vector space is n .

THEOREM 13.4: *The solution space of* (13.5) *is an n-dimensional vector space.*

Consider n solutions of (13.5):

$$z_j = \varphi_{jk}(x, y) , \qquad (j, k = 1, \cdots, n)$$

and the determinant obtained from these solutions:

$$\Delta = \begin{vmatrix} \varphi_{11} & \cdots\cdots & \varphi_{1n} \\ & \cdots\cdots & \\ \varphi_{n1} & \cdots\cdots & \varphi_{nn} \end{vmatrix} .$$

In the same way as for linear ordinary differential equations, we derive

$$d\Delta = \Big(\sum_{j=1}^{n} f_{jj}\, dx + \sum_{j=1}^{n} g_{jj}\, dy \Big) \Delta$$

or

$$d(\log \Delta) = \sum_{j=1}^{n} f_{jj}\, dx + \sum_{j=1}^{n} g_{jj}\, dy .$$

THEOREM 13.5: *We have*

(13.7) $\Delta(x,y) = \Delta(x_0,y_0)\exp[\int_{(x_0,y_0)}^{(x,y)} \sum_{j=1}^{n} f_{jj}(s,t)ds + \sum_{j=1}^{n} g_{jj}(s,t)dt]$.

We have completely similar theorems for the general system of total differential equations

$$dz_j = \sum_{k=1}^{m} f_{jk}(x_1,\cdots,x_m,z_1,\cdots,z_n)dx_k\ ,\quad j = 1,\cdots,n\ .$$

Exercise 13.1: Show that, if we write system (13.5) in the form

$$dZ = \Omega Z\ ,$$

then the integrability condition (13.6) can be written as

$$d\Omega = \Omega \wedge \Omega\ .$$

14. Transformation of the differential equations for F_j into systems of total differential equations. We saw that the systems of partial differential equations satisfied by F_j are written in the form

$$\begin{cases} A_1r + A_2s + A_3t + A_4p + A_5q + A_6z = 0\ , \\ B_1r + B_2s + B_3t + B_4p + B_5q + B_6z = 0\ , \end{cases}$$

where A_j and B_j are holomorphic in x and y. $A_3 = B_1 = 0$ for F_1, F_2 and F_3. These equations are solvable with respect to r and t so that we obtain

(14.1) $$\begin{cases} r = a_1s + a_2p + a_3q + a_4z\ , \\ t = b_1s + b_2p + b_3q + b_4z\ , \end{cases}$$

where a_j and b_j are rational functions in x and y.

Let us differentiate the first equation with respect to y:

$$\partial r/\partial y = a_1\partial s/\partial y + s\partial a_1/\partial y + a_2\partial p/\partial y + p\partial a_2/\partial y + a_3\partial q/\partial y + q\partial a_3/\partial y + a_4\partial z/\partial y + z\partial a_4/\partial y\ .$$

Thus we derive

(14.2) $$\partial s/\partial x - a_1\partial s/\partial y = (\partial a_1/\partial y + a_2)s + a_3t + (\partial a_2/\partial y)p + (\partial a_3/\partial y + a_4)q + (\partial a_4/\partial y)z$$

Similarly, by differentiating the second equation of (14.1) with respect to x, we obtain

(14.3) $$-b_1\partial s/\partial x + \partial s/\partial y = (\partial b_1/\partial x + b_3)s + b_2r + (\partial b_2/\partial x + b_4)p + (\partial b_3/\partial x)q + (\partial b_4/\partial x)z\ .$$

The left-hand members of (14.2) and (14.3) are linear forms in $\partial s/\partial x$ and $\partial s/\partial y$. The determinant obtained from the coefficients of these linear forms is

$$\begin{vmatrix} 1 & -a_1 \\ -b_1 & 1 \end{vmatrix} = 1 - a_1 b_1 .$$

We distinguish two cases:

Case I : $1 - a_1 b_1 = 0$,

Case II: $1 - a_1 b_1 \neq 0$.

Case I: In this case, we can eliminate $\partial s/\partial x$ and $\partial s/\partial y$ from (14.2) and (14.3) at the same time to obtain a linear relation between r, s, t, p, q, z :

$$c_1 r + c_2 s + c_3 t + c_4 p + c_5 q + c_6 z = 0 . \tag{14.4}$$

Suppose that from (14.1) and (14.4) we derive

$$r = \alpha_1 p + \alpha_2 q + \alpha_3 z ,$$
$$s = \beta_1 p + \beta_2 q + \beta_3 z ,$$
$$t = \gamma_1 p + \gamma_2 q + \gamma_3 z ,$$

where α_j, β_j and γ_j are rational functions in x and y. By definition we have

$$\partial z/\partial x = p , \quad \partial z/\partial y = q ,$$

and hence

$$\partial p/\partial x = \alpha_1 p + \alpha_2 q + \alpha_3 z , \quad \partial p/\partial y = \beta_1 p + \beta_2 q + \beta_3 z ,$$
$$\partial q/\partial x = \beta_1 p + \beta_2 q + \beta_3 z , \quad \partial q/\partial y = \gamma_1 p + \gamma_2 q + \gamma_3 z .$$

Thus we obtain a system of total differential equations:

$$(14.5) \quad \begin{cases} dz = p\,dx + q\,dy , \\ dp = (\alpha_1 p + \alpha_2 q + \alpha_3 z)dx + (\beta_1 p + \beta_2 q + \beta_3 z)dy , \\ dq = (\beta_1 p + \beta_2 q + \beta_3 z)dx + (\gamma_1 p + \gamma_2 q + \gamma_3 z)dy . \end{cases}$$

In vectorial notations, we can write this system as

$$d\begin{pmatrix} z \\ p \\ q \end{pmatrix} = \begin{pmatrix} 0 & dx & dy \\ \alpha_3 dx + \beta_3 dy & \alpha_1 dx + \beta_1 dy & \alpha_2 dx + \beta_2 dy \\ \beta_3 dx + \gamma_3 dy & \beta_1 dx + \gamma_1 dy & \beta_2 dx + \gamma_2 dy \end{pmatrix} \begin{pmatrix} z \\ p \\ q \end{pmatrix} .$$

In general, this system may not be completely integrable. If this system is completely integrable, then the solution space is a three-dimensional vector space.

Case II: In this case, we can express $\partial s/\partial x$ and $\partial s/\partial y$ as linear forms in r, s, t, p, q, z:

$$\begin{cases} \partial s/\partial x = c_1 r + c_2 s + c_3 t + c_4 p + c_5 q + c_6 z , \\ \partial s/\partial y = d_1 r + d_2 s + d_3 t + d_4 p + d_5 q + d_6 z , \end{cases}$$

where coefficients are rational functions of x and y. Inserting (14.1) into the right-hand members of these equations, we get

$$\begin{cases} \partial s/\partial x = \alpha_1 s + \alpha_2 p + \alpha_3 q + \alpha_4 z , \\ \partial s/\partial y = \beta_1 s + \beta_2 p + \beta_3 q + \beta_4 z . \end{cases}$$

By using formulas such as $r = \partial p/\partial x$, $s = \partial p/\partial y = \partial q/\partial x$ and $t = \partial q/\partial y$, we obtain a system of differential equations

$$\partial z/\partial x = p , \qquad \partial z/\partial y = q ,$$
$$\partial p/\partial x = a_1 s + a_2 p + a_3 q + a_4 z , \qquad \partial p/\partial y = s ,$$

$$\partial q/\partial x = s\ , \qquad \partial q/\partial y = b_1 s + b_2 p + b_3 q + b_4 z\ ,$$
$$\partial s/\partial x = \alpha_1 s + \alpha_2 p + \alpha_3 q + \alpha_4 z\ , \quad \partial s/\partial y = \beta_1 s + \beta_2 p + \beta_3 q + \beta_4 z\ .$$

This system can be written in the form:

$$(14.6)\ \begin{cases} dz = pdx + qdy\ , \\ dp = (a_1 s + a_2 p + a_3 q + a_4 z)dx + sdy\ , \\ dq = sdx + (b_1 s + b_2 p + b_3 q + b_4 z)dy\ , \\ ds = (\alpha_1 s + \alpha_2 p + \alpha_3 q + \alpha_4 z)dx + (\beta_1 s + \beta_2 p + \beta_3 q + \beta_4 z)dy\ , \end{cases}$$

or in vectorial notations

$$d\begin{pmatrix} z \\ p \\ q \\ s \end{pmatrix} = \begin{pmatrix} 0 & dx & dy & 0 \\ a_4 dx & a_2 dx & a_3 dx & a_1 dx + dy \\ b_4 dy & b_2 dy & b_3 dy & dx + b_1 dy \\ \alpha_4 dx + \beta_4 dy & \alpha_2 dx + \beta_2 dy & \alpha_3 dx + \beta_3 dy & \alpha_1 dx + \beta_1 dy \end{pmatrix} \begin{pmatrix} z \\ p \\ q \\ s \end{pmatrix}$$

This system of total differential equations is not necessarily completely integrable. If this system is completely integrable, then the solution space is a four-dimensional vector space.

Let us now consider the systems of partial differential equations satisfied by F_j $(j = 1, 2, 3, 4)$. For each of these systems, we shall find a_1 and b_1 so that we may determine which of Cases I and II occurs.

System for F_1: We know that F_1 satisfies (12.1). Therefore, in this case we have

$$a_1 = -y/x\ , \qquad b_1 = -x/y\ .$$

Hence we have $1 - a_1 b_1 \equiv 0$. This means that (12.1) belongs to Case I.

System for F_2: The function F_2 satisfies (12.2). Hence

$$a_1 = y/(1-x)\ , \quad b_1 = x/(1-y)\ , \quad 1 - a_1 b_1 \not\equiv 0\ .$$

System for F_3: In this case, from (12.3) we derive

$$a_1 = -y/x(1-x)\ , \quad b_1 = -x/y(1-y)\ , \quad 1 - a_1 b_1 \not\equiv 0\ .$$

System for F_4: From (12.4) we derive

$$a_1 = 2y/(1-x-y)\ , \quad b_1 = 2x/(1-x-y)\ , \quad 1 - a_1 b_1 \not\equiv 0\ .$$

This shows that (12.2), (12.3) and (12.4) belong to Case II. By using the procedure of deriving systems of total differential equations, we obtain such systems for F_1, F_2, F_3 and F_4. It can be verified also that these systems for F_j are all completely integrable. The proof of this fact will be left to the readers. Thus we come to the following conclusion.

THEOREM 14.1: The dimension of the solution space of the system for F_1 is three, while the dimensions of the solution spaces of the systems for F_2, F_3 and F_4 are all four.

Let us examine the system for F_1 in more detail. Differentiating the first equation of (12.1) with respect to y we obtain

$$x(1-x)\partial s/\partial x + y(1-x)\partial s/\partial y + (1-x)s + [\gamma - (\alpha+\beta+1)x]s - \beta yt - \beta q - \alpha\beta q = 0$$

or

$$x(1-x)\partial s/\partial x + y(1-x)\partial s/\partial y + [\gamma+1-(\alpha+\beta+2)x]s - \beta yt - (\alpha+1)\beta q = 0\ .$$

Similarly, differentiating the second equation of (12.1) we obtain

$$y(1-y)\partial s/\partial y + x(1-y)\partial s/\partial x + [\gamma+1-(\alpha+\beta'+2)y]s - \beta' xr - (\alpha+1)\beta' p = 0\ .$$

To eliminate $\partial s/\partial x$ and $\partial s/\partial y$ from these two equations, we multiply the first equation by $1-y$ and the second equation by $1-x$ and we subtract one from the other. Then we get

$$[(\gamma+1)(1-y)-(\alpha+\beta+2)x(1-y)-(\gamma+1)(1-x)+(\alpha+\beta'+2)(1-x)y]s$$
$$+\beta' x(1-x)r-\beta y(1-y)t+(\alpha+1)\beta'(1-x)p-(\alpha+1)\beta(1-y)q = 0 .$$

By solving this equation and (12.1) with respect to r, s and t, we get

$$(14.7)\quad \begin{cases} x(1-x)(x-y)r+[\gamma(x-y)-(\alpha+\beta+1)x^2+(\alpha+\beta-\beta'+1)xy+\beta' y]p \\ \qquad -\beta y(1-y)q-\alpha\beta(x-y)z = 0 , \\ (x-y)s-\beta' p+\beta q = 0 , \\ y(1-y)(y-x)t+[\gamma(y-x)-(\alpha+\beta'+1)y^2+(\alpha-\beta+\beta'+1)xy+\beta x]q \\ \qquad -\beta' x(1-x)p-\alpha\beta'(y-x)z = 0 . \end{cases}$$

In fact, rewriting $x(1-x)r$ and $y(1-y)t$ in the above equation by using (12.1), we get

$$[(\gamma-\alpha-\beta-1)x-(\gamma-\alpha-\beta'-1)y+(\beta-\beta')xy]s$$
$$-\beta' y(1-x)s-\beta'[\gamma-(\alpha+\beta+1)x]p+\beta\beta' yq+\alpha\beta\beta' z$$
$$+\beta x(1-y)s+\beta[\gamma-(\alpha+\beta'+1)y]q-\beta\beta' xp-\alpha\beta\beta' z$$
$$+(\alpha+1)\beta'(1-x)p-(\alpha+1)\beta(1-y)q = 0 ,$$

or

$$(\gamma-\alpha-1)(x-y)s-[\beta'\gamma-\beta'(\alpha+1)]p+[\beta\gamma-\beta(\alpha+1)]q = 0 .$$

Thus we derive the second equation of (14.7). The first equation of (14.7) is derived if we multiply the first equation of (12.1) by $(x-y)$ and if we replace $(x-y)s$ by $\beta' p-\beta q$. Similarly, if we multiply the second equation of (12.1) by $(y-x)$ and if we replace $(y-x)s$ by $\beta q-\beta' p$, we obtain the third equation of (14.7).

<u>Exercise</u> 1. Prove that

$$(x-y)s-\beta' p+\beta q = 0$$

has a solution of the form

$$\sum_{m,n=0}^{\infty} \frac{(\beta,m)(\beta',n)}{(1,m)(1,n)}\psi(m+n)\,x^m y^n ,$$

where ψ is an arbitrary function. In particular, if we take

$$\psi(m) = \frac{(\alpha,m)}{(\gamma,m)} ,$$

we find F_1 again.

<u>Exercise</u> 2. If we write the system (14.7) in the form

$$(14.7')\quad \begin{cases} r = \alpha_1 p+\alpha_2 q+\alpha_3 z , \\ s = \beta_1 p+\beta_2 q+\beta_3 z , \\ t = \gamma_1 p+\gamma_2 q+\gamma_3 z , \end{cases}$$

we can reduce the system (14.7) to a system of total differential equations for z, p, q by the procedure given in this section. Let (z_j, p_j, q_j) $(j = 1, 2, 3)$ be solutions of the systems of total differential equations. Prove that

$$\begin{vmatrix} z_1 & z_2 & z_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = \text{const.}\, x^{\beta'-\gamma} y^{\beta-\gamma}(1-x)^{\gamma-\alpha-\beta-1}(1-y)^{\gamma-\alpha-\beta'-1}(x-y)^{-\beta-\beta'} .$$

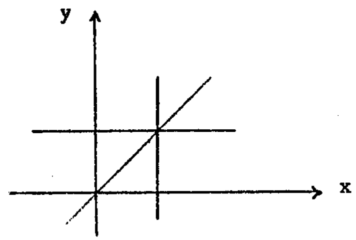
If we write (14.7) in the form (14.7'), the coefficients α_j, β_j and γ_j have poles on the lines in $\mathbb{C}\times\mathbb{C}$:

$$x = 0,\quad x = 1,\quad y = 0,\quad y = 1 \quad\text{and}\quad x = y .$$

This implies that any solution of the system for F_1 is regular analytic in

$$\mathbb{C} \times \mathbb{C} - \{x = 0\} \cup \{x = 1\} \cup \{y = 0\} \cup \{y = 1\} \cup \{x = y\}.$$

However, solutions are in general multiple-valued functions.



In the same way, if we write the systems for F_2, F_3 and F_4 in the form

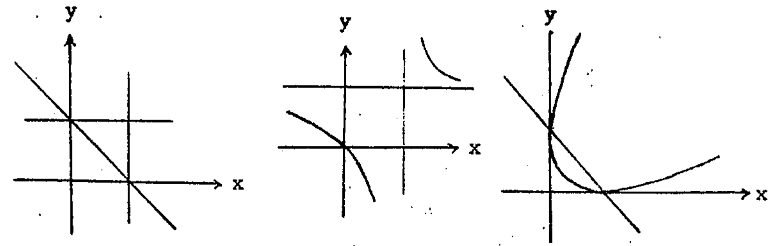
$$\begin{cases} r = a_1 s + a_2 p + a_3 q + a_4 z , \\ t = b_1 s + b_2 p + b_3 q + b_4 z , \\ \partial s/\partial x = \alpha_1 s + \alpha_2 p + \alpha_3 q + \alpha_4 z , \\ \partial s/\partial y = \beta_1 s + \beta_2 p + \beta_3 q + \beta_4 z , \end{cases}$$

then the coefficients have poles in $\mathbb{C} \times \mathbb{C}$ at

$$x = 0, \quad x = 1, \quad y = 0, \quad y = 1, \quad x + y = 1 \quad \text{for } F_2 ,$$

$$x = 0, \quad x = 1, \quad y = 0, \quad y = 1, \quad xy - x - y = 0 \quad \text{for } F_3 ,$$

$$x = 0, \quad y = 0, \quad x + y = 1, \quad (x-y)^2 - 2(x+y) + 1 = 0 \quad \text{for } F_4 .$$



F_2 F_3 F_4

Remark 1. A natural compactification of the complex plane $\mathbb{C}$ is the Riemann sphere which is the unit sphere of real dimension two, i.e. S^2. This can be regarded as the complex projective line $\mathbb{P}^1$. On the other hand, $\mathbb{C} \times \mathbb{C}$ has two natural compactifications. The projective plane $\mathbb{P}^2$ is one of them. The other is $\mathbb{P}^1 \times \mathbb{P}^1$. Both of them are compact complex manifolds, but they are not biholomorphically equivalent. This means that there is no biholomorphic mapping from one into the other. They have, however, a common modification and hence they are equivalent in a certain sense. Since coefficients of the systems for F_j are rational in x and y, these systems are well defined in the compactifications of $\mathbb{C} \times \mathbb{C}$. In various cases, a choice between two compactifications of $\mathbb{C} \times \mathbb{C}$ is not a serious problem.

Remark 2. The Gauss differential equation

$$x(1-y)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0$$

is reduced to a system

$$(14.8) \quad \begin{cases} y' = z/x , \\ z' = -\dfrac{\alpha\beta}{x-1}y + \left[\dfrac{1-\gamma}{x} + \dfrac{\gamma-\alpha-\beta-1}{x-1}\right] z , \end{cases}$$

if we put

$$z = xy' .$$

Note that

$$z' = xy'' + y'$$

and hence

$$\begin{aligned} x(1-x)z' &= x[x(1-x)y''] + (1-x)xy' \\ &= \alpha\beta xy - [\gamma-(\alpha+\beta+1)x]xy' + (1-x)xy' \\ &= \alpha\beta xy + [(1-\gamma)+(\alpha+\beta)x]xy' . \end{aligned}$$

The system (14.8) can be written in the form

$$(14.8') \qquad d\begin{bmatrix} y \\ z \end{bmatrix} = \left(A\frac{dx}{x} + B\frac{dx}{x-1}\right)\begin{bmatrix} y \\ z \end{bmatrix} ,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1-\gamma \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ -\alpha\beta & \gamma-\alpha-\beta-1 \end{bmatrix}$$

Note that coefficients of (14.8') have simple poles. We shall show now that the system (14.7) or (14.7') for F_1 can be reduced to a system of an analogous form. To do this, let us take

$$z , \quad x\,\partial z/\partial x , \quad y\,\partial z/\partial y$$

as unknown quantities. Then

$$\frac{\partial}{\partial x}[x\,\partial z/\partial x] = xr+p , \qquad \frac{\partial}{\partial y}[x\,\partial z/\partial x] = xs ,$$

$$\frac{\partial}{\partial x}[y\,\partial z/\partial y] = ys , \qquad \frac{\partial}{\partial y}[y\,\partial z/\partial y] = yt + q .$$

Therefore, in a way similar to that in the case of the Gauss equations, we derive from (14.7) the following system:

$$d\begin{bmatrix} z \\ x\,\partial z/\partial x \\ y\,\partial z/\partial y \end{bmatrix} = \left(A\frac{dx}{x} + B\frac{dy}{y} + C\frac{dx}{x-1} + D\frac{dy}{y-1} + E\frac{d(x-y)}{x-y}\right)\begin{bmatrix} z \\ x\,\partial z/\partial x \\ y\,\partial z/\partial y \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1-\gamma+\beta' & 0 \\ 0 & -\beta' & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -\beta \\ 0 & 0 & 1-\gamma+\beta \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 \\ -\alpha\beta & \gamma-\alpha-\beta-1 & -\beta \\ 0 & 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha\beta' & -\beta' & \gamma-\alpha-\beta'-1 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\beta' & \beta \\ 0 & \beta' & -\beta \end{bmatrix} .$$

Note that the coefficients of this system have simple poles at

$$x = 0 , \quad x = 1 , \quad y = 0 , \quad y = 1 \quad \text{and} \quad y = x .$$