CHAPTER III

Monodromy Group of the Gauss Differential Equation

15. Euler transform. In Section 3 (Chapter I) we derived the Euler integral representation of the function $F(\alpha, \beta, \gamma, x)$:

(3.1) $$F(\alpha, \beta, \gamma, x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}du .$$

The kernel of the integral is $(1-xu)^{-\alpha}$. Dropping the constant factor $\Gamma(\gamma)/\Gamma(\beta)\Gamma(\gamma-\beta)$, let us write this integral in a form

(15.1) $$y(x) = \int_C (1-xu)^{\lambda-1} \varphi(u)\, du ,$$

where $\lambda = -\alpha+1$, $\varphi(u) = u^{\beta-1}(1-u)^{\gamma-\beta-1}$ and $C = [0, 1]$.

This integral (15.1) is a solution of the hypergeometric differential equation

(15.2) $$L(y) = x(1-x)y'' + [\gamma-(\alpha+\beta+1)x]y' - \alpha\beta y = 0 .$$

We shall find other solutions of (15.2) also in a form (15.1) by choosing λ, φ, C in suitable ways.

Suppose that we can differentiate the function $y(x)$, given by (15.1), with respect to x under the integral to obtain

$$y'(x) = \int_C \frac{\partial}{\partial x} (1-xu)^{\lambda-1} \varphi(u)\, du ,$$

$$y''(x) = \int_C \frac{\partial^2}{\partial x^2} (1-xu)^{\lambda-1} \varphi(u)\, du .$$

Then we have

$$L(y(x)) = \int_C L((1-xu)^{\lambda-1})\, \varphi(u)\, du ,$$

where

$$L((1-xu)^{\lambda-1}) = x(1-x)[(\lambda-1)(\lambda-2)u^2(1-xu)^{\lambda-3}] + [\gamma-(\alpha+\beta+1)x][-(\lambda-1)u(1-xu)^{\lambda-2}] - \alpha\beta(1-xu)^{\lambda-1} .$$

Let us write $L((1-xu)^{\lambda-1})$ in the form

$$L((1-xu)^{\lambda-1}) = (1-xu)^{\lambda-3}H(x, u, \lambda) ,$$

where

$$H(x,u,\lambda) = (\lambda-1)(\lambda-2)u^2x(1-x) - (\lambda-1)u[\gamma-(\alpha+\beta+1)x](1-xu) - \alpha\beta(1-xu)^2 .$$

The function $H(x, u, \lambda)$ is quadratic in x, and the coefficient of x^2 in H is given by

$$[-(\lambda-1)(\lambda-2) - (\lambda-1)(\alpha+\beta+1) - \alpha\beta]u^2 .$$

Note that

$$(\lambda-1)(\lambda-2) + (\lambda-1)(\alpha+\beta+1) + \alpha\beta = (\lambda+\alpha-1)(\lambda+\beta-1) .$$

Therefore, if we choose $\lambda = -\alpha+1$ or $= -\beta+1$, the function H becomes linear in x. We shall fix λ in this manner. Since the hypergeometric differential equation (15.2) is symmetric with respect to α and β, we shall consider the case when $\lambda = -\alpha+1$.

In case when $\lambda = -\alpha+1$, we have

$$H(x, u, \lambda) = \alpha[\{(\alpha-\gamma+1)u^2 - (\alpha-\beta+1)u\}x + \gamma u - \beta] = \alpha[-(\alpha+1)xu(1-u) + (1-xu)(\gamma u-\beta)] ,$$

and hence

$$L(y(x)) = \alpha\int_C (1-xu)^{-\alpha-2}[-(\alpha+1)xu(1-u)+(1-xu)(\gamma u-\beta)]\,\varphi(u)\,du$$

$$= \alpha\int_C [-(\alpha+1)x(1-xu)^{-\alpha-2}u(1-u)\varphi(u)+(1-xu)^{-\alpha-1}(\gamma u-\beta)\varphi(u)]\,du$$

$$= \alpha\int_C [\frac{\partial}{\partial u}\{(1-xu)^{-\alpha-1}\}\{-u(1-u)\varphi(u)\}+(1-xu)^{-\alpha-1}(\gamma u-\beta)\varphi(u)]\,du$$

$$= [-\alpha(1-xu)^{-\alpha-1}u(1-u)\varphi(u)]_C$$

$$+\alpha\int_C (1-xu)^{-\alpha-1}[\frac{\partial}{\partial u}\{u(1-u)\varphi(u)\}+(\gamma u-\beta)\varphi(u)]\,du$$

Therefore, if we choose φ and C by the conditions

(15.3) $$\frac{\partial}{\partial u}[u(1-u)\varphi(u)]+(\gamma u-\beta)\varphi(u) = 0$$

and

(15.4) $$[(1-xu)^{-\alpha-1}u(1-u)\varphi(u)]_C = 0\ ,$$

then we have $L(y(x)) = 0$. The function

(15.5) $$\varphi(u) = u^{\beta-1}(1-u)^{\gamma-\beta-1}$$

satisfies the condition (15.3). Let us fix φ by (15.5). Then the condition (15.4) becomes

(15.4') $$[u^{\beta}(1-u)^{\gamma-\beta}(1-xu)^{-\alpha-1}]_C = 0\ .$$

This means that, if C is a closed path on the Riemann surface of

(15.6) $$u^{\beta}(1-u)^{\gamma-\beta}(1-xu)^{-\alpha-1}$$

or if (15.6) takes the same value at the starting point and the end point of C, the condition (15.4') is satisfied. For example, we can fix C in the following ways:

(i) the path joining 0 to 1 if $\mathrm{Re}\,\beta > 0$, $\mathrm{Re}(\gamma-\beta) > 0$;

(ii) the path joining $-\infty$ to 0 if $\mathrm{Re}\,\beta > 0$, $\mathrm{Re}(\alpha+1-\gamma) > 0$;

(iii) the path joining 1 to $+\infty$ if $\mathrm{Re}(\gamma-\beta) > 0$, $\mathrm{Re}(\alpha+1-\gamma) > 0$.

On the other hand, Jacobi showed that the following three curves satisfy our requirements under certain conditions on the parameters:

(iv) the path joining 0 to $1/x$;

(v) the path joining 1 to $1/x$;

(vi) the path joining ∞ to $1/x$.

In order to find the conditions on the parameters that these three curves satisfy our requirements, we must examine not only the condition (15.4'), but also the assumption that

$$y'(x) = \int_C \frac{\partial}{\partial x}(1-xu)^{\lambda-1}\varphi(u)\,du\ ,$$

$$y''(x) = \int_C \frac{\partial^2}{\partial x^2}(1-xu)^{\lambda-1}\varphi(u)\,du\ .$$

In deriving the conditions on λ , φ and C, we actually assumed that the order of integration and differentiation can be interchanged. This requirement must be satisfied by the three curves (iv), (v) and (vi). Note that we have

$$\frac{d}{dx}\int_a^{f(x)} F(x,u)du = \int_a^{f(x)} \frac{\partial}{\partial x}F(x,u)du + f'(x)F(x,f(x))\ .$$

Therefore, if $F(x,f(x)) = 0$, we get

$$\frac{d}{dx}\int_a^{f(x)} F(x,u)du = \int_a^{f(x)} \frac{\partial}{\partial x} F(x,u)du .$$

If we have $F(x,f(x)) = 0$ with $f(x) = 1/x$ and $F(x,u) = (1-xu)^{\lambda-1}\varphi(u)$, then the formula for $y'(x)$ is verified. On the other hand, if we have $F(x, f(x)) = 0$ for $f(x) = 1/x$ and $F(x,u) = \frac{\partial}{\partial x}(1-xu)^{\lambda-1}\varphi(u)$, the formula for $y''(x)$ is verified. Since $\lambda-1 = -\alpha$, these two conditions are satisfied, if

(15.7) $$\mathrm{Re}\,\alpha < -1.$$

In order that the integral (15.1) is well defined, it is sufficient to assume that

(15.8) $$\begin{cases} \mathrm{Re}\,\beta > 0, & \mathrm{Re}\,\alpha < 1 \quad \text{if } C \text{ is the path (iv)},\\ \mathrm{Re}(\gamma-\beta) > 0, & \mathrm{Re}\,\alpha < 1 \quad \text{if } C \text{ is the path (v)},\\ \mathrm{Re}(\alpha+1-\gamma) > 0, & \mathrm{Re}\,\alpha < 1 \quad \text{if } C \text{ is the path (vi)}. \end{cases}$$

Furthermore, the integral is holomorphic with respect to α, β and γ in the respective domain (15.8) for each case. In order that the condition (15.4') is satisfied, it is sufficient to assume that

(15.9) $$\begin{cases} \mathrm{Re}\,\beta > 0, & \mathrm{Re}\,\alpha < -1 \quad \text{if } C \text{ is the path (iv)},\\ \mathrm{Re}(\gamma-\beta) > 0, & \mathrm{Re}\,\alpha < -1 \quad \text{if } C \text{ is the path (v)},\\ \mathrm{Re}(\alpha+1-\gamma) > 0, & \mathrm{Re}\,\alpha < -1 \quad \text{if } C \text{ is the path (vi)}. \end{cases}$$

Therefore, the integral (15.1) is holomorphic for (15.8) and satisfies the hypergeometric differential equation for (15.9). Since the domain (15.8) contains the domain (15.9) for each case, the integral (15.1) is a solution of (15.2) for (15.8). Thus we

proved the following result:

THEOREM 15.1: If we put

(15.10) $$U(x,u) = u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha},$$

then the following integrals are solutions of the hypergeometric differential equation (15.2):

(15.11) $$\begin{cases} F_{01}(x) = \int_0^1 U(x,u)du & \text{if } \mathrm{Re}\,\beta>0,\ \mathrm{Re}(\gamma-\beta)>0,\\ F_{\infty 0}(x) = \int_\infty^0 U(x,u)du & \text{if } \mathrm{Re}\,\beta>0,\ \mathrm{Re}(\alpha+1-\gamma)>0,\\ F_{1\infty}(x) = \int_1^\infty U(x,u)du & \text{if } \mathrm{Re}(\gamma-\beta)>0,\ \mathrm{Re}(\alpha+1-\gamma)>0,\\ F_{0\frac{1}{x}}(x) = \int_0^{1/x} U(x,u)du & \text{if } \mathrm{Re}\,\beta>0,\ \mathrm{Re}\,\alpha<1,\\ F_{1\frac{1}{x}}(x) = \int_1^{1/x} U(x,u)du & \text{if } \mathrm{Re}(\gamma-\beta)>0,\ \mathrm{Re}\,\alpha<1,\\ F_{\frac{1}{x}\infty}(x) = \int_{1/x}^\infty U(x,u)du & \text{if } \mathrm{Re}(\alpha+1-\gamma)>0,\ \mathrm{Re}\,\alpha<1. \end{cases}$$

In particular, all the conditions on the parameters are satisfied if

(15.12) $$0 < \mathrm{Re}\,\beta < \mathrm{Re}\,\gamma < \mathrm{Re}(\alpha+1) < 2 .$$

The function $U(x,u)$ has singularities at $u = 0, 1, 1/x$ and ∞ as a function of u. Suppose that $\mathrm{Im}\,x > 0$. Then x is in the upper half-plane and $1/x$ is in the lower half-plane. More precisely, we make the convention:

$0 < \arg x < \pi$, $-\pi < \arg \frac{1}{x} < 0$, $-\pi < \arg(1-x) < 0$ and $-\pi < \arg(-x) < 0$.

Let us specify the paths of integration in the following manner:

(i) For F_{01}, the path is the line-segment joining $u = 0$ to $u = 1$ along the real-axis;

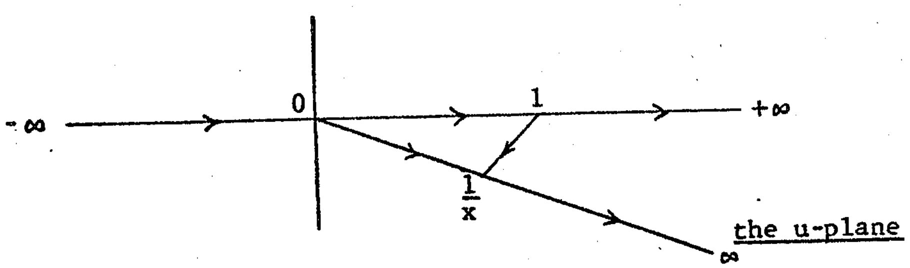
(ii) for $F_{\infty 0}$, the path is the negative real-axis which joins $u = -\infty$ to $u = 0$;

(iii) for $F_{1\infty}$, the path is the half-line joining $u = 1$ to $u = +\infty$ along the positive real-axis;

(iv) for $F_{0\frac{1}{x}}$, the path is the line-segment joining $u = 0$ to $u = 1/x$;

(v) for $F_{1\frac{1}{x}}$, the path is the line-segment joining $u = 1$ to $u = 1/x$;

(vi) for $F_{\frac{1}{x}\infty}$, the path is the half-line joining $u = 1/x$ to $u = \infty$ in the direction: $\arg(1/x)$.



In order to define the integrals (15.11), we must also specify the branches of the function $U(x,u)$ given by (15.10). Note that this function is multiple-valued with respect to u. To fix a

branch of $U(x,u)$, it is sufficient to determine the argument of each factor u, $1-u$ and $1-xu$ of the function U on the paths of integration. We shall fix those arguments in the following manner:

(i) For F_{01}, $\arg u = 0$, $\arg(1-u) = 0$, $-\pi \leq \arg(1-xu) \leq 0$;

(ii) for $F_{\infty 0}$, $\arg u = \pi$, $\arg(1-u) = 0$, $0 \leq \arg(1-xu) \leq \pi$;

(iii) for $F_{1\infty}$, $\arg u = 0$, $\arg(1-u) = -\pi$, $-\pi \leq \arg(1-xu) \leq 0$;

(iv) for $F_{0\frac{1}{x}}$, $-\pi \leq \arg u \leq 0$, $0 \leq \arg(1-u) \leq \pi$, $\arg(1-xu) = 0$;

(v) for $F_{1\frac{1}{x}}$, $-\pi \leq \arg u \leq 0$, $0 \leq \arg(1-u) \leq \pi$, $-\pi \leq \arg(1-xu) \leq 0$;

(vi) for $F_{\frac{1}{x}\infty}$, $-\pi \leq \arg u \leq 0$, $0 \leq \arg(1-u) \leq \pi$, $\arg(1-xu) = -\pi$.

As we have shown before, the integral $F_{01}(x)$ is the Euler integral representation of the function $F(\alpha, \beta, \gamma, x)$ multiplied by a constant:

$$\frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)}.$$

In other words, we have

$$F_{01}(x) = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)} F(\alpha, \beta, \gamma, x).$$

If we make the change of variable

$$u = v(v-1)^{-1} \quad (1 \geq v \geq 0),$$

the second integral $F_{\infty 0}$ becomes

$$F_{\infty 0}(x) = \int_1^0 [v(v-1)^{-1}]^{\beta-1}[1-v(v-1)^{-1}]^{\gamma-\beta-1}[1-xv(v-1)^{-1}]^{-\alpha}\frac{-dv}{(v-1)^2}.$$

Let us suppose that arg $v = 0$ and $\arg(1-v) = 0$ as v goes from 0 to 1. Then, since $\arg u = \pi$, we must have

$$v(v-1)^{-1} = e^{\pi i} v(1-v)^{-1},$$

and hence

$$[v(v-1)^{-1}]^{\beta-1} = e^{\pi i(\beta-1)} v^{\beta-1}(1-v)^{-\beta+1}.$$

On the other hand, $\arg(1-u) = 0$, $\arg(1-v) = 0$ and $1 - v(v-1)^{-1} = (1-v)^{-1}$ imply that

$$[1 - v(v-1)^{-1}]^{\gamma-\beta-1} = (1-v)^{-\gamma+\beta+1}.$$

If we suppose that

(15.13) $$0 \leq \arg[1-(1-x)v] \leq \pi,$$

then we get

$$[1 - xv(v-1)^{-1}]^{-\alpha} = [1-(1-x)v]^{-\alpha}(1-v)^{\alpha},$$

since

$$1 - xv(v-1)^{-1} = [1-(1-x)v](1-v)^{-1}$$

and

$$0 \leq \arg(1-xu) \leq \pi.$$

The assumption (15.13) is justified by the convention: $-\pi < \arg(1-x) < 0$. Note that we assumed at the beginning that $0 < \arg x < \pi$. Thus we have

$$F_{\infty 0}(x) = \int_0^1 e^{\pi i(\beta-1)} v^{\beta-1}(1-v)^{\alpha-\gamma}[1-(1-x)v]^{-\alpha}\, dv,$$

where

$$\arg v = 0, \quad \arg(1-v) = 0, \quad 0 \leq \arg[1-(1-x)v] \leq \pi.$$

This means that

$$F_{\infty 0}(x) = e^{\pi i(\beta-1)} \frac{\Gamma(\beta)\Gamma(\alpha+1-\gamma)}{\Gamma(\alpha+\beta+1-\gamma)} F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-x).$$

Similarly,

$$F_{1\infty}(x) = \int_1^0 (v^{-1})^{\beta-1}(1-v^{-1})^{\gamma-\beta-1}(1-xv^{-1})^{-\alpha}(-v^{-2})dv$$

$$= \int_0^1 v^{-\beta+1}[e^{-\pi i}(1-v)v^{-1}]^{\gamma-\beta-1}[(-xv^{-1})(1-x^{-1}v)]^{-\alpha}v^{-2}dv$$

$$= e^{-\pi i(\gamma-\beta-1)}(-x)^{-\alpha}\int_0^1 v^{\alpha-\gamma}(1-v)^{\gamma-\beta-1}(1-x^{-1}v)^{-\alpha}dv$$

$$= e^{-\pi i(\gamma-\beta-1)}\frac{\Gamma(\alpha-\gamma+1)\Gamma(\gamma-\beta)}{\Gamma(\alpha-\beta+1)}(-x)^{-\alpha}F(\alpha, \alpha-\gamma+1, \alpha-\beta+1, 1/x),$$

$$F_{0\frac{1}{x}}(x) = \int_0^1 (vx^{-1})^{\beta-1}(1-vx^{-1})^{\gamma-\beta-1}(1-v)^{-\alpha}x^{-1}dv$$

$$= \int_0^1 x^{-\beta}v^{\beta-1}(1-v)^{-\alpha}(1-x^{-1}v)^{\gamma-\beta-1}dv$$

$$= e^{-\pi i\beta}\frac{\Gamma(\beta)\Gamma(1-\alpha)}{\Gamma(\beta-\alpha+1)}(-x)^{-\beta}F(\beta-\gamma+1, \beta, \beta-\alpha+1, 1/x)$$

$$F_{1\frac{1}{x}}(x) = \int_1^0 [x^{-1}(1-(1-x)v)]^{\beta-1}[1-x^{-1}(1-(1-x)v)]^{\gamma-\beta-1}[1-(1-(1-x)v)]^{-\alpha}\left(-\frac{1-x}{x}\right)dv$$

$$= \int_0^1 [x^{-1}(1-(1-x)v)]^{\beta-1}[e^{\pi i}x^{-1}(1-x)(1-v)]^{\gamma-\beta-1}[(1-x)v]^{-\alpha}x^{-1}(1-x)dv$$

$$= e^{\pi i(\gamma-\beta-1)}\int_0^1 x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}v^{-\alpha}(1-v)^{\gamma-\beta-1}[1-(1-x)v]^{\beta-1}dv$$

$$= e^{\pi i(\gamma-\beta-1)}\frac{\Gamma(1-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha-\beta+1)}x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}F(1-\beta, 1-\alpha, \gamma-\alpha-\beta+1, 1-x)$$

$$= e^{\pi i(\gamma-\beta-1)} \frac{\Gamma(1-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha-\beta+1)}(1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-x) \ ,$$

$$F_{\frac{1}{x}\infty}(x) = \int_1^0 (v^{-1}x^{-1})^{\beta-1}(1-v^{-1}x^{-1})^{\gamma-\beta-1}(1-v^{-1})^{-\alpha}(-v^{-2}x^{-1})dv$$

$$= \int_0^1 (v^{-1}x^{-1})^{\beta-1}[e^{\pi i}(1-vx)v^{-1}x^{-1}]^{\gamma-\beta-1}[e^{-\pi i}(1-v)v^{-1}]^{-\alpha}v^{-2}x^{-1}dv$$

$$= \int_0^1 x^{1-\gamma}v^{\alpha-\gamma}(1-v)^{-\alpha}(1-xv)^{\gamma-\beta-1}e^{\pi i(\gamma+\alpha-\beta-1)}dv$$

$$= e^{\pi i(\gamma+\alpha-\beta-1)} \frac{\Gamma(\alpha+1-\gamma)\Gamma(1-\alpha)}{\Gamma(2-\gamma)} x^{1-\gamma} F(\beta-\gamma+1, \alpha+1-\gamma, 2-\gamma, x).$$

We shall summarize these results in Table 15.1.

The integral of the form (15.1) is transformed by the change of variable

$$u = 1/\xi$$

into

$$\int_{\Gamma}(x-\xi)^{\lambda-1}\psi(\xi)d\xi \ , \qquad (15.14)$$

where

$$\psi(\xi) = -(-\xi)^{1-\lambda}\varphi(\xi^{-1})\xi^{-2}$$

and Γ is the image of C by the change of variable $u = 1/\xi$. The integral operator which assigns to ψ the integral

$$\int_0^{\xi}(x-\xi)^{\lambda-1}\psi(\xi)d\xi$$

or

$$\frac{1}{\Gamma(\lambda)}\int_0^{\xi}(x-\xi)^{\lambda-1}\psi(\xi)d\xi$$

is called the Euler transform.

The integral

$$\int_a^b u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}du \ ,$$

where $a, b = 0, 1, 1/x, \infty$, is transformed by $u = 1/\xi$ into

$$\text{const.} \int_c^d \xi^{\alpha-\gamma}(1-\xi)^{\gamma-\beta-1}(x-\xi)^{-\alpha}d\xi \ ,$$

where $c, d = 0, 1, x, \infty$. From this we derive the following theorem.

THEOREM 15.2: If we put

$$\Xi(x,\xi) = \xi^{\alpha-\gamma}(1-\xi)^{\gamma-\beta-1}(x-\xi)^{-\alpha} \ ,$$

then

$$\int_0^1 \Xi(x,\xi)d\xi = \text{const.}\ (-x)^{-\alpha}F(\alpha, \alpha-\gamma+1, \alpha-\beta+1, 1/x) \ , \qquad (15.15\text{-}1)$$

$$\int_1^{\infty} \Xi(x,\xi)d\xi = \text{const.}\ F(\alpha, \beta, \gamma, x) \ , \qquad (15.15\text{-}2)$$

$$\int_{\infty}^0 \Xi(x,\xi)d\xi = \text{const.}\ F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x) \ , \qquad (15.15\text{-}3)$$

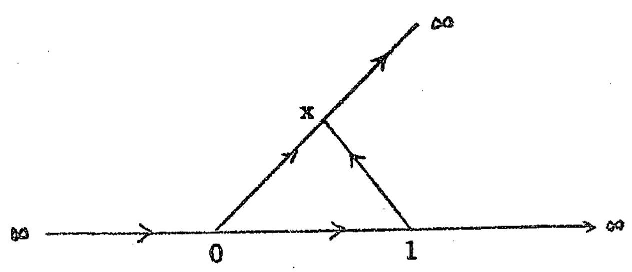
$$\int_0^x \Xi(x,\xi)d\xi = \text{const.}\ x^{1-\gamma}F(\beta-\gamma+1, \alpha-\gamma+1, 2-\gamma, x) \ , \qquad (15.15\text{-}4)$$

$$\int_1^x \Xi(x,\xi)d\xi = \text{const.}\ (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-x) \ , \qquad (15.15\text{-}5)$$

$$\int_x^{\infty} \Xi(x,\xi)d\xi = \text{const.}\ (-x)^{\beta}F(\beta, \beta-\gamma+1, \beta-\alpha+1, 1/x) \qquad (15.15\text{-}6)$$

are all solutions of the hypergeometric differential equation (15.2).

Supposing that $\operatorname{Im} x > 0$, taking the paths shown in the figure given below and fixing branches of $\Xi(x, \xi)$, we can verify the formulas (15.15).



16. <u>Connection formulas and monodromy group of the Gauss differential equation.</u> Suppose that

(16.1) $$0 < \operatorname{Re}\beta < \operatorname{Re}\gamma < \operatorname{Re}(\alpha+1) < 2$$

and

(16.2) $$0 < \arg x < \pi, \quad -\pi < \arg(1-x) < 0, \quad -\pi < \arg(-x) < 0.$$

Then the six integrals F_{01}, $F_{\frac{1}{x}\infty}$, $F_{\infty 0}$, $F_{1\frac{1}{x}}$, $F_{1\infty}$, $F_{0\frac{1}{x}}$ are well defined and we have three fundamental systems

$$\{F_{01}, F_{\frac{1}{x}\infty}\}, \quad \{F_{\infty 0}, F_{1\frac{1}{x}}\}, \quad \{F_{1\infty}, F_{0\frac{1}{x}}\}$$

of the hypergeometric differential equation

(16.3) $$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0 .$$

In this section, we shall find relations between these three fundamental systems.

Consider first three integrals $F_{\infty 0}$, F_{01} and $F_{1\infty}$. Consider also the following curves in the u-plane:

C_1 : a line $-R \leqq u \leqq -r$, where $R > r > 0$,

γ_1 : a semi-circle $u = re^{i(\pi-\theta)}$, where $0 \leqq \theta \leqq \pi$,

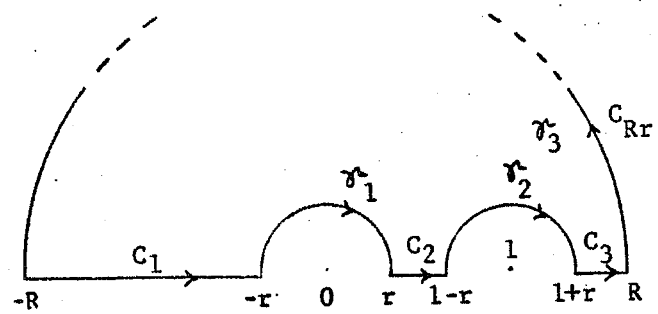
C_2 : a line $r \leqq u \leqq 1-r$,

γ_2 : a semi-circle $u = 1+re^{i(\pi-\theta)}$, where $0 \leqq \theta \leqq \pi$,

C_3 : a line $1+r \leqq u \leqq R$,

γ_3 : a semi-circle $u = Re^{i\theta}$, $0 \leqq \theta \leqq \pi$.

These six curves form a closed path in the u-plane. Let us denote by C_{Rr} this closed path. ($C_{Rr} = C_1\gamma_1C_2\gamma_2C_3\gamma_3$.)



We shall fix a branch $U_1(x, u)$ of the function $U(x, u)$ so that this branch is single-valued on C_{Rr} and its interior. This condition is equivalent to the condition that arg u, arg(1-u) and arg(1-xu) change continuously on the curve C_{Rr}. Hence $U_1(x, u)$ is uniquely determined by the conditions:

(16.4) $\quad \arg u = \pi , \quad \arg(1-u) = 0 , \quad 0 \leq \arg(1-xu) \leq \pi$

on the segment C_1. Under the assumptions (16.4), we observe that

(i) on γ_1, arg u decreases from π to 0,
arg(1-u) starts from 0 and comes back to 0 after taking negative values,
arg(1-xu) changes continuously from a positive value to a negative value;

(ii) on C_2, $\arg u = 0$, $\arg(1-u) = 0$, and arg(1-xu) varies between $-\pi$ and 0 and takes a negative value at $u = 1-r$;

(iii) on γ_2, $\arg u = 0$ at $u = 1-r$ and $1+r$,
arg(1-u) changes from 0 to $-\pi$,
$-\pi \leq \arg(1-xu) \leq 0$;

(iv) on C_3, $\arg u = 0$, $\arg(1-u) = -\pi$, $-\pi \leq \arg(1-xu) \leq 0$;

(v) on γ_3, arg u changes from 0 to π and comes back to the initial value (i.e. π),
arg(1-u) changes from $-\pi$ to 0 ,
arg(1-xu) increases and arrives at the initial positive value.

By virtue of Cauchy's theorem, we have

$$\int_{C_{Rr}} U_1(x, u)\, du = 0 .$$

Let r tend to zero and let R tend to infinity, then

$$\lim \int_{\gamma_j} U_1(x, u)\, du = 0 \qquad (j = 1, 2, 3)$$

and

$$\lim \int_{C_1} U_1(x, u)\, du = F_{\infty 0}(x) ,$$

$$\lim \int_{C_2} U_1(x, u)\, du = F_{01}(x) ,$$

and

$$\lim \int_{C_3} U_1(x, u)\, du = F_{1\infty}(x) .$$

Thus we obtain the following relation:

$$F_{\infty 0}(x) + F_{01}(x) + F_{1\infty}(x) = 0 .$$

Consider next three integrals $F_{\infty 0}$, $F_{0\frac{1}{x}}$ and $F_{\frac{1}{x}\infty}$. In this case, we shall use the following curves :

C_1 : a line $-R \leqq u \leqq -r$, where $R > r > 0$,

γ_1 : a circular arc $u = re^{i\theta}$, where $-\pi \leqq \theta \leqq \arg(1/x) < 0$,

C_2 : a line $u = \rho \exp(i \arg(1/x))$, $r \leqq \rho \leqq |x|^{-1} - r$,

γ_2 : a semi-circle $u = x^{-1} + re^{i\theta}$, $-\pi + \arg(x^{-1}) \leqq \theta \leqq \arg(x^{-1})$,

C_3 : a line $u = \rho \exp(i \arg(1/x))$, $|x|^{-1} + r \leqq \rho \leqq R$,

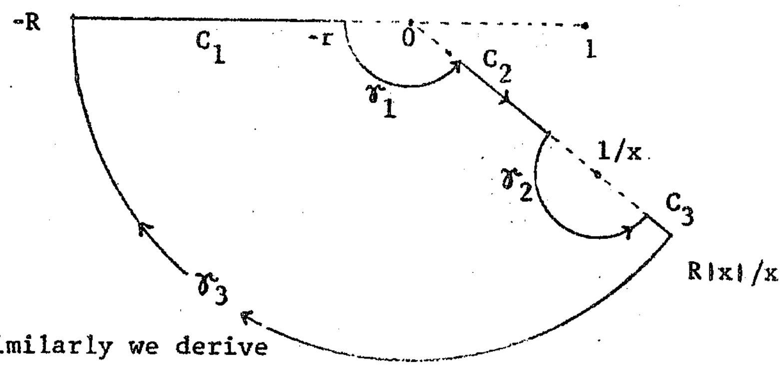
γ_3 : a circular arc $u = Re^{-i\theta}$, $-\arg(x^{-1}) \leqq \theta \leqq \pi$.

Determine $U_1(x, u)$ uniquely by taking

$$-\pi < \arg u = \arg(x^{-1}) < 0 ,$$
$$0 < \arg(1-u) < \pi ,$$
$$\arg(1-xu) = 0$$

on the line segment C_2. Letting $r \to 0$ and $R \to +\infty$, we obtain

$$e^{-2\pi i\beta} F_{\infty 0} + F_{0\frac{1}{x}} + e^{-2\pi i\alpha} F_{\frac{1}{x}\infty} = 0 .$$



Similarly we derive

$$F_{01} + F_{1\frac{1}{x}} - F_{0\frac{1}{x}} = 0$$

and

Table 15.1

integral	arg u	arg(1-u)	arg(1-xu)	transformation	identification
$F_{01}(x) = \int_0^1$	0	0	$[-\pi, 0]$		$\frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)} F(\alpha, \beta, \gamma, x)$
$F_{\infty 0}(x) = \int_\infty^0$	π	0	$[0, \pi]$	$u = v/(v-1)$	$e^{\pi i(\beta-1)} \frac{\Gamma(\beta)\Gamma(\alpha+1-\gamma)}{\Gamma(\alpha+\beta+1-\gamma)} F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-x)$
$F_{1\infty}(x) = \int_1^\infty$	0	$-\pi$	$[-\pi, 0]$	$u = 1/v$	$e^{-\pi i(\gamma-\beta-1)} \frac{\Gamma(\alpha-\gamma+1)\Gamma(\gamma-\beta)}{\Gamma(\alpha-\beta+1)} (-x)^{-\alpha} F(\alpha, \alpha-\gamma+1, \alpha-\beta+1, 1/x)$
$F_{0\frac{1}{x}}(x) = \int_0^{1/x}$	$[-\pi, 0]$	$[0, \pi]$	0	$u = v/x$	$e^{-\pi i\beta} \frac{\Gamma(\beta)\Gamma(1-\alpha)}{\Gamma(\beta-\alpha+1)} (-x)^{-\beta} F(\beta, \beta-\gamma+1, \beta-\gamma+1, 1/x)$
$F_{1\frac{1}{x}}(x) = \int_1^{1/x}$	$[-\pi, 0]$	$[0, \pi]$	$[-\pi, 0]$	$u = [1-(1-x)v]/x$	$e^{\pi i(\gamma-\beta-1)} \frac{\Gamma(1-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha-\beta+1)} (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-x)$
$F_{\frac{1}{x}\infty}(x) = \int_{1/x}^\infty$	$[-\pi, 0]$	$[0, \pi]$	$-\pi$	$u = 1/(vx)$	$e^{\pi i(\gamma+\alpha-\beta-1)} \frac{\Gamma(\alpha-\gamma+1)\Gamma(1-\alpha)}{\Gamma(2-\gamma)} x^{1-\gamma} F(\alpha-\gamma+1, \beta+1-\gamma, 2-\gamma, x)$

$$-F_{1\infty} + e^{-2\pi i(\gamma-\beta)} F_{1\frac{1}{x}} + e^{-2\pi i(\gamma-\beta)} F_{\frac{1}{x}\infty} = 0 .$$

THEOREM 16.1: The six integrals F_{01}, $F_{\frac{1}{x}\infty}$, $F_{\infty 0}$, $F_{1\frac{1}{x}}$, $F_{1\infty}$ and $F_{0\frac{1}{x}}$ satisfy the following four relations:

$$(16.5) \qquad \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & e^{-2\pi i\alpha} & e^{-2\pi i\beta} & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & e^{2\pi i(\beta-\gamma)} & 0 & e^{2\pi i(\beta-\gamma)} & -1 & 0 \end{pmatrix} \begin{pmatrix} F_{01} \\ F_{\frac{1}{x}\infty} \\ F_{\infty 0} \\ F_{1\frac{1}{x}} \\ F_{1\infty} \\ F_{0\frac{1}{x}} \end{pmatrix} = 0 .$$

It is easy to verify that every four-by-four submatrix of the matrix of (16.5) has rank four. Hence, if we select any two integrals from the six, then the other four integrals can be expressed as a linear combination of the two. Such relations are called <u>connection formulas</u>. For example, we have

$$F_{01} = \frac{e^{-2\pi i\alpha}-e^{-2\pi i\gamma}}{e^{-2\pi i(\gamma-\beta)}-e^{-2\pi i\alpha}} F_{\infty 0} + \frac{e^{-2\pi i(\gamma-\beta)}[e^{-2\pi i\alpha}-1]}{e^{-2\pi i(\gamma-\beta)}-e^{-2\pi i\alpha}} F_{1\frac{1}{x}}$$

$$= \frac{1-e^{-2\pi i(\gamma-\alpha)}}{e^{-2\pi i(\gamma-\alpha-\beta)}-1} F_{\infty 0} + \frac{e^{-2\pi i\alpha}-1}{1-e^{-2\pi i(\alpha+\beta-\gamma)}} F_{1\frac{1}{x}} .$$

On the other hand, Table 15.1 shows that

$$F_{01}(x) = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)} F(\alpha, \beta, \gamma, x) ,$$

$$F_{\infty 0}(x) = e^{\pi i(\beta-1)} \frac{\Gamma(\beta)\Gamma(\alpha+1-\gamma)}{\Gamma(\alpha+\beta+1-\gamma)} F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x) ,$$

and

$$F_{1\frac{1}{x}}(x) = e^{\pi i(\gamma-\beta-1)} \frac{\Gamma(1-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha-\beta+1)} (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1,1-x).$$

Therefore

$$(16.6) \quad F(\alpha,\beta,\gamma,x) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} F(\alpha,\beta,\alpha+\beta-\gamma+1,1-x)$$

$$+ \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1,1-x) .$$

Note that

$$\frac{1-e^{-2\pi i(\gamma-\alpha)}}{e^{-2\pi i(\gamma-\alpha-\beta)}-1} e^{\pi i(\beta-1)} = \frac{\sin(\pi(\gamma-\alpha))}{\sin(\pi(\gamma-\alpha-\beta))} ,$$

$$\frac{e^{-2\pi i\alpha}-1}{1-e^{-2\pi i(\alpha+\beta-\gamma)}} e^{\pi i(\gamma-\beta-1)} = \frac{\sin(\pi\alpha)}{\sin(\pi(\alpha+\beta-\gamma))} ,$$

and

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} .$$

The formula (16.6) can be obtained also in the following manner:

Set

$$F(\alpha,\beta,\gamma,x) = AF(\alpha,\beta,\alpha+\beta-\gamma+1,1-x)$$
$$+ B(1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1,1-x).$$

Let x tend to 1. Then by virtue of Theorem 3.3 (Section 3, Chapter I) we get

$$\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} = A$$

under a suitable restriction on the parameters α, β and γ. Next let x tend to 0. Then again by using Theorem 3.3, we get

$$1 = A \frac{\Gamma(\alpha+\beta-\gamma+1)\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)} + B \frac{\Gamma(\gamma-\alpha-\beta+1)\Gamma(1-\gamma)}{\Gamma(1-\alpha)\Gamma(1-\beta)} .$$

From this we obtain

$$B = \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} .$$

In Section 11 (Chapter II) we derived a connection formula (11.7) which represents $F(\alpha,\beta,\gamma,x)$ as a linear combination of

$$(-x)^{-\alpha}F(\alpha,\alpha+1-\gamma,\alpha+1-\beta,1/x)$$

and

$$(-x)^{-\beta}F(\beta,\beta+1-\gamma,\beta+1-\alpha,1/x) .$$

(Cf. Theorem 11.4, p.74.) As this connection formula holds under the condition that

$$(16.7) \qquad \alpha,\ \beta,\ \gamma,\ \alpha-\beta,\ \gamma-\alpha,\ \gamma-\beta,\ \gamma-\alpha-\beta \neq \text{integer} ,$$

all connection formulas of this kind also hold under the condition (16.7). If we know one of those connection formulas, then other formulas are derived by using changes of variables and by using the identification of twenty four solutions of Kummer.

17. Monodromy representations of the Gauss differential equation. This section is a continuation of Section 6 (Chapter I). In Section 6, we defined the monodromy representation of the Gauss differential equation (6.1) with respect to a fundamental system $\{\varphi, \psi\}$. We also stated Theorem 6.1 (p.39) without proof. In this section, by using the six integrals (15.11), we shall compute monodromy representations of the Gauss differential equation explicitly.

As in Section 6, let $D = \mathbf{C} - \{0, 1\}$ and let $x_0 \in D$. We denote by $\pi_1(D, x_0)$ the fundamental group of D with the base point x_0. By the definition of $\pi_1(D, x_0)$ which was given in Section 6, an element of $\pi_1(D, x_0)$ is a homotopy class of loops in D which start and terminate at x_0. Let ℓ_0 be a loop which encircles the point $x = 0$ once in the positive sense, but does not encircle the point $x = 1$. Denote also by ℓ_1 a loop which encircles the point $x = 1$ once in the positive sense, but does not encircle the point $x = 0$. The homotopy classes containing ℓ_0 and ℓ_1 are denoted respectively by $[\ell_0]$ and $[\ell_1]$. Then $\pi_1(D, x_0)$ is a free group generated by $[\ell_0]$ and $[\ell_1]$.

Let $\{\varphi, \psi\}$ be a fundamental system of the Gauss differential equation. Then the monodromy representation with respect to $\{\varphi, \psi\}$ is a homomorphism:

$$\rho : \pi_1(D, x_0) \longrightarrow GL(2, \mathbf{C}) .$$

This homomorphism ρ is uniquely determined by two matrices

$$A_0 = \rho([\ell_0]), \quad A_1 = \rho([\ell_1]). \tag{17.1}$$

The matrices (17.1) are called the circuit matrices around $x = 0$ and $x = 1$ with respect to $\{\varphi, \psi\}$, respectively. We shall present two methods for computing A_0 and A_1 in this section.

Method I: The first method is based on the connection formulas among the six integrals (15.11). Suppose that $\operatorname{Im} x_0 > 0$ and that

$$0 < \operatorname{Re}\beta < \operatorname{Re}\gamma < \operatorname{Re}(\alpha+1) < 2 .$$

We shall use the fundamental system $\{F_{01}, F_{\frac{1}{x}\infty}\}$. Note that

$$F_{01}(x) = \text{const.}\ F(\alpha, \beta, \gamma, x)$$

and

$$F_{\frac{1}{x}\infty}(x) = \text{const.}\ x^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x) .$$

The solution $F_{01}(x)$ is single-valued in the neighborhood of $x = 0$, while the solution $F_{\frac{1}{x}\infty}(x)$ goes to $e^{-2\pi i\gamma}F_{\frac{1}{x}\infty}(x)$ if it is continued analytically along ℓ_0. Therefore, in this case, we have

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2\pi i\gamma} \end{bmatrix}.$$

In order to compute A_1, we use the following connection formulas:

$$(17.2)\begin{cases} F_{01}(x) = \dfrac{e^{-2\pi i\alpha}-e^{-2\pi i\gamma}}{\Delta} F_{\infty 0}(x) + \dfrac{e^{-2\pi i(\gamma-\beta)}(e^{-2\pi i\alpha}-1)}{\Delta} F_{1\frac{1}{x}}(x) \\ F_{\frac{1}{x}\infty}(x) = \dfrac{e^{-2\pi i\beta}-1}{\Delta} F_{\infty 0}(x) + \dfrac{1-e^{-2\pi i(\gamma-\beta)}}{\Delta} F_{1\frac{1}{x}}(x) , \end{cases}$$

where

$$\Delta = e^{-2\pi i(\gamma-\beta)} - e^{-2\pi i\alpha} .$$

By using the fact that

$$F_{\infty 0}(x) = \text{const. } F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x) ,$$

$$F_{1\frac{1}{x}}(x) = \text{const. } (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-x) ,$$

we derive

$$A_1 = C \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i(\gamma-\alpha-\beta)} \end{bmatrix} C^{-1} ,$$

where C is the matrix of the linear relation (17.2).

Method II: Assume again that $\operatorname{Im} x_0 > 0$. We shall use the fundamental system $\{F_{01}, F_{1\frac{1}{x}}\}$, where

$$(17.3)\begin{cases} F_{01}(x) = \displaystyle\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}du , \\ F_{1\frac{1}{x}}(x) = \displaystyle\int_1^{1/x} u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}du . \end{cases}$$

The paths of integration are fixed in the same way as in Section 15.

Let us consider first the analytic continuation of F_{01} along

ℓ_0. We must investigate how the quantity $\arg(1-xu)$ changes along ℓ_0. We can assume that ℓ_0 consists of two parts: (i) a line-segment joining x_0 to ir_0 and (ii) the circle $|x| = r_0$, where r_0 is a small positive number. (See Fig. 17.1.)

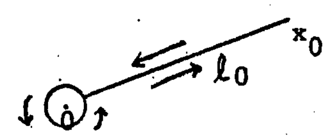


Fig. 17.1

Now it is easy to see that the change of $\arg(1-xu)$ along the loop ℓ_0 is zero. Hence $F_{01}(x)$ does not change by the analytic continuation along ℓ_0.

Let us next consider the analytic continuation of F_{01} along ℓ_1. As x moves along ℓ_1, the point $1/x$ moves along a loop encircling $x = 1$ in the positive sense. (See Fig. 17.2.) In order to keep the same branch of the integrant for the integral F_{01}, we must deform the path of integration so that the point $1/x$ is never on the path of integration. Let x start moving along ℓ_1 at $x = x_0$. As x moves along ℓ_1, we deform the path of integration. When x comes back to x_0, the path of integration becomes a curve shown by Fig. 17.3. This curve can be further deformed into a curve shown by Fig. 17.4. It is then not difficult to show that $F_{01}(x)$ changes into

$$F_{01}(x) + (1 - e^{-2\pi i\alpha})F_{1\frac{1}{x}}(x)$$

by the analytic continuation along ℓ_1.

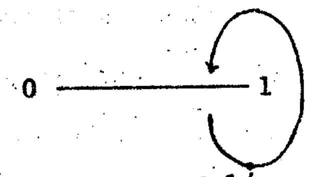


Fig. 17.2

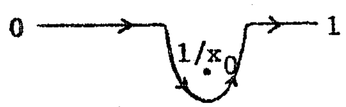


Fig. 17.3

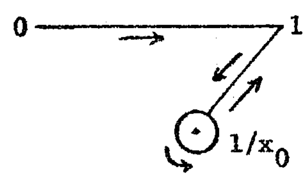


Fig. 17.4

Let us consider the analytic continuation of $F_{1\frac{1}{x}}$ along ℓ_0. As x moves along ℓ_0, the point $1/x$ moves along a loop encircling the points 0 and 1 in the negative sense. (See Fig. 17.5.) We must deform the path of integration for $F_{1\frac{1}{x}}$ so that the points 0 and 1 are never on the path. When x comes back to x_0 after moving along ℓ_0, the path of integration becomes a curve shown by Fig. 17.6. This curve can be further deformed into a curve given by Fig. 17.7. It is then not difficult to prove that $F_{1\frac{1}{x}}(x)$ changes into

$$(-1+e^{-2\pi i\beta})F_{01}(x) + e^{-2\pi i\gamma}F_{1\frac{1}{x}}(x)$$

by the analytic continuation along ℓ_0.

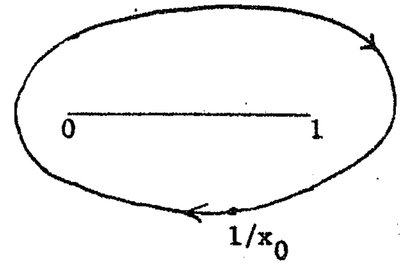


Fig. 17.5.

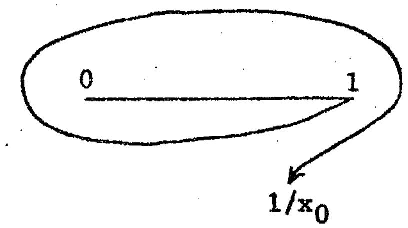


Fig. 17.6

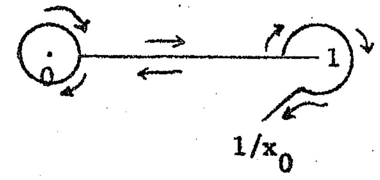


Fig. 17.7

It is easy to see that $F_{1\frac{1}{x}}(x)$ changes into $e^{2\pi i(\gamma-\alpha-\beta)}F_{1\frac{1}{x}}(x)$ by the analytic continuation along ℓ_1.

Thus we obtain

$$A_0 = \begin{bmatrix} 1 & 0 \\ -(1-e^{-2\pi i\beta}) & e^{-2\pi i\gamma} \end{bmatrix},$$

and

$$A_1 = \begin{bmatrix} 1 & 1-e^{-2\pi i\alpha} \\ 0 & e^{-2\pi i(\alpha+\beta-\gamma)} \end{bmatrix}.$$

(Cf. Theorem 6.1, Chapter I, on p.39.)