Hypergeometric Functions of Two Variables

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The research of the author was partially supported by NSF GP 27275 and N00014-67-A-0113-0019.

This note is based on my lecture given at University of Minnesota in the winter quarter of the academic year 1971-72.

The author thanks very much Professor Yasutaka Sibuya who made a manuscript. The author thanks also Professors and students who attended my lecture and gave me many advices.

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INTRODUCTION

The hypergeometric function $F(\alpha, \beta, \gamma, x)$ and the hypergeometric differential equation satisfied by $F(\alpha, \beta, \gamma, x)$ were studied by Euler and Gauss. It was Gauss who investigated this function and its differential equation extensively. Therefore, the hypergeometric differential equation is called either the Euler-Gauss equation or the Gauss equation. The work of Gauss was supplemented by Kummer. Nevertheless various formulas were not well understood until Riemann clarified the fundamental properties of the hypergeometric function and its equation from the function-theoretic point of view. Riemann found a larger class of differential equations which are called Riemann's differential equations. His study was followed by many mathematicians such as Jacobi, Schwarz, Goursat, Barnes and so on. Above all, the work of Schwarz was remarkable and important. Considering the ratio of two linearly independent solutions, he discovered a new class of automorphic functions other than elliptic modular functions. On the other hand, following Riemann's point of view, Fuchs established the theory of linear ordinary differential equations in the complex domain.

The function $F(\alpha, \beta, \gamma, x)$ has three important expressions:

(i) power series expression,

(ii) Euler integral representation,

and

(iii) Barnes integral representation.

By using these expressions, the hypergeometric function has been generalized to many new functions. In 1880 Appell introduced four functions of two variables as generalization of the hypergeometric function. Appell called these functions the hypergeometric functions of two variables. They satisfy systems of partial differential equations. The functions and the systems of differential equations are very similar to the function $F(\alpha, \beta, \gamma, x)$ and its differential equation.

The systems of differential equations satisfied by the hypergeometric functions of two variables are essentially systems of completely integrable total differential equations. Recently the foundation of the theory of analytic functions of several complex variables has been completed. Such a theory should be utilized to establish a general theory of total differential equations as well as partial differential equations in the complex domain. As a matter of fact, such a new study of total differential equations in the complex domain has just begun. It seems to me that we are passing through the stage quite similar to that transitional period from Riemann to Fuchs.

CHAPTER I

The Hypergeometric Function $F(\alpha, \beta, \gamma, x)$

1. The hypergeometric series and hypergeometric function. We start with the power series

$$(1.1) \qquad \sum_{m=0}^{\infty} \frac{(\alpha, m)(\beta, m)}{(\gamma, m)(1, m)} x^m$$

which is called the hypergeometric series, where α, β and γ are complex parameters and the notation (a, k) denotes the factorial function:

$$(a, k) = a(a+1)\cdots(a+k-1) \quad \text{for } k = 1, 2, \cdots ,$$

$$(a, 0) = 1 \quad \text{for } a \neq 0 ;$$

in particular, $(1, k) = k!$. Suppose that $\gamma \neq 0, -1, -2, \cdots$. Then series (1.1) is always meaningful.

Remark 1. Series (1.1) is symmetric with respect to α and β.

Remark 2. If one of α and β is equal to zero or a negative integer, then series (1.1) is reduced to a polynomial.

Remark 3. Since

$$(a, k) = \frac{\Gamma(a+k)}{\Gamma(a)} \quad \text{for } a \neq 0, -1, -2, \cdots ,$$

series (1.1) can be written as

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+m)\Gamma(\beta+m)}{\Gamma(\gamma+m)\Gamma(1+m)} x^m$$

if $\alpha, \beta \neq 0, -1, -2, \cdots$.

The ratio of the m-th coefficient to the (m-1)-th coefficient of (1.1) is

$$\frac{(\alpha+m-1)(\beta+m-1)}{(\gamma+m-1)m}$$

and we have

$$\lim_{m\to\infty}\frac{(\alpha+m-1)(\beta+m-1)}{(\gamma+m-1)m} = 1 .$$

Thus we obtain the following theorem.

THEOREM 1.1: The radius of convergence of the series (1.1) is one if neither α nor β is equal to 0, -1, -2, $\cdots$.

DEFINITION 1.1: The function defined by (1.1) and its analytic continuation is called the hypergeometric function and it is denoted by $F(\alpha, \beta, \gamma, x)$.

In general, the function $F(\alpha, \beta, \gamma, x)$ is an infinitely many valued function with singularities at $x = 0, 1$ and ∞ only. Therefore the analytic continuation of (1.1) into the domain $\mathbf{C} - [1, \infty)$ determines a branch of $F(\alpha, \beta, \gamma, x)$ which is holomorphic there. However, other branches of F have, in general, branch-points at $x = 0$. In this section and thereafter we keep the notation $F(\alpha, \beta, \gamma, x)$ to denote the branch defined directly by series (1.1) unless otherwise stated.

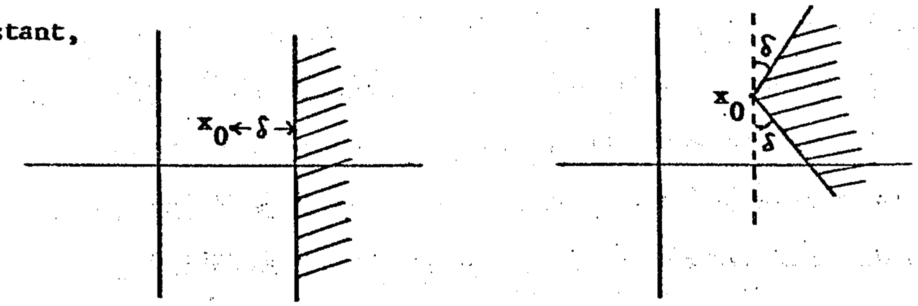
A series of the form

$$(1.2) \qquad \sum_{m=0}^{\infty}(-1)^m a_m \frac{(x-1)\cdots(x-m)}{m!} = \sum_{m=0}^{\infty}(-1)^m a_m \binom{x-1}{m}$$

is called a Newton series or a factorial series of the second kind. It is known that if (1.2) converges at a point $x = x_0$ and if x_0 is not a positive integer, then

(a) the series (1.2) converges for $\mathrm{Re}\, x > \mathrm{Re}\, x_0$,

(b) the series (1.2) converges uniformly for $\mathrm{Re}\, x > \mathrm{Re}\, x_0 + \delta$ and for $|\arg(x - x_0)| < \frac{\pi}{2} - \delta$, where δ is an arbitrary positive constant,



(c) the series (1.2) represents a function holomorphic in $\mathrm{Re}\, x > \mathrm{Re}\, x_0$

The series (1.1) can be considered as a Newton series with respect to $1-\alpha$ and $1-\beta$. In fact, (1.1) is rewritten as

$$\sum_{m=0}^{\infty}(-1)^m \frac{(\beta, m)x^m}{(\gamma, m)(1, m)}(1-\alpha-1)(1-\alpha-2)\cdots(1-\alpha-m)$$

and

$$\sum_{m=0}^{\infty}(-1)^m \frac{(\alpha, m)x^m}{(\gamma, m)(1, m)}(1-\beta-1)(1-\beta-2)\cdots(1-\beta-m) .$$

Since (1.1) is convergent for any α and β provided $|x| < 1$ and $\gamma \neq 0, -1, -2, \cdots$, the function F is holomorphic with respect to α and β for $|\alpha| < \infty$ and $|\beta| < \infty$.

A series of the form

$$(1.3) \qquad \sum_{m=0}^{\infty}\frac{m!\, a_m}{x(x+1)\cdots(x+m)}$$

is called a factorial series or a factorial series of the first kind. It is known that if (1.3) converges at $x = x_0$, then

(a') (1.3) converges for $\mathrm{Re}\, x > \mathrm{Re}\, x_0$ possibly except at $x = 0, -1, -2, \cdots$, (b') (1.3) converges uniformly for $\operatorname{Re} x > \operatorname{Re} x_0 + \delta$, $|x-k| > \delta$, $k = 0, -1, -2, \cdots$ and for $|\arg(x-x_0)| < \frac{\pi}{2} - \delta$, $|x-k| > \delta$, $k = 0, -1, -2, \cdots$, where δ is an arbitrary positive constant,

(c') (1.3) represents a function meromorphic in the domain $\operatorname{Re} x > \operatorname{Re} x_0$ possibly with simple poles at $x = 0, -1, -2, \cdots$.

The series (1.1) omitted by the constant term can be considered as a factorial series in γ. Therefore the function F is a function meromorphic with respect to γ possibly with simple poles at $\gamma = 0, -1, -2, \cdots$. Thus we obtain the following theorem.

THEOREM 1.2: The function $F(\alpha, \beta, \gamma, x)$ is a function which is holomorphic in α, β, γ and x in the domain

$$\mathbb{C} \times \mathbb{C} \times (\mathbb{C} - \{0, -1, -2, \cdots\}) \times \mathbb{D} \quad (\mathbb{D} = \{x; |x| < 1\})$$

and whose poles are at $\gamma = 0, -1, -2, \cdots$ and are all simple.

Remark 4. As for the convergence of (1.1) on $|x| = 1$, it is known that (1.1) is

absolutely convergent if $\operatorname{Re}(\gamma - \alpha - \beta) > 0$,

conditionally convergent except at $x = 1$ if $0 \geq \operatorname{Re}(\gamma - \alpha - \beta) > -1$

and

divergent if $-1 \geq \operatorname{Re}(\gamma - \alpha - \beta)$.

Exercise 1. Prove Theorem 1.2 directly without using the properties of Newton series and factorial series.

Hint. Consider the series $F(\alpha, \beta, \gamma, x)/\Gamma(\gamma)$ and, using the formulas

$$\lim_{k \to \infty} \frac{(a,k)}{(k-1)!\,k^a} = 1 \quad \text{or} \quad \frac{\Gamma(a+k)}{\Gamma(a)\Gamma(k)} \sim k^a \quad (k \to \infty),$$

show that

$$\left\{ \frac{\Gamma(\alpha+m)\,\Gamma(\beta+m)\,x^{\frac{m}{2}}}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma+m)\,\Gamma(1+m)} ; \quad m = 0, 1, 2, \cdots \right\}$$

is bounded.

References: Milne-Thompson, The calculus of finite differences, 271-321,

and

Nörlund, Leçons sur les séries d'interpolation, 1926, 99-227.

2. Contiguous functions. In this section, we shall start with the following theorem.

THEOREM 2.1: F and its derivative satisfy the following relations

$$xF'(\alpha,\beta,\gamma,x)+\alpha F(\alpha,\beta,\gamma,x) = \alpha F(\alpha+1,\beta,\gamma,x),$$

$$x(1-x)F'(\alpha,\beta,\gamma,x)+(\gamma-\alpha-\beta x)F(\alpha,\beta,\gamma,x) = (\gamma-\alpha)F(\alpha-1,\beta,\gamma,x)$$

$$\gamma(1-x)F'(\alpha,\beta,\gamma,x)+\gamma(\gamma-\alpha-\beta)F(\alpha,\beta,\gamma,x) = (\gamma-\alpha)(\gamma-\beta)F(\alpha,\beta,\gamma+1,x),$$

$$xF'(\alpha,\beta,\gamma,x)+(\gamma-1)F(\alpha,\beta,\gamma,x) = (\gamma-1)F(\alpha,\beta,\gamma-1,x).$$

Remark 1. We can derive two formulas from the first and the second by interchanging α and β. (See Remark 1 in §1.)

Proof: Let us put

$$A_m = \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)}$$

to write F as

(2.1) $$F(\alpha,\beta,\gamma,x) = \sum_{m=0}^{\infty} A_m x^m .$$

Clearly we have

(2.2) $$xF'(\alpha,\beta,\gamma,x) = \sum_{m=0}^{\infty} mA_m x^m .$$

Next from the definition of the hypergeometric series we derive

$$F(\alpha+1,\beta,\gamma,x) = \sum_{m=0}^{\infty} \frac{(\alpha+1,m)(\beta,m)}{(\gamma,m)(1,m)} x^m = \sum_{m=0}^{\infty} \frac{\alpha+m}{\alpha}\frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)} x^m .$$

Hence

(2.3) $$F(\alpha+1,\beta,\gamma,x) = \sum_{m=0}^{\infty} \frac{\alpha+m}{\alpha} A_m x^m .$$

In a similar manner, we obtain

(2.4) $$F(\alpha-1,\beta,\gamma,x) = \sum_{m=0}^{\infty} \frac{\alpha-1}{\alpha+m-1} A_m x^m .$$

From (2.1) and (2.2) we derive

$$xF'(\alpha,\beta,\gamma,x)+\alpha F(\alpha,\beta,\gamma,x) = \sum_{m=0}^{\infty}(m+\alpha)A_m x^m .$$

This formula together with (2.3) yields the first relation of the theorem.

Now let us prove the second relation. From (2.2) we derive

$$x(1-x)F'(\alpha,\beta,\gamma,x) = (1-x)\sum_{m=0}^{\infty} mA_m x^m = \sum_{m=0}^{\infty} mA_m x^m - \sum_{m=0}^{\infty} mA_m x^{m+1}$$

$$= \sum_{m=0}^{\infty} mA_m x^m - \sum_{m=1}^{\infty}(m-1)A_{m-1}x^m .$$

By the definition we have

$$A_{m-1} = \frac{(\alpha,m-1)(\beta,m-1)}{(\gamma,m-1)(1,m-1)} = \frac{(\gamma+m-1)m}{(\alpha+m-1)(\beta+m-1)} A_m \quad (m = 1,2,\cdots)$$

and we make the convention $A_{-1} = 0$. Then

$$\sum_{m=1}^{\infty}(m-1)A_{m-1}x^m = \sum_{m=0}^{\infty}(m-1)A_{m-1}x^m$$

and we get

$$x(1-x)F'(\alpha,\beta,\gamma,x) = \sum_{m=0}^{\infty}\left(m - \frac{(\gamma+m-1)(m-1)m}{(\alpha+m-1)(\beta+m-1)}\right)A_m x^m .$$

On the other hand,

$$(\gamma-\alpha-\beta x)F(\alpha,\beta,\gamma,x) = \sum_{m=0}^{\infty}(\gamma-\alpha)A_m x^m - \sum_{m=0}^{\infty}\beta A_m x^{m+1}$$

$$= \sum_{m=0}^{\infty}(\gamma-\alpha)A_m x^m - \sum_{m=0}^{\infty}\beta A_{m-1}x^m .$$

Therefore

$$(\gamma-\alpha-\beta x)F(\alpha,\beta,\gamma,x) = \sum_{m=0}^{\infty}\left(\gamma-\alpha-\frac{\beta(\gamma+m-1)m}{(\alpha+m-1)(\beta+m-1)}\right)A_m x^m .$$

Hence

$$x(1-x)F'(\alpha,\beta,\gamma,x)+(\gamma-\alpha-\beta x)F(\alpha,\beta,\gamma,x)$$

$$= \sum_{m=0}^{\infty}\left(m-\frac{(\gamma+m-1)(m-1)m}{(\alpha+m-1)(\beta+m-1)}+\gamma-\alpha-\frac{\beta(\gamma+m-1)m}{(\alpha+m-1)(\beta+m-1)}\right)A_m x^m$$

$$= \sum_{m=0}^{\infty}\frac{(\gamma-\alpha)(\alpha-1)}{\alpha+m-1}A_m x^m .$$

This result together with (2.4) yields the second relation of the theorem. We can prove the other relations in the same way.

Remark 2. The identities in Theorem 2.1 can be written as follows:

$$x^{-\alpha+1}\frac{d}{dx}\left(x^{\alpha}F(\alpha,\beta,\gamma,x)\right) = \alpha F(\alpha+1,\beta,\gamma,x) ,$$

$$x^{-\gamma+\alpha+1}(1-x)^{\gamma-\alpha-\beta+1}\frac{d}{dx}\left(x^{\gamma-\alpha}(1-x)^{-\gamma+\alpha+\beta}F(\alpha,\beta,\gamma,x)\right)$$

$$= (\gamma-\alpha)F(\alpha-1,\beta,\gamma,x) ,$$

$$\gamma(1-x)^{\gamma-\alpha-\beta+1}\frac{d}{dx}\left((1-x)^{-\gamma+\alpha+\beta}F(\alpha,\beta,\gamma,x)\right)$$

$$= (\gamma-\alpha)(\gamma-\beta)F(\alpha,\beta,\gamma+1,x) ,$$

$$x^{-\gamma+2}\frac{d}{dx}\left(x^{\gamma-1}F(\alpha,\beta,\gamma,x)\right) = (\gamma-1)F(\alpha,\beta,\gamma-1,x) .$$

On the right-hand members of the formulas in Theorem 2.1 there appear functions obtained from $F(\alpha,\beta,\gamma,x)$ by increasing or decreasing one of the parameters by unity.

DEFINITION 2.1: The six functions obtained from $F(\alpha,\beta,\gamma,x)$ by increasing or decreasing one of the parameters by unity: $F(\alpha+1,\beta,\gamma,x)$, $F(\alpha-1,\beta,\gamma,x)$, $F(\alpha,\beta+1,\gamma,x)$, $F(\alpha,\beta-1,\gamma,x)$, $F(\alpha,\beta,\gamma+1,x)$ and $F(\alpha,\beta,\gamma-1,x)$ are called contiguous to $F(\alpha,\beta,\gamma,x)$.

THEOREM 2.2: The function $F(\alpha,\beta,\gamma,x)$ and its two contiguous functions are connected by a homogeneous linear relation whose coefficients are polynomials in α, β, γ and x. For example,

$$\alpha(1-x)F(\alpha+1,\beta,\gamma,x)+(\gamma-2\alpha+(\alpha-\beta)x)F(\alpha,\beta,\gamma,x) - (\gamma-\alpha)F(\alpha-1,\beta,\gamma,x) = 0 , \tag{2.5}$$

$$(\gamma-\alpha)(\gamma-\beta)xF(\alpha,\beta,\gamma+1,x) + \gamma(\gamma-1-(2\gamma-\alpha-\beta-1)x)F(\alpha,\beta,\gamma,x) - \gamma(\gamma-1)(1-x)F(\alpha,\beta,\gamma-1,x) = 0 . \tag{2.6}$$

In order to prove these formulas, eliminate $F'(\alpha,\beta,\gamma,x)$ from the corresponding two identities of Theorem 2.1. For example, in order to derive (2.5), eliminate F' from the first two identities. The number of such relations between F and its contiguous functions is $\binom{6}{2} = 15$. On the other hand, (2.5) and (2.6) are regarded as linear difference equations with respect to α and γ respectively.

COROLLARY: $F(\alpha,\beta,\gamma,x)$, considered as a function of one of three parameters α, β, γ, satisfies a linear difference equation of the second order whose coefficients are polynomials.

Notice that the coefficients in (2.5) are linear in α. A difference equation of a form

$$(a_0'z+b_0')w(z+1)+(a_1'z+b_1')w(z)+(a_2'z+b_2')w(z-1) = 0$$

or in a more usual form

$$(2.7)\qquad (a_0z+b_0)w(z+2)+(a_1z+b_1)w(z+1)+(a_2z+b_2)w(z) = 0$$

is called a hypergeometric difference equation. As to the hypergeometric difference equation, see

Batchelder: Introduction to the theory of linear difference equations.

A function $F(\alpha+p,\beta+q,\gamma+r,x)$, where p, q, r are integers, is often called associated to $F(\alpha,\beta,\gamma,x)$. It is known that every function associated to $F(\alpha,\beta,\gamma,x)$ can be expressible as a linear combination of F and F' whose coefficients are rational functions in α, β, γ and x. Three functions associated to $F(\alpha,\beta,\gamma,x)$ are connected by a linear relation whose coefficients are polynomials in α, β, γ and x.

Exercise 1. Suppose in (2.7) that $a_0a_2 \neq 0$ and $a_1^2 - 4a_0a_2 \neq 0$. Then show that difference equation (2.7) admits a solution

$$w(z) = e^{cz}F(z+d,\beta,\gamma,x),$$

where c, d, β, γ and x are constants. (It is not necessary to assume that $|x| < 1$.)

Hint. Make transformations of forms $z = \xi + d$, $w(z) = e^{cz}u(z)$.

3. Euler integral representation. There are many functions which can be defined by definite integrals. For example, the Γ-function is given by

$$\Gamma(x) = \int_0^{+\infty} e^{-u}u^{x-1}du .$$

The hypergeometric function is also defined by a definite integral.

THEOREM 3.1: If $|x|<1$ and $\mathrm{Re}\,\gamma > \mathrm{Re}\,\beta > 0$, then we have

$$(3.1)\qquad F(\alpha,\beta,\gamma,x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}du,$$

where the integration is taken along the segment $0 \leq u \leq 1$, and branches of many-valued functions $u^{\beta-1}$, $(1-u)^{\gamma-\beta-1}$ and $(1-xu)^{-\alpha}$ are determined by

$$\arg u = 0,\quad \arg(1-u) = 0 \quad\text{and}\quad |\arg(1-xu)| \leq \frac{\pi}{2}$$

respectively.

Proof: For a fixed x such that $|x| < 1$, we have $|xu| \leq |x| < 1$ for $0 \leq u \leq 1$, and hence the function $(1-xu)^{-\alpha}$ is expanded into the series

$$(1-xu)^{-\alpha} = \sum_{m=0}^{\infty}\frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-m+1)}{m!}(-xu)^m = \sum_{m=0}^{\infty}\frac{(\alpha,m)}{(1,m)}u^mx^m$$

which converges uniformly in u for $0 \leq u \leq 1$. Therefore we have

$$I = \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}du = \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}\sum_{m=0}^{\infty}\frac{(\alpha,m)}{(1,m)}u^mx^m du$$

$$= \int_0^1\sum_{m=0}^{\infty}\frac{(\alpha,m)}{(1,m)}x^mu^{\beta+m-1}(1-u)^{\gamma-\beta-1}du .$$

Interchanging the order of the integration and the summation, we have

$$I = \sum_{m=0}^{\infty} \frac{(\alpha,m)}{(1,m)} x^m \int_0^1 u^{\beta+m-1}(1-u)^{\gamma-\beta-1}\, du .$$

Applying the definition of the Beta-function:

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \int_0^1 u^{r-1}(1-u)^{s-1}\, du \quad (\mathrm{Re}\, r,\ \mathrm{Re}\, s > 0),$$

we obtain

$$I = \sum_{m=0}^{\infty} \frac{(\alpha,m)}{(1,m)} x^m \frac{\Gamma(\beta+m)\Gamma(\gamma-\beta)}{\Gamma(\gamma+m)} = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)} \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)} x^m .$$

Therefore Theorem 3.1 is proved.

Remark 1. The Euler integral

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}\, du$$

is well defined for $x \in \mathbb{C} - [1, \infty)$ if $\mathrm{Re}\, \gamma > \mathrm{Re}\, \beta > 0$. This implies that the hypergeometric series is analytically continuable into the domain $\mathbb{C} - [1, \infty)$. The branch of $F(\alpha, \beta, \gamma, x)$ thus obtained is called the principal branch.

THEOREM 3.2: The hypergeometric function has the following three formulas of transformation:

$$\begin{aligned} F(\alpha, \beta, \gamma, x) &= (1-x)^{-\alpha} F(\alpha, \gamma-\beta, \gamma, x/(x-1)) \\ &= (1-x)^{-\beta} F(\gamma-\alpha, \beta, \gamma, x/(x-1)) \\ &= (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x), \end{aligned}$$

which are valid for $|\arg(1-x)| < \pi$ (or $\mathbb{C} - [1, \infty)$), i.e. these formulas are valid for the principal branch.

Proof: It should be noted that by the mapping $y = x/(x-1)$, the domain $\mathbb{C} - [1, \infty)$ is mapped into itself in the one-to-one manner. Make the change of variable $u = 1-v$. Then $1-u = v$, $1-xu = 1-x+xv = (1-x)(1-\frac{x}{x-1}v)$, $du = -dv$. Hence we obtain

$$\begin{aligned} F(\alpha, \beta, \gamma, x) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}\, du \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_1^0 (1-v)^{\beta-1} v^{\gamma-\beta-1}(1-x)^{-\alpha}(1-xv/(x-1))^{-\alpha}(-dv) \\ &= (1-x)^{-\alpha} \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 v^{\gamma-\beta-1}(1-v)^{\beta-1}(1-xv/(x-1))^{-\alpha} dv \\ &= (1-x)^{-\alpha} F(\alpha, \gamma-\beta, \gamma, x/(x-1)) . \end{aligned}$$

This proves the first identity of Theorem 3.2. The changes of variables

$$u = v/(1-x+xv) \quad \text{and} \quad u = (1-v)/(1-xv)$$

will yield the second and third identities respectively.

COROLLARY: If $\gamma - \alpha$ or $\gamma - \beta$ is zero or a negative integer and if $\gamma \neq 0, -1, -2, \cdots$, then $F(\alpha, \beta, \gamma, x)$ is reduced to a polynomial multiplied by $(1-x)^{\gamma-\alpha-\beta}$.

Proof: This corollary follows directly from the third identity of Theorem 3.2.

Consider the series

$$\sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)} . \tag{3.2}$$

As it was previously mentioned, the series (3.2) is absolutely convergent if $\mathrm{Re}(\gamma - \alpha - \beta) > 0$ and $\gamma \neq 0, -1, -2, \cdots$. This implies that $\lim_{x\to 1} F(\alpha, \beta, \gamma, x)$ exists and

$$F(\alpha, \beta, \gamma, 1) = \lim_{x\to 1} F(\alpha, \beta, \gamma, x) = \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)}$$

if $\mathrm{Re}(\gamma - \alpha - \beta) > 0$ and $\gamma \neq 0, -1, -2, \cdots$.

It is also known that (3.2) converges uniformly in every compact set in the domain

$$\mathrm{Re}(\gamma - \alpha - \beta) > 0, \quad \gamma \neq 0, -1, -2, \cdots .$$

This means that $F(\alpha, \beta, \gamma, 1)$ is holomorphic in α, β and γ in this domain. On the other hand, for $x = 1$, $\mathrm{Re}\,\beta > 0$, $\mathrm{Re}(\gamma - \alpha - \beta) > 0$, the Euler integral is well defined and equal to

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 u^{\beta-1}(1-u)^{\gamma-\alpha-\beta-1}\,du .$$

Furthermore,

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 u^{\beta-1}(1-u)^{\gamma-\alpha-\beta-1}\,du$$

for $\mathrm{Re}\,\gamma > \mathrm{Re}\,\beta > 0$, $\mathrm{Re}(\gamma - \alpha - \beta) > 0$. This integral is equal to

$$\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} .$$

Hence

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} .$$

By virtue of the theorem of identity in the function theory, this relation holds for $\mathrm{Re}(\gamma - \alpha - \beta) > 0$, $\gamma \neq 0, -1, -2, \cdots$. Thus we obtained the following theorem.

THEOREM 3.3: We have

$$F(\alpha, \beta, \gamma, 1) = \sum_{m=0}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)}$$

is holomorphic in the domain: $\mathrm{Re}(\gamma - \alpha - \beta) > 0$, $\gamma \neq 0, -1, -2, \cdots$ and equal to

$$\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} .$$

<u>Exercise</u> 1. Prove that

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} = \frac{1}{2}\pi F\left(\frac{1}{2}, \frac{1}{2}, 1, k^2\right) .$$

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4. Barnes integral representation. There is another type of integral representation of $F(\alpha, \beta, \gamma, x)$ which is called the Barnes integral representation. We shall make a general and introductory remark on the construction of such a representation.

Let $f(x)$ be a function defined by a power series

$$f(x) = \sum_{m=0}^{\infty} a_m x^m .$$

Suppose that there exists a function $g(s)$ satisfying the following conditions:

(a) $g(s)$ is meromorphic in $|s| < \infty$,

(b) $g(s)$ is holomorphic at $s = 0, 1, 2, \cdots$,

(c) $g(m) = a_m$.

Let us put

$$h(s) = g(s) \frac{\pi}{\sin(\pi s)} (-x)^s .$$

Condition (b) implies that the points $s = 0, 1, 2, \cdots$ are at most simple poles of $h(s)$ and condition (c) implies that

$$\operatorname*{Res}_{s=m} h(s) = a_m x^m \quad \text{for } m = 0, 1, 2, \cdots .$$

Therefore, if C' is a loop which surrounds the poles $s = 0, 1, 2, \cdots, N$ in the positive sense and only those poles, then we have

$$\frac{1}{2\pi i} \int_{C'} h(s) ds = \sum_{m=0}^{N} a_m x^m .$$

If C denotes the inverse loop of C', we have

$$-\frac{1}{2\pi i} \int_{C} h(s) ds = \sum_{m=0}^{N} a_m x^m .$$

After this preliminary remark, let us go back to the hypergeometric series

$$F(\alpha, \beta, \gamma, x) = \sum_{m=0}^{\infty} \frac{(\alpha, m)(\beta, m)}{(\gamma, m)(1, m)} x^m \quad (\gamma \neq 0, -1, -2, \cdots).$$

As it was previously mentioned, this series can be written in terms of the Γ-function as follows:

$$F(\alpha, \beta, \gamma, x) = \sum_{m=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+m)\Gamma(\beta+m)}{\Gamma(\gamma+m)\Gamma(1+m)} x^m$$

if neither α nor β nor γ is zero or a negative integer. In such a case we can naturally take as $g(s)$ the function

$$g(s) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)\Gamma(1+s)} .$$

This function $g(s)$ is meromorphic in $|s| < \infty$ and hence condition (a) is satisfied. Poles of $g(s)$ are at

$$s = -\alpha, -\alpha-1, \cdots$$

and

$$s = -\beta, -\beta-1, \cdots .$$

If $\alpha, \beta \neq 0, -1, -2, \cdots$, then none of these points is zero or a positive integer. This means that condition (b) is satisfied. Clearly

$$g(m) = \frac{(\alpha, m)(\beta, m)}{(\gamma, m)(1, m)} \quad \text{for } m = 0, 1, 2, \cdots .$$

This implies that condition (c) is satisfied.

Note that the function

$$h(s) = g(s)\frac{\pi}{\sin(\pi s)}(-x)^s = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)\Gamma(1+s)}\frac{\pi}{\sin(\pi s)}(-x)^s$$

has its poles just at

$$s = -\alpha,\ -\alpha-1,\ \cdots;\ -\beta,\ -\beta-1,\ \cdots;\ 0,\ 1,\ 2,\ \cdots .$$

Recall a well-known formula for the Γ-function

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} .$$

From this formula we can derive

$$\Gamma(-s)\Gamma(1+s) = \frac{\pi}{\sin(-\pi s)} = -\frac{\pi}{\sin(\pi s)},$$

and hence

$$-\frac{\pi}{\Gamma(1+s)\sin(\pi s)} = \Gamma(-s) .$$

Therefore we have

$$-h(s) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-x)^s .$$

Now we shall prove the following theorem:

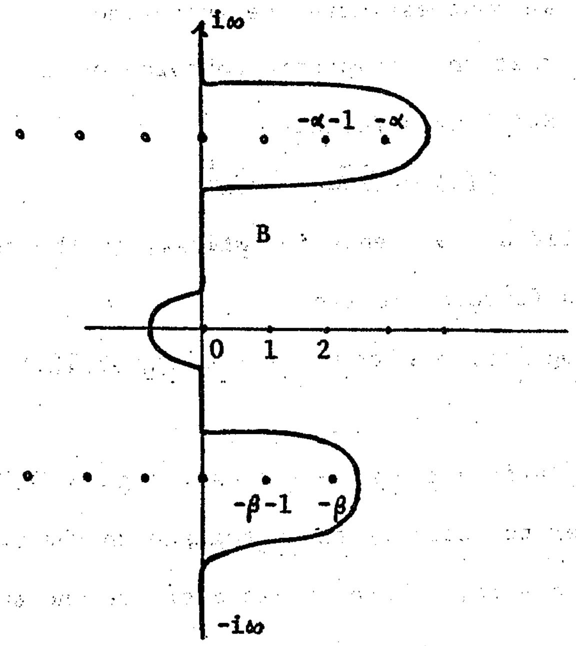
THEOREM 4.1: Suppose that $\alpha,\ \beta,\ \gamma \neq 0, -1, -2, \cdots$. Then

$$F(\alpha,\beta,\gamma,x) = -\frac{1}{2\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\int_B \frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)\Gamma(1+s)}\frac{\pi}{\sin(\pi s)}(-x)^s\,ds$$

$$= \frac{1}{2\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\int_B \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-x)^s\,ds$$

for $|x| < 1$, $|\arg(1-x)| < \pi$, where the path of integration B starts from $-i\infty$ and goes to $+i\infty$ in the complex s-plane, run-

ning along the imaginary axis, but curving around in such a way that the poles $s = 0, 1, 2, \cdots$ lie to the right of B, while the poles $s = -\alpha,\ -\alpha-1,\ \cdots,\ -\beta,\ -\beta-1,\ \cdots$ lie to the left of B.



Proof: The proof is divided into two steps.

I) If $\alpha,\ \beta,\ \gamma \neq 0, -1, -2, \cdots$, then the integral

$$\int_B \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-x)^s\,ds$$

is holomorphic in x for $|\arg(-x)| < \pi$.

Proof of I: Let δ be an arbitrarily small positive number It is sufficient to show that this integral converges uniformly in x for $|\arg(-x)| \leq \pi - \delta$. In other words, for any $\varepsilon > 0$, we shall find a $t_0 > 0$ such that t_0 is independent of x and that we have

$$\left|\int_{it_0}^{i\infty} " \, ds\right| < \varepsilon \quad \text{and} \quad \left|\int_{-i\infty}^{-it_0} " \, ds\right| < \varepsilon .$$

To do this, we must estimate the integrand.

Recall that the asymptotic behavior of the Γ-function $\Gamma(s)$ is given by Stirling's formula

$$\Gamma(s) \sim \sqrt{2\pi}\, e^{-s} s^{s-\frac{1}{2}}$$

which is valid as s tends to infinity in the sector $|\arg s| \leq \pi - \delta$. This formula implies

$$\log \Gamma(s) = s \log s - s - \tfrac{1}{2} \log s + O(1)$$

and hence

$$\log \Gamma(a+s) = s \log s - s + (a-\tfrac{1}{2}) \log s + O(1) .$$

In order to estimate the integrand on the upper part of B, let us put $s = it$, where t is positive and sufficiently large. Then

$$\log \Gamma(\alpha+s) = it \log(it) - it + (\alpha-\tfrac{1}{2}) \log(it) + O(1) .$$

Observing that

$$\log(it) = \log t + i\tfrac{1}{2}\pi,$$

we obtain

$$\log \Gamma(\alpha+s) = -\frac{1}{2}\pi t + (\operatorname{Re}\alpha - \frac{1}{2})\log t - \frac{1}{2}\pi \operatorname{Im}\alpha + PI + O(1)$$

$$= -\frac{1}{2}\pi t + (\operatorname{Re}\alpha - \frac{1}{2})\log t + PI + O(1) ,$$

where PI denotes a purely imaginary quantity. Similarly, we obtain

$$\log \Gamma(\beta+s) = -\frac{1}{2}\pi t + (\operatorname{Re}\beta - \frac{1}{2}) \log t + PI + O(1) ,$$

$$\log \Gamma(\gamma+s) = -\frac{1}{2}\pi t + (\operatorname{Re}\gamma - \frac{1}{2}) \log t + PI + O(1)$$

and

$$\log \Gamma(-s) = -it(\log t - i\tfrac{1}{2}\pi) + it - \tfrac{1}{2}\log(-it) + O(1)$$

$$= -\tfrac{1}{2}\pi t - \tfrac{1}{2}\log t + PI + O(1) .$$

On the other hand,

$$\log(-x)^s = s \log(-x) = it(\log|x| + i \arg(-x))$$

$$= -t \arg(-x) + PI .$$

From these formulas, we can derive

$$\log \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-x)^s = \operatorname{Re}(\alpha+\beta-\gamma-1)\log t - t(\pi + \arg(-x)) + PI + O(1)$$

and hence

$$\left|\frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-x)^s\right| = O(t^{\operatorname{Re}(\alpha+\beta-\gamma-1)} e^{-t(\pi+\arg(-x))})$$

as t tends to $+\infty$, where $s = it$. In the same way we obtain

$$\left|\frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-x)^s\right| = O(t^{\operatorname{Re}(\alpha+\beta-\gamma-1)} e^{-t(\pi-\arg(-x))})$$

as t tends to $+\infty$ in case when $s = -it$.

By virtue of the hypothesis: $|\arg(-x)| \leq \pi - \delta$, we get

$$\pi \pm \arg(-x) \geq \delta > 0 .$$

Therefore the quantity

$$t^{\operatorname{Re}(\alpha+\beta-\gamma-1)} e^{-t(\pi \pm \arg(-x))}$$

decays exponentially as t tends to $+\infty$. From this fact it is easily concluded that, for any $\varepsilon > 0$, there exists a $t_0 > 0$ such that

$$\left|\int_{it_0}^{i\infty} \text{"} \; ds\right| < \varepsilon \quad \text{and} \quad \left|\int_{-i\infty}^{-it} \text{"} \; ds\right| < \varepsilon \; .$$

This completes the proof of I).

II) If $\alpha, \beta, \gamma \neq 0, -1, -2, \cdots,$ then

$$F(\alpha, \beta, \gamma, x) = \frac{1}{2\pi i} \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \int_B \frac{\Gamma(\alpha+s)\,\Gamma(\beta+s)}{\Gamma(\gamma+s)\,\Gamma(1+s)} \frac{(-\pi)}{\sin(\pi s)} (-x)^s \, ds$$

for $|x| < 1$, $|\arg(-x)| < \pi$.

<u>Proof</u>: Let N be a positive integer which is sufficiently large. Consider the closed path $B_N + C_N$, where B_N is the arc of B in the strip between $\operatorname{Im} s = N+\frac{1}{2}$ and $\operatorname{Im} s = -(N+\frac{1}{2})$ and C_N is a semi-circle defined by $s = (N+\frac{1}{2})e^{i\theta}$, $|\theta| \leq \frac{1}{2}\pi$.

As it was explained at the beginning of this section, we have

$$\frac{1}{2\pi i} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \int_{B_N} \frac{\Gamma(\alpha+s)\,\Gamma(\beta+s)\,\Gamma(-s)}{\Gamma(\gamma+s)} (-x)^s \, ds$$

$$+ \frac{1}{2\pi i} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \int_{C_N} \frac{\Gamma(\alpha+s)\,\Gamma(\beta+s)\,\Gamma(-s)}{\Gamma(\gamma+s)} (-x)^s \, ds$$

$$= \sum_{m=0}^{N} \frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)} x^m \, .$$

By utilizing the result I), we can show that the first integral approaches the Barnes integral

$$\frac{1}{2\pi i} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \int_B \frac{\Gamma(\alpha+s)\,\Gamma(\beta+s)\,\Gamma(-s)}{\Gamma(\gamma+s)} (-x)^s \, ds$$

as N tends to $+\infty$. Therefore all we have to do is to show that

the second integral tends to zero as N tends to $+\infty$ if $|x| < 1$, $|\arg(-x)| < \pi$. This step is left to the readers as Exercise 1.

5. Hypergeometric differential equation. We shall derive a differential equation satisfied by the hypergeometric function. Let us consider the differential operator:

$$\vartheta = x\,d/dx .$$

It is easily seen that, for a power series $\sum_{m=0}^{\infty} a_m x^m$, we have

$$\vartheta\Big(\sum_{m=0}^{\infty} a_m x^m\Big) = \sum_{m=0}^{\infty} m\,a_m x^m = \sum_{m=1}^{\infty} m\,a_m x^m .$$

Therefore, we obtain

$$\begin{aligned}\vartheta(\vartheta+\gamma-1)F(\alpha,\beta,\gamma,x) &= \sum_{m=0}^{\infty} m(m+\gamma-1)\frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)}x^m \\ &= \sum_{m=1}^{\infty} \frac{(\alpha,m)(\beta,m)}{(\gamma,m-1)(1,m-1)}x^m \\ &= \sum_{m=0}^{\infty} \frac{(\alpha,m+1)(\beta,m+1)}{(\gamma,m)(1,m)}x^{m+1} \\ &= x\sum_{m=0}^{\infty} (\alpha+m)(\beta+m)\frac{(\alpha,m)(\beta,m)}{(\gamma,m)(1,m)}x^m \\ &= x(\vartheta+\alpha)(\vartheta+\beta)F(\alpha,\beta,\gamma,x).\end{aligned}$$

This means that

$$y = F(\alpha,\beta,\gamma,x)$$

is a solution of the differential equation

$$\vartheta(\vartheta+\gamma-1)y - x(\vartheta+\alpha)(\vartheta+\beta)y = 0 .$$

THEOREM 5.1: $y = F(\alpha,\beta,\gamma,x)$ satisfies the differential equation

$$x(1-x)d^2y/dx^2 + [\gamma - (\alpha+\beta+1)x]dy/dx - \alpha\beta\, y = 0 \tag{5.1}$$

or

$$d^2y/dx^2 + \left(\frac{\gamma}{x} + \frac{\alpha+\beta+1-\gamma}{x-1}\right)dy/dx + \frac{\alpha\beta}{x(x-1)}y = 0 . \tag{5.2}$$

The equation (5.1) is derived by rewriting the differential equation

$$\vartheta(\vartheta+\gamma-1)y - x(\vartheta+\alpha)(\vartheta+\beta)y = 0 .$$

Dividing (5.1) by $x(1-x)$, we obtain (5.2).

DEFINITION 5.1: The differential equation (5.1) or (5.2) is called the hypergeometric (or Gauss or Euler-Gauss) differential equation.

This equation has been subjects of many mathematical works. Furthermore, a considerable part of these works were done along the development of the classical analysis and summed up to the general theory of linear ordinary differential equations in the complex domain.

The hypergeometric differential equation is susceptible of several changes of variables, independent or dependent. First consider a change of variable

$$y = x^{\rho}(1-x)^{\sigma} z ,$$

where ρ and σ are complex constants. By this transformation, the hypergeometric differential equation (5.2) is taken to an equation

$$(5.3)\quad d^2z/dx^2 + \left[\frac{2\rho+\gamma}{x} + \frac{2\sigma+\alpha+\beta+1-\gamma}{x-1}\right] dz/dx$$

$$+\left[\frac{\rho(\rho+\gamma-1)}{x^2} + \frac{\sigma(\sigma+\alpha+\beta-\gamma)}{(x-1)^2} + \frac{\rho(\sigma+\alpha+\beta+1-\gamma)+\sigma(\rho+\gamma)+\alpha(}{x(x-1)}\right.$$

$$= 0 .$$

In fact, we have

$$dy/dx = x^{\rho}(1-x)^{\sigma}\left[dz/dx + \left(\frac{\rho}{x} + \frac{\sigma}{x-1}\right) z\right]$$

and

$$d^2y/dx^2 = x^{\rho}(1-x)^{\sigma}\left[d^2z/dx^2 + 2\left(\frac{\rho}{x} + \frac{\sigma}{x-1}\right) dz/dx + \left(\frac{\rho(\rho-1)}{x^2} + \frac{2\rho\sigma}{x(x-1)} + \frac{\sigma(\sigma-1)}{(x-1)^2}\right) z\right].$$

Inserting these into (5.2), we obtain (5.3). In particular, if we tal

$$\rho = 0 \quad \text{or} \quad 1-\gamma ,$$

$$\sigma = 0 \quad \text{or} \quad \gamma-\alpha-\beta ,$$

(5.3) becomes a hypergeometric differential equation again. In other words, the transformations

(5.4) $$y = (1-x)^{\gamma-\alpha-\beta} z ,$$

(5.5) $$y = x^{1-\gamma} z$$

and

(5.6) $$y = x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta} z$$

take (5.3) respectively to

$$(5.4')\quad d^2z/dx^2 + \left(\frac{\gamma}{x} + \frac{\gamma-\alpha-\beta+1}{x-1}\right) dz/dx + \frac{(\gamma-\alpha)(\gamma-\beta)}{x(x-1)} z = 0 ,$$

$$(5.5')\quad d^2z/dx^2 + \left(\frac{2-\gamma}{x} + \frac{\alpha+\beta+1-\gamma}{x-1}\right) dz/dx + \frac{(\alpha+1-\gamma)(\beta+1-\gamma)}{x(x-1)} z = 0$$

and

$$(5.6')\quad d^2z/dx^2 + \left(\frac{2-\gamma}{x} + \frac{\gamma-\alpha-\beta+1}{x-1}\right) dz/dx + \frac{(1-\alpha)(1-\beta)}{x(x-1)} z = 0 .$$

These three equations have solutions

(5.4") $$F(\gamma-\alpha, \gamma-\beta, \gamma, x) ,$$

(5.5") $$F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x)$$

and

(5.6") $$F(1-\alpha, 1-\beta, 2-\gamma, x)$$

respectively. Therefore the equation (5.2) has the following four solutions:

$$y = F(\alpha, \beta, \gamma, x) ,$$

$$y = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x) ,$$

$$y = x^{1-\gamma} F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x)$$

$$y = x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta} F(1-\alpha, 1-\beta, 2-\gamma, x).$$

Note that, as we showed in Theorem 3.2, the hypergeometric function $F(\alpha, \beta, \gamma, x)$ has three different expressions:

$$F(\alpha, \beta, \gamma, x) = (1-x)^{-\alpha} F(\alpha, \gamma-\beta, \gamma, x/(x-1))$$

$$= (1-x)^{-\beta} F(\gamma-\alpha, \beta, \gamma, x/(x-1))$$

$$= (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x) .$$

Let us consider next the change of the independent variable

$$\xi = 1 - x .$$

This transformation takes the points $x = 0, 1, \infty$ to the points $\xi = 1, 0, \infty$ respectively, and the transformed equation is

$$(5.7)\quad d^2y/d\xi^2 + \left(\frac{\alpha+\beta+1-\gamma}{\xi} + \frac{\gamma}{\xi-1}\right) dy/d\xi + \frac{\alpha\beta}{\xi(\xi-1)} y = 0$$

which has a solution $F(\alpha, \beta, \alpha+\beta+1-\gamma, \xi)$. Hence the equation (5.2) has a solution $F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-x)$. If we make the change of variable

$$y = \xi^{\rho}(1-\xi)^{\sigma} z ,$$

where $\rho = 0$ or $\gamma-\alpha-\beta$ and $\sigma = 0$ or $1-\gamma$, we obtain the following four solutions of (5.2):

$$F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-x)$$
$$x^{1-\gamma}F(\alpha+1-\gamma, \beta+1-\gamma, \alpha+\beta+1-\gamma, 1-x),$$
$$(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma+1-\alpha-\beta, 1-x),$$
$$x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}F(1-\alpha, 1-\beta, \gamma+1-\alpha-\beta, 1-x).$$

Finally consider the combination of changes of the independent variable and the dependent variable:

$$x = 1/\xi , \quad y = \xi^{\alpha} z .$$

Then the equation (5.2) is taken into an equation

$$\text{(5.8)} \qquad d^2z/d\xi^2 + \left(\frac{\alpha+1-\beta}{\xi} + \frac{\alpha+\beta+1-\gamma}{\xi-1}\right) dz/d\xi + \frac{\alpha(\alpha+1-\gamma)}{\xi(\xi-1)} z = 0 ,$$

which has a solution

$$z = F(\alpha, \alpha+1-\gamma, \alpha+1-\beta, \xi).$$

Moreover the equation (5.8) has three other solutions:

$$(1-\xi)^{\gamma-\alpha-\beta}F(1-\beta, \gamma-\beta, \alpha+1-\beta, \xi),$$
$$\xi^{\beta-\alpha}F(\beta+1-\gamma, \beta, \beta+1-\alpha, \xi),$$
$$\xi^{\beta-\alpha}(1-\xi)^{\gamma-\alpha-\beta}F(1-\alpha, \gamma-\alpha, \beta+1-\alpha, \xi).$$

Going back to the original variables, we obtain the following four solutions of (5.2):

$$x^{-\alpha}F(\alpha, \alpha+1-\gamma, \alpha+1-\beta, 1/x),$$
$$x^{\beta-\gamma}(1-x)^{\gamma-\alpha-\beta}F(1-\beta, \gamma-\beta, \alpha+1-\beta, 1/x),$$
$$x^{-\beta}F(\beta+1-\gamma, \beta, \beta+1-\alpha, 1/x),$$
$$x^{\alpha-\gamma}(1-x)^{\gamma-\alpha-\beta}F(1-\alpha, \gamma-\alpha, \beta+1-\alpha, 1/x).$$

Among linear fractional transformations:

$$\xi = \frac{ax+b}{cx+d},$$

only the following six transformations:

$$\text{(5.9)} \qquad \xi = x, \ \xi = 1-x, \ \xi = 1/x, \ \xi = 1/(1-x), \ \xi = (x-1)/x, \ \xi = x/(x-1)$$

map the set $\{0, 1, \infty\}$ onto itself. These six transformations form a group under the composition of functions. This group is generated by two transformations $\xi = 1-x$ and $\xi = 1/x$. To each of these six transformations of the independent variable combined with a suitable change of the dependent variable there corresponds a hypergeometric differential equation. Thus we obtain six hypergeometric differential equations, each of which has four expressions of solutions.

THEOREM 5.2: Assume that γ, $\gamma-\alpha-\beta$, $\alpha-\beta$ are not integers. Then the hypergeometric differential equation (5.1) has twenty four expressions of solutions which are written in the form

$$x^{\rho_i}(1-x)^{\sigma_i}F(\alpha_i, \beta_i, \gamma_i, x_i),$$

where ρ_i, σ_i, α_i, β_i and γ_i are linear in α, β and γ, and x_i is one of the six transformations (5.9).

The twenty four expressions of solutions are called Kummer's solutions. We supposed that $\gamma \neq 0, -1, -2, \cdots$ for $F(\alpha, \beta, \gamma, x)$.

The condition that γ, $\gamma-\alpha-\beta$, $\alpha-\beta$ are not integers guarantees that the third parameter in each of the twenty four expressions satisfies the corresponding condition.

It can be shown that any one of Kummer's solutions is essentially equal to one of the following six solutions:

$$\begin{cases} F(\alpha, \beta, \gamma, x) , \\ x^{1-\gamma}F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x) , \end{cases}$$

$$\begin{cases} F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-x) , \\ (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma+1-\alpha-\beta, 1-x) , \end{cases}$$

$$\begin{cases} x^{-\alpha}F(\alpha, \alpha+1-\gamma, \alpha+1-\beta, 1/x) , \\ x^{-\beta}F(\beta, \beta+1-\gamma, \beta+1-\alpha, 1/x). \end{cases}$$

Exercise 1. Find all solutions of Kummer.

6. Monodromy representation. The hypergeometric function satisfies the hypergeometric differential equation:

$$d^2y/dx^2 + \left(\frac{\gamma}{x} + \frac{\alpha+\beta+1-\gamma}{x-1}\right) dy/dx + \frac{\alpha\beta}{x(x-1)} y = 0 . \tag{6.1}$$

The coefficients are simple rational functions whose poles are just at $x = 0$ and 1. The point at infinity is a zero of the coefficients, but solutions of (6.1) are not necessarily holomorphic at $x = \infty$. Therefore we must study the equation (6.1) in the neighborhood of $x = \infty$. To do this, we make the change of independent variable

$$x = 1/\xi$$

and we shall investigate the transformed equation around $\xi = 0$. The equation (6.1) is taken to

$$d^2y/d\xi^2 + \left(\frac{1-\alpha-\beta}{\xi} + \frac{\alpha+\beta+1-\gamma}{\xi-1}\right) dy/d\xi - \frac{\alpha\beta}{\xi^2(\xi-1)} y = 0 .$$

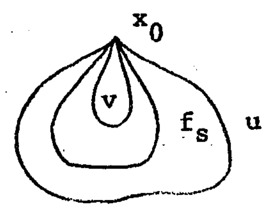
Clearly $\xi = 0$ is a pole of the coefficients of this equation. Thus we have found that the hypergeometric differential equation (6.1) has three singularities at $x = 0, 1, \infty$ in the extended complex plane $\mathbb{C} \cup \{\infty\}$ or the Riemann sphere S.

Let x_0 be any point $\neq 0, 1, \infty$. The general theory of linear differential equations guarantees that every solution of (6.1) is holomorphic at x_0 and that every solution of (6.1) can be analytically continued along any path which does not pass through the points $x = 0, 1$ and ∞. Therefore singularities of solutions of (6.1) are only at $x = 0, 1$ and ∞. Since the domain $D = S - \{0, 1, \infty\} = \mathbb{C} - \{0, 1\}$ is not simply connected, solutions of (6.1) are not necessarily single-valued. This means that $x = 0$, 1 and ∞ are in general branch-points of solutions. This fact is one of the reasons why the group theory is useful in the study of linear differential equations.

Let us keep x_0 fixed, and consider the set of all loops starting and terminating at x_0 in the domain D. Such a loop can be represented by a continuous map u from the unit interval $I = [0, 1]$ into D satisfying $u(0) = u(1) = x_0$. Let us put

$$\mathcal{L} = \{u;\ u : I \longrightarrow D,\ \text{continuous and } u(0) = u(1) = x_0\}.$$

We shall introduce an equivalence-relation $\sim$ in $\mathcal{L}$ by defining $u \sim v$ <u>if and only if there exists a mapping</u> F <u>from</u> $I \times I$ <u>into</u> D <u>such that</u>



(1) F <u>is continuous</u>,

(2) $F(t, 0) = u(t)$ <u>for</u> $t \in I$,

(3) $F(t, 1) = v(t)$ <u>for</u> $t \in I$,

<u>and</u>

(4) $F(0, s) = F(1, s) = x_0$ <u>for</u> $s \in I$.

The condition (4) implies that $f_s(t) = F(t, s)$ for any fixed $s \in I$ is a loop belonging to $\mathcal{L}$. On the other hand, the conditions (2) and (3) mean that $f_0 = u$ and $f_1 = v$ respectively. Therefore we can say that $u \sim v$ if and only if u can be deformed continuously to v in D.

We shall state five propositions concerning the relation $\sim$. Proofs of these propositions will be left to readers as <u>Exercise</u>.

PROPOSITION 6.1: The relation $\sim$ is an equivalence-relation.

Let us denote by $[u]$ the equivalence-class containing u and by $\pi_1(D, x_0)$ the set of all equivalence-classes: i.e.

$$\pi_1(D, x_0) = \mathcal{L}/\sim \quad \text{(the quotient set of } \mathcal{L} \text{ by } \sim).$$

We shall next define a multiplication in $\pi_1(D, x_0)$, under which $\pi_1(D, x_0)$ becomes a group. To do this, let u and v be two loops in $\mathcal{L}$. Define the product $u \cdot v$ in this order by

$$u\cdot v(t) = \begin{cases} u(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ v(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Clearly $u \cdot v$ is a loop in $\mathcal{L}$ which is composed of u and v in this order.

PROPOSITION 6.2: $u' \cdot v' \sim u \cdot v$ if $u \sim u'$ and $v \sim v'$.

By virtue of Proposition 6.2, we can define a multiplication in $\pi_1(D, x_0)$ by

$$[u][v] = [u \cdot v].$$

PROPOSITION 6.3: Let us denote by e the constant map

$$e : I \longrightarrow D,\quad e(t) = x_0 \quad \text{for } t \in I.$$

Then

$$e \cdot u \sim u \quad \text{and} \quad u \cdot e \sim u \quad \text{for every } u \text{ in } \mathcal{L}.$$

Hence

$$[e][u] = [u][e] = [u].$$

This proposition means that $[e]$ is a unit-element in

$\pi_1(D, x_0)$ with respect to the multiplication defined above.

PROPOSITION 6.4: Let us denote by u^{-1} the inverse loop of u defined by

$$u^{-1} : I \longrightarrow D, \quad u^{-1}(t) = u(1-t) \quad \text{for} \quad t \in I.$$

Then

$$u^{-1} \cdot u \sim e \quad \text{and} \quad u \cdot u^{-1} \sim e .$$

Hence

$$[u^{-1}][u] = [u][u^{-1}] = [e].$$

This proposition means that

$$[u]^{-1} = [u^{-1}].$$

PROPOSITION 6.5: $([u][v])[w] = [u]([v][w])$.

These five propositions prove that $\pi_1(D, x_0)$ is a group. This group is called <u>the fundamental group of</u> D <u>with the base point</u> x_0.

Let us go back to the hypergeometric differential equation (6.1). Let φ and ψ be linearly independent solutions of (6.1) which are defined in a neighborhood U of x_0. This means that $\{\varphi, \psi\}$ is a fundamental system of solutions in U. Assume that $u \in \mathcal{L}$ and continue φ and ψ analytically along u. Then φ and ψ will be changed into other solutions defined in U, since φ and ψ are not necessarily single-valued. Let $\bar{\varphi}$ and $\bar{\psi}$ be the analytic continuations of φ and ψ along u respectively. Since φ and ψ are linear independent, the solutions $\bar{\varphi}$ and $\bar{\psi}$ are expressible as linear forms in φ and ψ:

$$\bar{\varphi} = a\varphi + b\psi ,$$
$$\bar{\psi} = c\varphi + d\psi ,$$

which can be written as

$$\begin{bmatrix} \bar{\varphi} \\ \bar{\psi} \end{bmatrix} = A \begin{bmatrix} \varphi \\ \psi \end{bmatrix},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a two-by-two matrix, and $\begin{bmatrix} \varphi \\ \psi \end{bmatrix}$ and $\begin{bmatrix} \bar{\varphi} \\ \bar{\psi} \end{bmatrix}$ are column vectors.

From the hypothesis that φ and ψ are linearly independent, it follows that $\bar{\varphi}$ and $\bar{\psi}$ are also linearly independent. In fact, if there were a linear relation $c_1\bar{\varphi} + c_2\bar{\psi} = 0$, then this relation would hold for φ and ψ because of the permanence of functional relations for analytic functions. Hence A must be non-singular (or, $A \in GL(2, \mathbb{C})$). It is well known that $GL(2, \mathbb{C})$ (the set of all non-singular two-by-two matrices with complex entries) is a group under the usual multiplication of matrices. Moreover it follows from the general theory of analytic continuations that, if $u' \sim u$ and if $\bar{\varphi}'$ and $\bar{\psi}'$ denote respectively the analytic continuations of φ and ψ along u', then $\bar{\varphi}' = \bar{\varphi}$ and $\bar{\psi}' = \bar{\psi}$. This means that the matrix A is determined by $[u]$ rather than u. The matrix A does not depend on the choice of representatives of [u]. Thus we can define a map

$$\rho : \pi_1(D, x_0) \longrightarrow GL(2, \mathbb{C}), \quad \rho([u]) = A .$$

Let v be another loop in $\mathcal{L}$. If φ and ψ are continued analytically along v, then

$$\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \text{ goes to } B\begin{bmatrix} \varphi \\ \psi \end{bmatrix},$$

where $B = \rho([v]) \in GL(2, \mathbb{C})$. Consider the product $u \cdot v$ and denote by $\bar{\bar{\varphi}}$ and $\bar{\bar{\psi}}$ the analytic continuations of φ and ψ along $u \cdot v$ respectively. We have, then,

$$\begin{bmatrix} \bar{\bar{\varphi}} \\ \bar{\bar{\psi}} \end{bmatrix} = C\begin{bmatrix} \varphi \\ \psi \end{bmatrix}, \quad C = \rho([u \cdot v]) = \rho([u][v]).$$

Since $\bar{\bar{\varphi}}$ and $\bar{\bar{\psi}}$ are the analytic continuations of $\bar{\varphi}$ and $\bar{\psi}$ along v, and since

$$\begin{bmatrix} \bar{\varphi} \\ \bar{\psi} \end{bmatrix} = A\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \quad \text{and} \quad B = \rho([v]),$$

we have

$$\begin{bmatrix} \bar{\bar{\varphi}} \\ \bar{\bar{\psi}} \end{bmatrix} = A\begin{bmatrix} \bar{\varphi} \\ \bar{\psi} \end{bmatrix} = AB\begin{bmatrix} \varphi \\ \psi \end{bmatrix},$$

and hence

$$C = AB ,$$

or

$$\rho([u][v]) = \rho([u])\,\rho([v]) .$$

Therefore ρ is a homomorphism of the group $\pi_1(D, x_0)$ into the group $GL(2, \mathbb{C})$. In other words, ρ is a representation of $\pi_1(D, x_0)$ on $\mathbb{C}^2$. The representation ρ is called <u>the monodromy representation</u> of (6.1) with respect to a fundamental system $\{\varphi, \psi\}$. The image $\rho(\pi_1(D, x_0))$ is called the monodromy group with respect to $\{\varphi, \psi\}$.

The following theorem will be proved in Section 17 (Chapter III).

THEOREM 6.1: Suppose that none of α, β, $\gamma-\alpha$ and $\gamma-\beta$ is an integer. Then there exists a fundamental system of (6.1) such that the monodromy group with respect to this system is generated by

$$\begin{bmatrix} 1 & 0 \\ -(1-e^{-2\pi i\beta}) & e^{-2\pi i\gamma} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1-e^{-2\pi i\alpha} \\ 0 & e^{-2\pi i(\alpha+\beta-\gamma)} \end{bmatrix},$$

and there exists a fundamental system of (6.1) such that the monodromy group with respect to this system is generated by

$$\begin{bmatrix} e^{-2\pi i\gamma} & 0 \\ 1-e^{-2\pi i(\gamma-\alpha)} & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & e^{-2\pi i\alpha}(1-e^{-2\pi i(\gamma-\beta)}) \\ 0 & e^{-2\pi i(\alpha+\beta-\gamma)} \end{bmatrix}$$