Incomplete $\mathcal{A}$-Hypergeometric Systems

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1 Introduction

The incomplete gamma function and incomplete beta function are important special functions in statistics. In modern theory of special functions, we regard hypergeometric functions as pairings of twisted cycles and twisted cocycles [1, Chap 2]. However, domains of integrations of these incomplete functions are not cycles. Solutions of $\mathcal{A}$-hypergeometric systems ([3], [7, p.221]) have integral representations and domains of integrations are cycles. Some important integrals with parameters such as marginal likelihood functions in statistics, e.g., [4], look like $\mathcal{A}$-hypergeometric, but domains of integrations are not cycles. Being motivated with these functions, we want to generalize the theory of $\mathcal{A}$-hypergeometric systems to that for incomplete functions. We also hope that our theorems and formulas are useful for numerical and asymptotic evaluations of incomplete $\mathcal{A}$-hypergeometric functions.

This paper is a first step toward this direction. We make general discussions as well as detailed discussions on $\Delta_1 \times \Delta_1$-incomplete hypergeometric functions. A definition of incomplete generalized hypergeometric functions is given by Chardhry and Qadir [2]. A study of relations of these two definitions is a future problem.

2 General definition

Let D be the Weyl algebra in n variables. A multi-valued holomorphic function f defined on a Zariski open set in $\mathbf{C}^n$ is called a *holonomic function* if there exists a left ideal I of D such that (1) D/I is holonomic and (2) $I \bullet f = 0$.

We denote by $A = (a_{ij})$ a $d \times n$-matrix whose elements are integers. We suppose that the set of the column vectors of A spans $\mathbf{Z}^d$.

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Definition 1 We call the following system of differential equations $H_A(\beta, g)$ an *incomplete $\mathcal{A}$-hypergeometric system*:

$$\begin{aligned} \left(\sum_{j=1}^{n} a_{ij}x_j\partial_j - \beta_i\right) \bullet f &= g_i, \qquad (i = 1, \ldots, d) \\ \left(\prod_{i=1}^{n} \partial_i^{u_i} - \prod_{j=1}^{n} \partial_j^{v_j}\right) \bullet f &= 0 \end{aligned}$$

with $u, v \in \mathbf{N}_0^n$ running over all u, v such that $Au = Av$.

Here, $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$, and $\beta = (\beta_1, \ldots, \beta_d) \in \mathbf{C}^d$ are parameters and $g = (g_1, \ldots, g_d)$ where g_i are given holonomic functions which may depend on parameters β. We call solutions of the incomplete $\mathcal{A}$-hypergeometric system *incomplete $\mathcal{A}$-hypergeometric functions*.

Although we have introduced the incomplete $\mathcal{A}$-hypergeometric system for arbitrary holonomic functions g_i, g_i are often also solutions of smaller incomplete $\mathcal{A}$-hypergeometric system or well-known special functions in interesting cases.

Example 1 The *incomplete beta function* is defined as

$$B(\alpha, \beta; y) = \int_0^y s^{\alpha-1}(1-s)^{\beta-1}ds$$

Replace $s = yt$. Then, we have $B(\alpha, \beta; y) = y^\alpha \int_0^1 t^{\alpha-1}(1-yt)^{\beta-1}dt$. Put $B(\alpha, \beta; y) = y^\alpha \tilde{B}(\beta-1, \alpha-1; 1, -y)$ where $\tilde{B}(\beta-1, \alpha-1; x_1, x_2) = \int_0^1 t^{\alpha-1}(x_1 + x_2t)^{\beta-1}dt$. The function $\tilde{B}$ is a solution of an incomplete $\mathcal{A}$-hypergeometric system $H_A(\beta, g)$ for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\beta = (\beta-1, \alpha-1)$, $g_1 = 0, g_2 = t^\alpha(x_1+x_2t)^{\beta-1}$. We will revisit this example in Example 3.

Example 2 The *incomplete elliptic integral of the first kind* is defined as

$$F(z; k) = \int_0^z \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Replacing x^2 by z^2t, we obtain

$$F(z; k) = \frac{1}{2}z \int_0^1 t^{-\frac{1}{2}}(1-z^2t)^{-\frac{1}{2}}(1-k^2z^2t)^{-\frac{1}{2}}dt.$$

Hence, the incomplete elliptic integral of the first kind can be regarded as a solution restricted to a subvariety of incomplete $\mathcal{A}$-hypergeometric system $H_A(\beta, g)$ for $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$, $\beta = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. See the Section 4 on the incomplete $\Delta_1 \times \Delta_1$-hypergeometric system on g_i. We will revisit this example in Example 4. **Add a picture of $\Delta_1 \times \Delta_1$ in $\mathbf{R}^3$.**

Theorem 1 *Solutions of the incomplete $\mathcal{A}$-hypergeometric system are holonomic functions.*

Proof. Since the $\mathcal{A}$-hypergeometric left ideal $H_A(\beta) \subset D$ is holonomic ideal, the function f satisfies the ordinary differential equation of the form

$$\left(\sum_{k=0}^{r_i} a_k(\beta, x) \partial_i^k \right) \bullet f = \sum_{i=1}^{d} \ell_i \bullet g_i$$

where $\ell_i \in D$. Since $g := \sum_{i=1}^{d} \ell_i \bullet g_i$ is also a holonomic function, there exists an ordinary differential operator such that $\left(\sum_{k=0}^{r_i'} b_k(\beta, x) \partial_i^k \right) \bullet g = 0$. Hence, f is annihilated by an ordinary differential operator with parameters of the order $r_i r_i'$ with respect to x_i and then f is annihilated by a zero dimensional ideal J in $\mathbf{C}(x_1, \ldots, x_n)\langle \partial_i, \ldots, \partial_n \rangle$ [7, p.33]. Since $D/(J \cap D)$ is a left holonomic D-module (see, e.g., [9, Th 2.4], [7, p.34]), we have the conclusion. We note that this construction can be done algorithmically [5].

3 Algorithms deriving contiguity relations

In this section, we will give algorithms to obtain contiguity relations for incomplete $\mathcal{A}$-hypergeometric functions under the condition that for all i $(1 \leq i \leq d)$ there exists a constant $c_k \in \mathbf{C} \setminus \{0\}$ such that

$$\partial_k \bullet g_i(\beta) = c_k^{-1} g_i(\beta - a_k). \tag{1}$$

This condition holds in many interesting cases. We mean by a contiguity relation, a relation among two incomplete $\mathcal{A}$-hypergeometric functions $\Phi(\beta; x)$ and $\Phi(\beta'; x)$ where $\beta - \beta' \in \mathbf{Z}^d$ and $\Phi(\beta; x)$ is a solution of $H_A(\beta, g)$. Under the condition (1), we note that

$$c_k^{-1} \partial_k \bullet \Phi(\beta; x) =: \Phi(\beta - a_k; x)$$

is a solution of $H_A(\beta - a_k, g)$. In other words, the operator ∂_k gives a contiguity relation for the parameter shift $-a_k$. This contiguity relation follows from

$$\begin{aligned} & \partial_k \left(\sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i \right) \bullet f \\ = & \left(\sum_{j=1}^{n} a_{ij} x_j \partial_j - (\beta_i + a_{ik}) \right) \partial_k \bullet f = \partial_k \bullet g_i(\beta) \end{aligned}$$

and the condition (1).

If we can find the inverse operator for ∂_k, then it means that we find "generators" of the all contiguity relations because the matrix A is the full rank.

In [6] and [7], they give algorithms to find the inverse operator for ∂_k in case of $\mathcal{A}$-hypergeometric system. We utilize their algorithms to find the inverse operator. Let us recall Definition 1. Let h_i $(1 \leq i \leq d)$ be $\sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i$ respectively and h_i $(d+1 \leq i \leq s)$ be generators of the toric ideal I_A of the form $\prod_{i=1}^{n} \partial_i^{u_i} - \prod_{j=1}^{n} \partial_j^{v_j}$. It follows from [6] and [7] that there exist $r, r_i \in D(\beta)$ such that

$$r\partial_k + \sum_{i=1}^{s} r_i h_i = 1.$$

Applying the operator of the left hand side to $\Phi(\beta; x)$, we obtain the contiguity relation

$$c_k r \bullet \Phi(\beta - a_k; x) + \sum_{i=1}^{d} r_i \bullet g_i = \Phi(\beta; x)$$

which contains the functions g_i.

Our second algorithm gives contiguity relations which do not contain the functions g_i. Suppose that we are given operators $\tilde{h}_i$ such that $\tilde{h}_i \bullet g_i = 0$ and operators ∂_k, $\tilde{h}_i h_i$ $(1 \leq i \leq d)$, and h_i $(d+1 \leq i \leq s)$ generate a trivial ideal in $D(\beta)$. Then, we can construct operators $r, r_i \in D(\beta)$ such that

$$r\partial_k + \sum_{i=1}^{d} r_i \tilde{h}_i h_i + \sum_{i=d+1}^{s} r_i h_i = 1$$

by the Gröbner basis method. The operator $c_k r$ is the inverse of ∂_k.

We will apply these algorithms to obtain a complete list of contiguity relations of $\Delta_1 \times \Delta_1$-hypergeometric functions in Section 4.

4 Incomplete $\Delta_1 \times \Delta_1$-hypergeometric functions

We assume $0 < a < b$ for simplicity. We consider the integral

$$\int_a^b t^{\gamma} (x_{11} + x_{21} t)^{\alpha_1} (x_{12} + x_{22} t)^{\alpha_2} dt \tag{2}$$

for $x_{ij} > 0$ and $\operatorname{Re}\gamma, \operatorname{Re}\alpha_i > -1$. The integral and its analytic continuations satisfy the following incomplete $\mathcal{A}$-hypergeometric system.

$$\begin{cases} \left(\theta_{11}\theta_{22} - \frac{x_{11}x_{22}}{x_{21}x_{12}}\theta_{21}\theta_{12}\right) \bullet f & = 0 \\ (\theta_{11} + \theta_{21} - \alpha_1) \bullet f & = 0 \\ (\theta_{12} + \theta_{22} - \alpha_2) \bullet f & = 0 \\ (\theta_{21} + \theta_{22} + \gamma + 1) \bullet f & = [g(t,x)]_{t=a}^{t=b} \end{cases} \tag{3}$$

Here, $g(t, x) = t^{\gamma+1}(x_{11} + x_{21}t)^{\alpha_1}(x_{12} + x_{22}t)^{\alpha_2}$. This fact can be shown by exchanging the integral and differentiations (see, e.g., [7, p.221]). When $[g(t, x)]_{t=a}^{t=b} = 0$, our system is essentially the Gauss hypergeometric equation (see, e.g., [7, Ch 1]).

Since the matrix $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ associated to the system can be regarded as $\Delta_1 \times \Delta_1$, we call it incomplete $\Delta_1 \times \Delta_1$-hypergeometric system and its solutions are incomplete $\Delta_1 \times \Delta_1$-hypergeometric functions.

In order to make a rigorous discussion, we need to specify branches of multi-valued functions appearing in our discussion. In the sequel, z^α denotes the unique analytic continuation of the function z^α defined on $z > 0$ to the upper and the lower half plane as long as we make no annotation.

Remark 1 Under this definition, we have $(zw)^\alpha = e^{2\pi\alpha} z^\alpha w^\alpha$ for $\operatorname{Im} z < 0, \operatorname{Im} w < 0$ and $\operatorname{Im} zw > 0$, $(zw)^\alpha = e^{-2\pi\alpha} z^\alpha w^\alpha$ for $\operatorname{Im} z > 0, \operatorname{Im} w > 0$ and $\operatorname{Im} zw < 0$, and $(zw)^\alpha = z^\alpha w^\alpha$ for other cases.

We take real numbers $x^*_{ij} > 0$ such that

$$0 < \frac{x^*_{21}}{x^*_{22}} < \frac{x^*_{11}}{x^*_{21}} < a < b < \frac{x^*_{21}}{x^*_{11}} < \frac{x^*_{22}}{x^*_{21}}$$

We consider the simply connected domain define by

$$\begin{aligned} &(-1)^{d_1} \operatorname{Im} x_{11} > 0, (-1)^{d_2} \operatorname{Im} x_{21} > 0,\ (-1)^{d_3} \operatorname{Im} x_{12} > 0,\ (-1)^{d_4} \operatorname{Im} x_{22} > 0, \\ &(-1)^{d_5} \operatorname{Im} x_{21}/x_{11} > 0,\ (-1)^{d_6} \operatorname{Im} x_{22}/x_{12} > 0,\ (-1)^{d_7} \operatorname{Im} \frac{x_{21}x_{12}}{x_{11}x_{22}} > 0 \end{aligned}$$

Here, d_i takes the values 0 or 1. Since we assume $a, b \in \mathbf{R}$ and the singular locus of the homogeneous $\mathcal{A}$-hypergeometric system is $x_{11}x_{21}x_{12}x_{22}(x_{11}x_{22} - x_{21}x_{12}) = 0$, the solutions of our system are holomorphic on each of these domains. We denote by D_d where $d = (d_1, \dots, d_7) \in \{0, 1\}^7$ the simply connected domain standing for d.

We define the four domains as follows

$$\begin{aligned} D^{11}_{12} &= \{x_{ij} \mid |x_{21}b/x_{11}| < 1, |x_{21}a/x_{11}| < 1, |x_{22}b/x_{12}| < 1, |x_{22}a/x_{12}| < 1\} \\ D^{11}_{22} &= \{x_{ij} \mid |x_{21}b/x_{11}| < 1, |x_{21}a/x_{11}| < 1, |x_{12}/(x_{22}b)| < 1, |x_{12}/(x_{22}a)| < 1\} \\ D^{21}_{12} &= \{x_{ij} \mid |x_{11}/(x_{21}b)| < 1, |x_{11}/(x_{21}a)| < 1, |x_{22}b/x_{12}| < 1, |x_{22}a/x_{12}| < 1\} \\ D^{21}_{22} &= \{x_{ij} \mid |x_{11}/(x_{21}b)| < 1, |x_{11}/(x_{21}a)| < 1, |x_{12}/(x_{22}b)| < 1, |x_{12}/(x_{22}a)| < 1\} \end{aligned}$$

The point (x^*_{ij}) belongs to the last domain D^{21}_{22}. It is easy to see that each of these domains and D_d has an open intersection since $a, b \in \mathbf{R}$.

Remark 2 We note that

$$(x_{1i} + x_{2i}t)^{\alpha_i} = x_{1i}^{\alpha_i} \left(1 + \frac{x_{2i}t}{x_{1i}}\right)^{\alpha_i} = (x_{2i}t)^{\alpha_i} \left(\frac{x_{1i}}{x_{2i}t} + 1\right)^{\alpha_i}$$

for $x_{ij} > 0$ and $t > 0$. This relation will be used to specify branches of $[g(t,x)]_{t=a}^{t=b}$. For example, when $(x_{ij}) \in D^{21}_{22}$, we regard $[g(t,x)]_{t=a}^{t=b}$ as

$$\left[t^{\gamma+1} (x_{21}t)^{\alpha_1} \left(\frac{x_{11}}{x_{21}t} + 1\right)^{\alpha_1} (x_{22}t)^{\alpha_2} \left(\frac{x_{12}}{x_{22}t} + 1\right)^{\alpha_2} \right]_{t=a}^{t=b}$$

of which series expansion converges on D_{22}^{21}. Since the domain D_{22}^{21} has an open intersection with D_d, the function $[g(t,x)]_{t=a}^{t=b}$ has a unique analytic continuation to D_d.

4.1 Homogeneous system

As we have proved in Theorem 1, solutions of incomplete $\mathcal{A}$-hypergeometric systems are holonomic functions. The advantage of this point of view is that we can apply some algorithms for holonomic systems to study solutions of our system. For example, we can apply the algorithm given in the Chapter 2 of [7] to find candidates of series solutions. Holonomic systems which annihilate these functions can be obtained in an algorithmic way. However, outputs by the algorithm are sometimes tedious.

In the case of $\Delta_1 \times \Delta_1$-hypergeometric system, solutions satisfy the following relatively simple holonomic system.

$$\begin{cases} \left(\theta_{11}\theta_{22} - \frac{x_{11}x_{22}}{x_{21}x_{12}}\theta_{21}\theta_{12}\right) \bullet f & = 0 \\ (\theta_{11} + \theta_{21} - \alpha_1) \bullet f & = 0 \\ (\theta_{12} + \theta_{22} - \alpha_2) \bullet f & = 0 \\ (\partial_{22} - a\partial_{21})(\partial_{12} - b\partial_{11})(\theta_{21} + \theta_{22} + \gamma + 1) \bullet f & = 0 \end{cases} \tag{4}$$

4.2 Contiguity relations

We will derive contiguity relations of incomplete $\Delta_1 \times \Delta_1$-hypergeometric functions by applying our two algorithms. In this section, we put $\beta = -\gamma - 1$ to make formulas of contiguity relations of the incomplete $\Delta_1 \times \Delta_1$-hypergeometric function simpler forms. We put

$$\Phi(\alpha_1, \alpha_2, \beta; x) = \int_a^b t^{-\beta-1}(x_{11} + x_{21}t)^{\alpha_1}(x_{12} + x_{22}t)^{\alpha_2}dt.$$

Theorem 2 *The incomplete $\Delta_1 \times \Delta_1$-hypergeometric function $\Phi(\alpha_1, \alpha_2, \beta; x)$ satisfies the following contiguity relations.*

- *Shifts with respect to $a_1 = (1, 0, 0)$*

$$S(\alpha_1, \alpha_2, \beta; -a_1)\Phi(\alpha_1, \alpha_2, \beta) = \alpha_1\Phi(\alpha_1 - 1, \alpha_2, \beta)$$
$$\hat{S}(\alpha_1 - 1, \alpha_2, \beta; +a_1)\Phi(\alpha_1 - 1, \alpha_2, \beta) = (\alpha_1 + \alpha_2 - \beta)\Phi(\alpha_1, \alpha_2, \beta) - [g(t,x)]_{t=a}^{t=b}$$
$$S(\alpha_1 - 1, \alpha_2, \beta; +a_1)\Phi(\alpha_1 - 1, \alpha_2, \beta) = \alpha_2(\alpha_1 + \alpha_2 - \beta)\Phi(\alpha_1, \alpha_2, \beta)$$

where

$S(\alpha_1, \alpha_2, \beta; -a_1) = \partial_{11}$,
$\hat{S}(\alpha_1 - 1, \alpha_2, \beta; +a_1) = (x_{21}x_{12} - x_{11}x_{22})\partial_{22} + (\alpha_1 + \alpha_2)x_{11}$,
$S(\alpha_1 - 1, \alpha_2, \beta; +a_1) = x_{12}x_{21}\{(a + b)x_{21}\partial_{21}\partial_{22} + x_{22}\partial_{22}^2 + (1 - \beta)\partial_{22} - ab(x_{21}\partial_{11}\partial_{22} + \partial_{12}\partial_{22} - \beta\partial_{12}) + x_{11}\partial_{21}\partial_{22} + x_{12}\partial_{22}^2\} + (\alpha_1 + \alpha_2 - \beta)(\alpha_1\alpha_2 x_{11} + x_{21}x_{12}\partial_{22})$

- *Shifts with respect to* $a_2 = (1,0,1)$

$$S(\alpha_1,\alpha_2,\beta;-a_2)\Phi(\alpha_1,\alpha_2,\beta) = \alpha_1\Phi(\alpha_1-1,\alpha_2,\beta-1)$$
$$\hat{S}(\alpha_1-1,\alpha_2,\beta-1;+a_2)\Phi(\alpha_1-1,\alpha_2,\beta-1) = \beta\Phi(\alpha_1,\alpha_2,\beta) + [g(t,x)]_{t=a}^{t=b}$$
$$S(\alpha_1-1,\alpha_2,\beta-1;+a_2)\Phi(\alpha_1-1,\alpha_2,\beta-1) = ab\alpha_2\beta\Phi(\alpha_1,\alpha_2,\beta)$$

where

$S(\alpha_1,\alpha_2,\beta;-a_2) = \partial_{21}$,
$\hat{S}(\alpha_1-1,\alpha_2,\beta-1;+a_2) = x_{11}x_{22}\partial_{12} + x_{21}x_{22}\partial_{22} + \alpha_1 x_{21}$,
$S(\alpha_1-1,\alpha_2,\beta-1;+a_2) = x_{11}(x_{11}x_{22}\partial_{21}+x_{21}x_{12}\partial_{22})\partial_{22}+(a+b)x_{11}x_{22}(x_{21}\partial_{21}+x_{22}\partial_{22}) + abx_{22}(x_{21}^2\partial_{21} - x_{11}\partial_{12}) + ab(\alpha_2+\beta-1)x_{11}x_{22}\partial_{12} + (\alpha_1-\beta+1)x_{11}x_{12}\partial_{22} - \alpha_2(x_{11}^2 + (a+b)x_{11}x_{21} + abx_{21}^2)\partial_{21} + (a+b)\alpha_2(\beta-1)x_{11} + ab\alpha_2(\alpha_1+\beta+1)x_{21}$

- *Contiguity relations with respect to* $a_3 = (0,1,0)$ *are obtained from those with respect to* a_1 *by the permutations* $\alpha_1 \leftrightarrow \alpha_2$, $x_{i1} \leftrightarrow x_{i2}$, $\partial_{i1} \leftrightarrow \partial_{i2}$.

- *Contiguity relations with respect to* $a_4 = (0,1,1)$ *are obtained from those with respect to* a_2 *by the same permutations as above.*

Proof. Since the function $[g(t,x)]_{t=a}^{t=b}$ satisfies the condition (1), we can apply the first algorithm for $A = (a_1,a_2,a_3,a_4) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ to obtain the contiguity relations containing the function $[g(t,x)]_{t=a}^{t=b}$. Contiguity relations which do not contain the function $[g(t,x)]_{t=a}^{t=b}$ is obtained from (4). The generation condition of the trivial ideal generated by 1 is checked for (4) by a computer and then we can apply the second algorithm in section 3.

Theorem 2 gives contiguity relations for $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, but it does not give these for $e_3 = (0,0,1)$. The set of vectors $\{e_1,e_2,e_3\}$ is the standard basis of $\mathbf{Z}^3$. The contiguity relations for e_3 can be obtained from Theorem 2 as follows.

Corollary 1 *The incomplete* $\Delta_1 \times \Delta_1$*-hypergeometric function* $\Phi(\alpha_1,\alpha_2,\beta;x)$ *satisfies the following contiguity relations.*

- *Shifts with respect to* $e_3 = (0,0,1)$

$$S(\alpha_1,\alpha_2,\beta+1;-e_3)\Phi(\alpha_1,\alpha_2,\beta+1) = \alpha_1\alpha_2(\alpha_1+\alpha_2-\beta)\Phi(\alpha_1,\alpha_2,\beta)$$
$$S(\alpha_1,\alpha_2,\beta-1;+e_3)\Phi(\alpha_1,\alpha_2,\beta-1) = ab\alpha_1\alpha_2\beta\Phi(\alpha_1,\alpha_2,\beta)$$
$$\hat{S}(\alpha_1,\alpha_2,\beta+1;-e_3)\Phi(\alpha_1,\alpha_2,\beta+1) = \alpha_1(\alpha_1+\alpha_2-\beta)\Phi(\alpha_1,\alpha_2,\beta) - \alpha_1[g(t,x)]_{t=a}^{t=b}$$
$$\hat{S}(\alpha_1,\alpha_2,\beta-1;+e_3)\Phi(\alpha_1,\alpha_2,\beta-1) = \alpha_1\beta\Phi(\alpha_1,\alpha_2,\beta) + \alpha_1[g(t,x)]_{t=a}^{t=b}$$

where

$$\begin{aligned}
S(\alpha_1, \alpha_2, \beta + 1; -e_3) &= S(\alpha_1 - 1, \alpha_2, \beta; +a_1)S(\alpha_1, \alpha_2, \beta + 1; -a_2),\\
S(\alpha_1, \alpha_2, \beta - 1; +e_3) &= S(\alpha_1 - 1, \alpha_2, \beta - 1; +a_2)S(\alpha_1, \alpha_2, \beta - 1; -a_1),\\
\hat{S}(\alpha_1, \alpha_2, \beta + 1; -e_3) &= \hat{S}(\alpha_1 - 1, \alpha_2, \beta; +a_1)S(\alpha_1, \alpha_2, \beta + 1; -a_2),\\
\hat{S}(\alpha_1, \alpha_2, \beta - 1; +e_3) &= \hat{S}(\alpha_1 - 1, \alpha_2, \beta - 1; +a_2)S(\alpha_1, \alpha_2, \beta - 1; -a_1).
\end{aligned}$$

Example 3 Let $a = 0, b = 1$. We consider the following degenerated incomplete $\Delta_1 \times \Delta_1$-hypergeometric function.

$$\Psi(\alpha_1, \beta; x) = \int_0^1 t^{-\beta-1}(x_{11} + x_{21}t)^{\alpha_1} dt$$

Then the last contiguity relation of Corollary 1 for this function is

$$x_{21}\partial_{11}\Psi(\alpha_1, \beta - 1; x) = \beta\Psi(\alpha_1, \beta; x) + (x_{11} + x_{21})^{\alpha_1}.$$

Multiplying both sides by x_{11} and by using the relation of the incomplete $\mathcal{A}$-hypergeometric system:

$$x_{11}\partial_{11}\Psi(\alpha_1, \beta - 1; x) = (\alpha_1 - \beta + 1)\Psi(\alpha_1, \beta - 1; x) - (x_{11} + x_{21})^{\alpha_1},$$

we have

$$(\alpha_1 - \beta + 1)x_{21}\Psi(\alpha_1, \beta - 1; x) = \beta x_{11}\Psi(\alpha_1, \beta; x) + (x_{11} + x_{21})^{\alpha_1+1}.$$

Put $x_{11} = 1, x_{21} = -y$ and replace β by $-\alpha$, α_1 by $\beta - 1$, we have

$$(\alpha + \beta)(-y)\Psi(\beta - 1, -\alpha - 1; y) = -\alpha\Psi(\beta - 1, -\alpha; y) + (1 - y)^{\beta}.$$

Multiplying both sides by $-y^{\alpha}$, we obtain

$$(\alpha + \beta)B(\alpha + 1, \beta; y) = \alpha B(\alpha, \beta; y) - y^{\alpha}(1 - y)^{\beta}.$$

This is a well-known relation of the incomplete beta function.

4.3 Series solutions

We define the following 4 series.

$$f_{12}^{11} = x_{11}^{\alpha_1} x_{12}^{\alpha_2} \sum_{k,m\geq 0} \frac{(-1)^{k+m}}{\gamma+k+m+1} \cdot \frac{(-\alpha_1)_k(-\alpha_2)_m}{(1)_k(1)_m} \cdot (b^{\gamma+k+m+1} - a^{\gamma+k+m+1}) \left(\frac{x_{21}}{x_{11}}\right)^k \left(\frac{x_{22}}{x_{12}}\right)^m$$

$$f_{22}^{11} = x_{11}^{\alpha_1} x_{22}^{\alpha_2} \sum_{k,m\geq 0} \frac{(-1)^{k+m}}{\gamma+\alpha_2+k-m+1} \cdot \frac{(-\alpha_1)_k(-\alpha_2)_m}{(1)_k(1)_m} \cdot (b^{\gamma+\alpha_2+k-m+1} - a^{\gamma+\alpha_2+k-m+1}) \left(\frac{x_{21}}{x_{11}}\right)^k \left(\frac{x_{12}}{x_{22}}\right)^m$$

$$f_{12}^{21} = x_{21}^{\alpha_1} x_{12}^{\alpha_2} \sum_{k,m\geq 0} \frac{(-1)^{k+m}}{\gamma+\alpha_1-k+m+1} \cdot \frac{(-\alpha_1)_k(-\alpha_2)_m}{(1)_k(1)_m} \cdot (b^{\gamma+\alpha_1-k+m+1} - a^{\gamma+\alpha_1-k+m+1}) \left(\frac{x_{11}}{x_{21}}\right)^k \left(\frac{x_{22}}{x_{12}}\right)^m$$

$$f_{22}^{21} = x_{21}^{\alpha_1} x_{22}^{\alpha_2} \sum_{k,m\geq 0} \frac{(-1)^{k+m}}{\gamma+\alpha_1+\alpha_2-k-m+1} \cdot \frac{(-\alpha_1)_k(-\alpha_2)_m}{(1)_k(1)_m} \cdot (b^{\gamma+\alpha_1+\alpha_2-k-m+1} - a^{\gamma+\alpha_1+\alpha_2-k-m+1}) \left(\frac{x_{11}}{x_{21}}\right)^k \left(\frac{x_{12}}{x_{22}}\right)^m$$

Theorem 3 *We assume that $\gamma \notin \mathbf{Z}$.*

1. *The series $f_{ij}^{k\ell}$ converges on the domain $D_{ij}^{k\ell}$ and has a unique analytic continuation to D_d. Here, $D_{ij}^{k\ell}$ is defined in the beginning of this section.*
2. *The function $f_{ij}^{k\ell}$ defined on D_d as above satisfies the incomplete $\Delta_1 \times \Delta_1$-hypergeometric system for the branch of $[g(t,x)]_{t=a}^{t=b}$ given in the Remark 2.*
3. *f_{12}^{11} can be expressed in terms of the Appell function F_1 as*

$$\begin{aligned} f_{12}^{11} &= x_{11}^{\alpha_1} x_{12}^{\alpha_2} \left(\frac{b^{\gamma+1}}{\gamma+1} F_1\left(\gamma+1, -\alpha_1, -\alpha_2, \gamma+2, \frac{-x_{21}b}{x_{11}}, \frac{-x_{22}b}{x_{12}}\right) \right. \\ & \left. - \frac{a^{\gamma+1}}{\gamma+1} F_1\left(\gamma+1, -\alpha_1, -\alpha_2, \gamma+2, \frac{-x_{21}a}{x_{11}}, \frac{-x_{22}a}{x_{12}}\right)\right) \end{aligned}$$

Proof. (Sketch) The item 1 is proved by utilizing majorant series. There exists a constant C such that

$$\begin{aligned} & C\left(\sum_{k=0}^{\infty} \frac{|(-\alpha_1)_k|}{k!} \left|\frac{bx_{21}}{x_{11}}\right|^k\right) \left(\sum_{m=0}^{\infty} \frac{|(-\alpha_2)_m|}{m!} \left|\frac{bx_{22}}{x_{12}}\right|^m\right) \\ + \quad & C\left(\sum_{k=0}^{\infty} \frac{|(-\alpha_1)_k|}{k!} \left|\frac{ax_{21}}{x_{11}}\right|^k\right) \left(\sum_{m=0}^{\infty} \frac{|(-\alpha_2)_m|}{m!} \left|\frac{ax_{22}}{x_{12}}\right|^m\right) \end{aligned}$$

is a majorant series of f_{12}^{11}. Other cases can be shown analogously.

The item 2 is proved by applying the algorithm to find series solutions for (4) given in the Chapter 2 of [7]. The Gröbner cone consists of 8 maximal dimensional cones. After constructing series solutions of the homogeneous system (4), we check if they satisfy the inhomogeneous system (3) and we find these four solutions.

The item 3 can be proved by utilizing the relation $\frac{1}{\gamma+k+m+1} = \frac{1}{\gamma+1}\frac{(\gamma+1)_{k+m}}{(\gamma+2)_{k+m}}$.

Example 4 As we have seen in Example 2, the incomplete elliptic integral of the first kind can be regarded as incomplete $\Delta_1 \times \Delta_1$-hypergeometric function. Let us apply Theorem 3 to obtain an expression in term of the Appell function F_1.

Put $x_{11} = 1, x_{21} = -z^2, x_{12} = 1, x_{22} = -k^2 z^2$ and $\alpha_1 = \alpha_2 = \gamma = -\frac{1}{2}$, $a = 0, b = 1$. Then we have

$$\begin{aligned} F(z;k) &= \frac{1}{2} z \cdot \frac{1}{-\frac{1}{2}+1} F_1\left(-\frac{1}{2}+1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}+2; \frac{z^2}{1}, \frac{k^2 z^2}{1}\right) \\ &= z F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2, k^2 z^2\right). \end{aligned}$$

This expression of the incomplete elliptic integral seems to be well-known [10].

Q. The common refinement of the Gröbner fan of $H_A(\beta)$ and that of $\mathrm{Ann}\,[g(t,x)]_{t=a}^{t=b}$ is our fan with four cones? See `Prog-ihg/gf1.sm1`.

4.4 Connection formulas

Connection formulas for the Gauss hypergeometric functions are given on the upper half plane and on the lower half plane. We will give connection formulas of our series solutions in an analogous way.

The domain of convergence of our series solution f_{ij}^{pq} and D_d is non-empty and open set for any d, then there exists a unique analytic continuation of the series f_{ij}^{pq} to the domain D_d. We will give connection formulas among our 4 series solutions on D_d.

Theorem 4 *We suppose* $0 < a < b$ *and exponents* $\alpha_1, \alpha_2, \gamma$ *are generic.*

1.

$$\begin{aligned} f_{12}^{11} &= e^{2\pi i \alpha_1} f_{12}^{21} \quad \textit{on } D_{(1,1,*,*,0,*,*)} \\ f_{12}^{11} &= e^{-2\pi i \alpha_1} f_{12}^{21} \quad \textit{on } D_{(0,0,*,*,1,*,*)} \\ f_{12}^{11} &= f_{12}^{21} \quad \textit{on other } D_d\textit{'s} \end{aligned}$$

2.

$$\begin{aligned} f_{12}^{11} &= e^{2\pi i \alpha_2} f_{22}^{11} \quad \textit{on } D_{(*,*,0,1,*,0,*)} \\ f_{12}^{11} &= e^{-2\pi i \alpha_2} f_{22}^{11} \quad \textit{on } D_{(*,*,1,0,*,1,*)} \\ f_{12}^{11} &= f_{22}^{11} \quad \textit{on other } D_d\textit{'s} \end{aligned}$$

3.

$$\begin{aligned} f^{21}_{12} &= e^{2\pi i\alpha_2} f^{21}_{22} \quad \textit{on } D_{(*,*,0,1,*,0,*)} \\ f^{21}_{12} &= e^{-2\pi i\alpha_2} f^{21}_{22} \quad \textit{on } D_{(*,*,1,0,*,1,*)} \\ f^{21}_{12} &= f^{21}_{22} \quad \textit{on other } D_d\textit{'s} \end{aligned}$$

I have not yet calculated the constants for other cases.

Intuitively speaking, the series $f^{k\ell}_{ij}$ are different expansions of the same integral (2) in different domains and hence they will agree with some adjustments of constant factor as in the Theorem. Here, we will give a proof without using the integral representation. The advantage of this discussion is that we can avoid topological discussions about choices of branches of the integrant. Analogous discussion is used to study global behavior of solutions of the Euler-Darboux equation [8].

Proof. We note $\frac{1}{\gamma+m+k+1} = \frac{1}{\gamma+m+1}\frac{(\gamma+m+1)_k}{(\gamma+m+2)_k}$. Then, the series f^{11}_{12} can be expressed as a superposition of contiguous family of Gauss hypergeometric functions as follows.

$$\begin{aligned} &x_{11}^{\alpha_1} x_{12}^{\alpha_2}\left(\sum_{m=0}^{\infty}\left(\frac{-x_{22}}{x_{12}}\right)^m \frac{b^{\gamma+m+1}(-\alpha_2)_m}{(\gamma+m+1)m!} F\left(\begin{matrix}-\alpha_1, & \gamma+m+1 \\ & \gamma+m+2\end{matrix}; \frac{-x_{21}b}{x_{11}}\right)\right. \\ &\left. - \sum_{m=0}^{\infty}\left(\frac{-x_{22}}{x_{12}}\right)^m \frac{a^{\gamma+m+1}(-\alpha_2)_m}{(\gamma+m+1)m!} F\left(\begin{matrix}-\alpha_1, & \gamma+m+1 \\ & \gamma+m+2\end{matrix}; \frac{-x_{21}a}{x_{11}}\right)\right) \qquad (5) \end{aligned}$$

The Gauss hypergeometric function has a unique analytic continuation to $\mathrm{Im}\, x_{21}/x_{11} > 0$ and $\mathrm{Im}\, x_{21}/x_{11} < 0$. We replace the Gauss hypergeometric functions in (5) with their series expansions around $x_{21}/x_{11} = \infty$. In other words, we make replacements by using the connection formula of the Gauss hypergeometric function in (5). For the first hypergeometric function in (5), we utilize

$$\begin{aligned} & F\left(\begin{matrix}-\alpha_1, & \gamma+m+1 \\ & \gamma+m+2\end{matrix}; \frac{-x_{21}b}{x_{11}}\right) \\ = & \frac{\Gamma(\gamma+m+2)\Gamma(\gamma+m+1+\alpha_1)}{\Gamma(\gamma+m+1)\Gamma(\gamma+m+2+\alpha_1)}\left(\frac{x_{21}b}{x_{11}}\right)^{\alpha_1} F\left(\begin{matrix}-\alpha_1, & 1-\alpha_1-\gamma-m-2 \\ & 1-\alpha_1-\gamma-m-1\end{matrix}; \frac{-x_{11}}{x_{21}b}\right) \\ & + \frac{\Gamma(\gamma+m+2)\Gamma(-\alpha_1-\gamma-m-1)}{\Gamma(-\alpha_1)\Gamma(\gamma+m+2-\gamma-m-1)}\left(\frac{x_{21}b}{x_{11}}\right)^{-\gamma-m-1} \\ & \qquad \cdot F\left(\begin{matrix}\gamma+m+1, & 1+\gamma+m+1-\gamma-m-2 \\ & 1+\gamma+m+1+\alpha_1\end{matrix}; \frac{-x_{11}}{x_{21}b}\right) \\ = & \frac{\gamma+m+1}{\gamma+\alpha_1+m+1}\left(\frac{x_{21}b}{x_{11}}\right)^{\alpha_1} F\left(\begin{matrix}-\alpha_1, & 1-\alpha_1-\gamma-m-2 \\ & 1-\alpha_1-\gamma-m-1\end{matrix}; \frac{-x_{11}}{x_{21}b}\right) \\ & + \frac{\Gamma(\gamma+m+2)\Gamma(-\alpha_1-\gamma-m-1)}{\Gamma(-\alpha_1)}\left(\frac{x_{21}b}{x_{11}}\right)^{-\gamma-m-1} \end{aligned}$$

and the analogous formula for the second Gauss hypergeometric function in (5). The terms obtained from the second terms of the connection formulas of the Gauss hypergeometric functions are canceled and we obtain

$$x_{11}^{\alpha_1}x_{12}^{\alpha_2}\left(\frac{x_{21}}{x_{11}}\right)^{\alpha_1}\left(\sum_{m=0}^{\infty}\left(\frac{-x_{22}}{x_{12}}\right)^m\frac{b^{\alpha_1+\gamma+m+1}(-\alpha_2)_m}{(\gamma+\alpha_1+m+1)m!}F\left(\begin{matrix}-\alpha_1, & 1-\alpha_1-\gamma-m-2\\ & -\alpha_1-\gamma-m\end{matrix};\frac{-x_{11}}{x_{12}b}\right)\right.$$
$$\left.-\sum_{m=0}^{\infty}\left(\frac{-x_{22}}{x_{12}}\right)^m\frac{a^{\alpha_2+\gamma+m+1}(-\alpha_2)_m}{(\gamma+\alpha_1+m+1)m!}F\left(\begin{matrix}-\alpha_1, & 1-\alpha_1-\gamma-m-2\\ & -\alpha_1-\gamma-m\end{matrix};\frac{-x_{11}}{x_{12}a}\right)\right)$$

Expanding the Gauss hypergeometric functions, we see that the above sum equals to $x_{11}^{\alpha_1}x_{12}^{\alpha_2}(x_{21}/x_{11})^{\alpha_1}x_{21}^{-\alpha_1}x_{12}^{-\alpha_2}f_{12}^{21}$. Applying the formulas in Remark 1, we obtain the first result 1. Other cases can be obtained analogously.

4.5 Monodromy formula

We study analytic continuation (monodromy) of the function $f_{ij}^{k\ell}$. We only give formula for f_{12}^{11}. Formulas for other $f_{ij}^{k\ell}$ can be obtained analogously by symmetry. In order to give formulas, we define

$$f_{12}^{11}(p,q;x)=x_{11}^{\alpha_1}x_{12}^{\alpha_2}\sum_{k,m\geq 0}\frac{(-1)^{k+m}}{\gamma+k+m+1}\cdot\frac{(-\alpha_1)_k(-\alpha_2)_m}{(1)_k(1)_m}$$
$$\cdot\left(qb^{\gamma+k+m+1}-pa^{\gamma+k+m+1}\right)\left(\frac{x_{21}}{x_{11}}\right)^k\left(\frac{x_{22}}{x_{12}}\right)^m,$$
$$\tilde{f}(x)=x_{11}^{\alpha_1}x_{12}^{\alpha_2}\left(-\frac{x_{11}}{x_{21}}\right)^{\gamma+1}F\left(\begin{matrix}-\alpha_2, & \gamma+1\\ & \gamma+\alpha_1+2\end{matrix};\frac{x_{11}x_{22}}{x_{12}x_{21}}\right).$$

We note that $\tilde{f}(x)$ is a solution of the homogeneous system $H_A(\beta)$.

Theorem 5 *We fix x_{12},x_{21},x_{22} to real numbers and regarded the function as a function in one variable. Let γ_a be a path which encircles the point $-ax_{21}$ in the positive direction and γ_b be a path which encircles the point $-bx_{21}$ in the positive direction. We also suppoese that exponents are generic. The analytic continuations of f_{12}^{11} along γ_a and γ_b are*

$$f_{12}^{11}(1,1;x)\quad\looparrowright_{\gamma_b^*}\quad f_{12}^{11}(1,e^{2\pi i\alpha_1};x)+\frac{-2\pi ie^{\pi i(\alpha_1+1)}}{\Gamma(-\alpha_1)}\tilde{f}(x)$$
$$f_{12}^{11}(1,1;x)\quad\looparrowright_{\gamma_a^*}\quad f_{12}^{11}(e^{2\pi i\alpha_1},1;x)-\frac{-2\pi ie^{\pi i(\alpha_1+1)}}{\Gamma(-\alpha_1)}\tilde{f}(x)$$

Proof. We replace the Gauss hypergeometric function in (5) with analytic continuations of them; we utilize the following formula of the analytic continuation of the Gauss hypergeometric function $F(a,b,c;x)$ along a path which encircles $x=1$ positively.

$$F(a,b,c;x)\looparrowright(1-A)F(a,b,c;x)+Bx^{1-c}F(a-c+1,b-c+1,2-c;x)$$

Here, we put

$$A = \frac{(1-e^{-2\pi i a})(1-e^{-2\pi i b})}{1-e^{-2\pi i c}}, B = \frac{2\pi i}{1-c}\cdot\frac{\Gamma(c)^2}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}\cdot e^{\pi i(c-a-b)}.$$

We note that

$$\begin{aligned}1-A &= 1-\frac{(1-e^{-2\pi i(-\alpha_1)})(1-e^{-2\pi i(\gamma+m+1)})}{1-e^{-2\pi i(\gamma+m+2)}}\\ &= 1-(1-e^{2\pi i\alpha_1})\\ &= e^{2\pi i\alpha_1}.\end{aligned}$$

Then, we obtain the first term $f^{11}_{12}(1, e^{2\pi i\alpha_1}; x)$ of the right hand side of the first formula by the replacement of the type $(1-A)F(a,b,c;x)$. We make replacements of the type $x^{1-c}F(a-c+1, b-c+1, 2-c; x)$. Then, we have

$$F\left(\begin{matrix}-\alpha_1-\gamma-m-1, & 0 \\ & -\gamma-m\end{matrix}; -\frac{x_{21}b}{x_{11}}\right) = 1.$$

Therefore, we have

$$\begin{aligned}
& x_{11}^{\alpha_1}x_{12}^{\alpha_2}\sum_{m=0}^{\infty}\left(-\frac{x_{12}}{x_{22}}\right)^m \frac{b^{\gamma+m+1}(-\alpha_2)_m}{(\gamma+m+1)m!}\\
& \frac{2\pi i}{-(\gamma+m+1)}\cdot\frac{\Gamma(\gamma+m+2)^2}{\Gamma(-\alpha_1)\Gamma(\gamma+m+1)\Gamma(\gamma+\alpha_1+m+2)\Gamma(1)}\cdot e^{\pi i(\alpha_1+1)}\left(-\frac{x_{21}b}{x_{11}}\right)^{-(\gamma+m+1)}\\
= {} & x_{11}^{\alpha_1}x_{12}^{\alpha_2}\sum_{m=0}^{\infty}\left(-\frac{x_{12}}{x_{22}}\right)^m \frac{b^{\gamma+m+1}(-\alpha_2)_m}{(\gamma+m+1)m!}\\
& \frac{2\pi i}{-(\gamma+m+1)}\cdot\frac{(\gamma+m+1)^2\Gamma(\gamma+m+1)^2}{\Gamma(-\alpha_1)\Gamma(\gamma+m+1)\Gamma(\gamma+\alpha_1+m+2)}\cdot e^{\pi i(\alpha_1+1)}b^{-(\gamma+m+1)}\left(-\frac{x_{11}}{x_{21}}\right)^{\gamma+m+1}\\
= {} & \frac{-2\pi i e^{\pi i(\alpha_1+1)}}{\Gamma(-\alpha_1)}\left(-\frac{x_{11}}{x_{21}}\right)^{\gamma+1} x_{11}^{\alpha_1}x_{12}^{\alpha_2}\sum_{m=0}^{\infty}\left(-\frac{x_{12}}{x_{22}}\right)^m\frac{(-\alpha_2)_m}{m!}\cdot\frac{\Gamma(\gamma+m+1)}{\Gamma(\gamma+\alpha_1+m+2)}\left(-\frac{x_{11}}{x_{21}}\right)^m\\
= {} & \frac{-2\pi i e^{\pi i(\alpha_1+1)}}{\Gamma(-\alpha_1)}\tilde{f}(x)
\end{aligned}$$

which is the second term of the right hand side of the first formula. The second formula in the Theorem is obtained analogously by exchanging the role of a and b.

Remark 3 The function $[g(t,x)]_{t=a}^{t=b}$ is analytically continued as follows.

$$\begin{aligned}
[g(t,x)]_{t=a}^{t=b} &\looparrowright_{\gamma_b^*} e^{2\pi i\alpha_1}g(b,x)-g(a,x)\\
[g(t,x)]_{t=a}^{t=b} &\looparrowright_{\gamma_a^*} g(b,x)-e^{2\pi i\alpha_1}g(a,x)
\end{aligned}$$

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