A-Hypergeometric Functions

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1.1 A-hypergeometric equations

Let A be a $d \times n$ matrix with integer entries. We denote by a_i the i-th column vector of A. We suppose that a_i's generate the lattice $\mathbf{Z}^d$, in other words, we have $\sum_{i=1}^{n} \mathbf{Z} a_i = \mathbf{Z}^d$. Let $\beta = (\beta_1, \ldots, \beta_d) \in \mathbf{C}^d$ be a vector of parameters. The ring of differential operators

$$\mathbf{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle,\ x_j x_j = x_j x_i, \partial_i \partial_j = \partial_j \partial_i, \partial_i x_j = x_j \partial_i + \delta_{ij}$$

is denoted by D or by D_n. The action of $x^p \partial^q$ to a function $f(x)$ is defined by $x^p \partial^q \bullet f = x^p \frac{\partial^{|q|} f}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}}$.

Definition 1 [18] We call the following system of differential equations an *A-hypergeometric system* or a GKZ hypergeometric system:

$$(E_i - \beta_i) \bullet f = 0, \quad \text{where } E_i - \beta_i = \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i, \qquad (i = 1, \ldots, d)$$

$$\Box_u \bullet f = 0, \quad \text{where } \Box_u = \prod_{\{i \,|\, 1 \le i \le n, u_i > 0\}} \partial_i^{u_i} - \prod_{\{j \,|\, 1 \le j \le n, u_j < 0\}} \partial_j^{-u_j}$$

with $u \in \mathbf{Z}^n$ running over all u such that $Au = 0, u \neq 0$.

We denote by I_A the affine toric ideal generated by $\Box_u$ for all $u \in \mathbf{Z}^n$ such that $Au = 0$ in $S_n = \mathbf{C}[\partial_1, \ldots, \partial_n]$. The left ideal in D generated by $E_i - \beta_i$, $i = 1, \ldots, d$ and I_A is denoted by $H_A(\beta)$ and is called the A-hypergeometric ideal. The quotient left D-module $D/H_A(\beta)$ is denoted by $M_A(\beta)$ and called the A-hypergeometric D-module.

Several invariants of the D-module can be described in terms of the set of points $\{a_1, \ldots, a_n\}$ like the theory of toric varieties. We also denote the set of points by A in this chapter; the symbol A stands for a matrix or a set of points. When the meaning of A is clear in the context, we do not say which it means. $\mathbf{N}_0 A$ and $\mathbf{Z}A$ mean $\sum_{i=1}^n \mathbf{N}_0 a_i$ and $\sum_{i=1}^n \mathbf{Z} a_i$ respectively.

Although the A-hypergeometric system can be defined for any matrix A, there are nice classes of matrices A (or sets of points a_i). And solutions of the associated A-hypergeometric systems are deserved to be special functions. Let us introduce some of them. Take integers k and k' satisfying $1 \leq k \leq k'$. Put $e_1 = (1, 0, \ldots, 0)^T \in \mathbf{Z}^{k+1}$, $e_2 = (0, 1, 0, \ldots, 0)^T \in \mathbf{Z}^{k+1}$, $\ldots$, and $e'_1 = (1, 0, \ldots, 0)^T \in \mathbf{Z}^{k'+1}$, $e'_2 = (0, 1, 0, \ldots, 0)^T \in \mathbf{Z}^{k'+1}$, $\ldots$. Let $A(k, k')$ be a $(k+k'+1) \times (k+1)(k'+1)$ matrix of which columns consist of $p(e_i \oplus e'_j)$ where p is the projection to the first $k + k' + 1$ coordinates (the projection which removes the last coordinate). $A(1, 1)$, $A(1, 2)$, $A(2, 2)$ are given in Table 1.1.

The columns of $A(k, k')$ generate $\mathbf{Z}^{k+k'+1}$ and they lie on the hyperplane $\sum_{j=1}^{k+1} y_j = 1$ in $\mathbf{R}^{k+k'+1}$. Since the convex hull of $e_1, \ldots, e_{k+1}$ is the simplex Δ_k and that of $e'_1, \ldots, e'_{k'+1}$ is the simplex $\Delta_{k'}$, we call this A-hypergeometric system $\Delta_k \times \Delta_{k'}$-hypergeometric system or the hypergeometric system $E'(k + 1, k + k' + 2)$. The latter naming comes from a relation of this system with the hypergeometric system $E(k, n)$ (Section 1.4). For this hypergeometric system, we often denote the variable x_p by x_{ij} where $p = (i - 1)k' + (j - 1) + 1$. This double index notation is convinient. We also regard a vector of length $(k + 1)(k' + 1)$ as a matrix under this double index notation. For example, for a vector e, the condition $A(k, k')e = \beta$ means that the row sums and the column sums of e expressed in terms of the $(k + 1) \times (k' + 1)$ matrix are $(\beta_1, \ldots, \beta_{k+1})$ and $(\beta_{k+2}, \ldots, \beta_{k+k'+1}, \sum_{i=1}^{k+1} \beta_i - \sum_{j=k+2}^{k+k'+1} \beta_j)$ respectively.

The ideal I_A for $A = A(k, k')$ is generated by

$$\partial_{iq}\partial_{jp} - \partial_{ip}\partial_{jq}, 1 \leq i < j \leq k + 1, 1 \leq p < q \leq k' + 1.$$

More precisely, it is the reduced Gröbner basis with respect to the graded reverse lexicographic order $\succ$ with $\partial_{1,1} \succ \partial_{1,2} \succ \cdots \succ \partial_{1,k} \succ \partial_{2,1} \succ \cdots$ [48, Prop 5.4]. For any A, generators of I_A can be obtained by a Gröbner basis computation [48, Alg. 4.5]. Generators of I_A is called the Markov basis in algebraic statistics. There are theoretical and computational efforts to find explicit Markov basis. We have a database of Markov bases for several matrix A ([1] or [23]).

The matrix $A(1, k')$ stands for the Lauricella function F_D of k' variables (see Example 2 for the correspondence). In particular, when $k' = 1$, it stands for the Gauss hypergeometric function. Let us give matrices A for other Lauricella functions (see Chapter 2 on these functions). Let $e_0, e_1, \ldots, e_{2m}$ be the standard basis of $\mathbf{Z}^{2m+1}$. Put $A = \{e_0, e_1, \ldots, e_{2m}, e_0 + e_1 - e_{m+1}, e_0 +$

$$A(1,1) = \begin{pmatrix} x_{11} & x_{12} & x_{21} & x_{22} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, A(1,2) = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A(2,2) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A(F_A,2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}, \quad A(F_C,2) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A(0134) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}, \quad A_s = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad A(P_4) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix}$$

Suppose that $a_i \in \mathbf{Z}^m$. For $A_1 = (a_1, \ldots, a_{n_1}), \ldots, A_k = (a_{n_{k-1}+1}, \ldots, a_{n_k})$, we define $A(A_1, \ldots, A_k)$ by

$$\begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ . & \cdots & . & . & \cdots & . & & . & \cdots & . \\ . & \cdots & . & . & \cdots & . & \cdot\ \cdot\ \cdot & . & \cdots & . \\ . & \cdots & . & . & \cdots & . & & . & \cdots & . \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 1 \\ a_1 & \cdots & a_{n_1} & a_{n_1+1} & \cdots & a_{n_2} & & a_{n_{k-1}+1} & \cdots & a_{n_k} \end{pmatrix}$$

Table 1.1 A

$e_2 - e_{m+2}, \ldots, e_0 + e_m - e_{2m}\}$. Then A is a $(2m+1) \times (3m+1)$ matrix, which stands for the Lauricella function F_A of m variables [38]. They lie on the hyperplane $y_0 + y_1 + \cdots + y_{2m} = 1$ in $\mathbf{R}^{2m+1}$. We denote the matrix by $A(F_A, m)$. The associated toric ideal I_A is generated by $\partial_0\partial_j - \partial_{m+j}\partial_{2m+j}$, $j = 1, \ldots, m$. Here, we use the variables $u_0, u_1, \ldots, u_{3m}$ as independent variables instead of $x_1, \ldots, x_n$. When $m = 2$, it is the Appell function F_2; the matrix is given in Table 1.1.

Let $e_1, \ldots, e_{m+1}, e_{m+2}$ be the standard basis of $\mathbf{Z}^{m+2}$. Put $A = \{e_1 + e_{m+2}, e_2 + e_{m+2}, \ldots, e_{m+1} + e_{m+2}, -e_1 + e_{m+2}, -e_2 + e_{m+2}, \ldots, -e_{m+1} + e_{m+2}\}$. Then A is an $(m+2) \times 2(m+1)$ matrix, which stands for the Lauricella function F_C of m variables [38]. They lie on the hyperplane $z_{m+2} = 1$ in $\mathbf{R}^{m+2}$. We denote the matrix by $A(F_C, m)$. Note that the lattice generated by the columns of $A(F_C, m)$ is a proper sublattice of $\mathbf{Z}^{m+2}$. Then, we need to regard the sublattice as $\mathbf{Z}^{m+2}$. The associated toric ideal I_A is generated by $\partial_j\partial_{-j} - \partial_{m+1}\partial_{-(m+1)}$, $j = 1, \ldots, m$. Here, we use the variables $u_1, \ldots, u_{m+1}, u_{-1}, \ldots, u_{-(m+1)}$ as independent variables. When $m = 2$, it is the Appell function F_4; the matrix is given in Table 1.1. The notion of binomial D-modules is proposed and studied in [14]. Binomial D-modules are generalizations of A-hypergeometric equations and they fit to study Appell-Horn equations and their generalizations to several variables in algebraic methods.

A-hypergeometric systems associated to smooth fano polytopes have importance in studies of period maps for $K3$ and Calabi-Yau varieties (see, e.g., [26], [27], [47] and their references). For example, the matrix $A(P_4)$ [34] appears in this context.

Let us discuss on integral representations of solutions of A-hypergeometric equations. Consider $A_1, \ldots, A_k$ in Table 1.1 and define k polynomials $f_j(x,t) = \sum_{i=n_{j-1}+1}^{n_j} x_i t^{a_i}$. Take complex numbers α_j, $\gamma = (\gamma_1, \ldots, \gamma_m)$. We consider the integral

$$\Phi(\alpha, \gamma; x) \quad = \quad \int_C \prod_{j=1}^{k} f_j(x,t)^{\alpha_j} t^{\gamma} dt_1 \cdots dt_m,$$

where C is a twisted m-cycle defined for $\prod_{j=1}^{k} f_j(x,t)^{\alpha_j} t^{\gamma}$. The function $\Phi(\alpha, \gamma; x)$ satisfies the A-hypergeometric system $H_A(\beta)$ for $A = A(A_1, \ldots, A_m)$ and $\beta = (\alpha_1, \ldots, \alpha_k, -\gamma_1 - 1, \ldots, -\gamma_m - 1)^T$. When f_j are linear with respect to the variable t, we call the function Φ hypergeometric function for hyperplane arrangements. Note that when $A_1 = \ldots = A_k = \Delta_{k'}$, $A(A_1, \ldots, A_k) = A(k, k')$. As to studies on these hypergeometric functions in terms of twisted cohomology groups, see [4], [5], [3], [36].

For general A, the integral

$$\Phi(\gamma; x) \quad = \quad \int_C \exp\left(\sum_{i=1}^{n} x_i t^{a_i}\right) t^{\gamma} dt_1 \cdots dt_d$$

satisfies $H_A(\beta)$ with $\beta = (-\gamma_1 - 1, \ldots, -\gamma_d - 1)^T$ for a "suitable" d-cycle formally. However, a homological study of such cycles is not performed in a full general form.

1.2 Some definitions from combinatorics, polytopes and Gröbner basis

The matrix A is said to be pointed when $a_1, \ldots, a_n$ lie in a single open half-space. For example, $A = (-1, 1)$ is not pointed and all A's in Table 1.1 are pointed. The set of points A is called normal, when A satisfies $(\sum \mathbf{R}_{\geq 0} a_k) \cap \mathbf{Z}^n = \sum \mathbf{Z}_{\geq 0} a_k$.

For a facet σ of the cone $\mathrm{pos}(A) = \mathbf{R}_{\geq 0} A$, F_σ is a linear function on $\mathbf{R}A = \mathbf{R}^d$ uniquely determined by the conditions:
1. $F_\sigma(\mathbf{Z}A) = \mathbf{Z}$, 2. $F_\sigma(a_i) \geq 0$ for all $i = 1, \ldots, n$, 3. $F_\sigma(a_i) = 0$ for all $a_i \in \sigma$.
We call F_σ the *primitive integral support function* of σ.

For $\Delta_k \times \Delta_{k'}$ embedded in $\mathbf{R}^{k+1} \times \mathbf{R}^{k'+1} = \{(x_1, \ldots, x_{k+1}; y_1, \ldots, y_{k'+1})\}$, the support functions are x_i and y_j. When we project the points to $\mathbf{R}^{k+1} \times \mathbf{R}^{k'}$, the primitive integral support functions are x_i $(i = 1, \ldots, k+1)$, and y_j $(j = 1, \ldots, k')$, and $1 - \sum_{j=1}^{k'} y_j$.

The Supporting functions for $A(F_A, m)$ are s_j, $s_j + s_{m+1}$, $1 \leq j \leq m$ and $s_0 + \sum_{j \in J} s_{m+j}$, $J \subseteq [1, m]$ where $\{s_i\}$ is the dual basis of $\{e_i\}$. Those for $A(F_C, m)$ are $(1/2)(s_{m+2} + \sum_{j \in J} s_j - \sum_{j \notin J} s_j)$, $J \subseteq [1, m+1]$ [38].

Let $\mathbf{Z}A$ be the lattice generated by the columns of A. Let us set the volume of the convex hull U of the lattice base and the origin to 1. The volume of polytopes in $\mathbf{R}A$ normalized with the U is called the *normalized volume*. The normalized volume of the convex hull of A and the origin is denoted by $\mathrm{vol}(A)$. The normalized volume of $A(k, k')$ is known to be equal to $\binom{k+k'}{k}$. For given A, it can be evaluated by geometry software systems like polymake, or by computer algebra systems which use a formula $\mathrm{degree}(I_A) = \mathrm{vol}(A)$.

Example 1 Macaulay2 [22] commands to evaluate the volume (the degree) of $A(0134)$. Here, `o5` is I_A.

```
loadPackage "FourTiTwo"
M=matrix "1,1,1,1; 0,1,3,4"
R=QQ[a..d]
I=toricGroebner(M,R)
   o5 = ideal (b^3  - a^2*c, b*c - a*d, - a*c^2  + b^2*d, c^3  - b*d^2 )
degree(I)
   o6 = 4
```

For a given weight vector $w \in \mathbf{R}^n$ (`Weights` below), consider points $\{(a_i, w_i)\}$ in $\mathbf{R}^{d+1}$ and the convex hull of them. The projection of the convex hull to the first d coordinates naturally induces a triangulation of the set of points A for a generic weight w, which is called a regular triangulation [21], [48]. We compute a regular triangulation of $\Delta_1 \times \Delta_2$ for $w = (4, 2, 0, 10, 8, 6)$ by the computer algebra system Macaulay2

```
i1 : loadPackage "FourTiTwo"
i2 : M=matrix "1,1,1,0,0,0; 0,0,0,1,1,1; 1,0,0,1,0,0; 0,1,0,0,1,0"
i3 : R=QQ[x11,x12,x13,x21,x22,x23, MonomialOrder=>{Weights=>{4,2,0,10,8,6}}]
i4 : I=toricGroebner(M,R)
  o4 = ideal (x13*x21 - x11*x23, x12*x21 - x11*x22, x13*x22 - x12*x23)
i5 : J=leadTerm(I)
  o8 = | x13x22 x13x21 x12x21 |
i6 : associatedPrimes(ideal(J))
  o12 = {ideal (x22, x21), ideal (x13, x12), ideal (x13, x21)}
```

By taking the complements of the indices of each associated primes, we get a regular triangulation $(11, 12, 13, 23)$, $(11, 21, 22, 23)$, $(11, 12, 22, 23)$.

1.3 A-hypergeometric series

Let us introduce A-hypergeometric series following [18] and [42, 3.4]. Let $v = (v_1, \ldots, v_n)$ be a vector in $\mathbf{C}^n$ and $u = (u_1, \ldots, u_n)$ a vector in $\mathbf{Z}^n$. We decompose u into positive and negative parts, $u = u_+ - u_-$, where u_+ and u_- are non-negative vectors with disjoint support. Consider the following two scalars in $\mathbf{C}$, which can be expressed by falling factorials:

$$[v]_{u_-} = \prod_{i:u_i<0} \prod_{j=1}^{-u_i} (v_i - j + 1),$$
$$[u + v]_{u_+} = \prod_{i:u_i>0} \prod_{j=1}^{u_i} (u_i + v_i - j + 1) = \prod_{i:u_i>0} \prod_{j=1}^{u_i} (v_i + j).$$

For example, when $v = (v_1, v_2, 0, v_4)$ and $u = (-2, 2, 2, -2)$, we have $\frac{[v]_{u_-}}{[v+u]_{u_+}} = \frac{v_1(v_1-1)v_4(v_4-1)}{(v_2+2)(v_2+1)2!}$. Note that when $v \in (\mathbf{C} \setminus \mathbf{Z}_{<0})^n$, we have $[u + v]_{u_+} \neq 0$. We set $L = \mathrm{Ker}(\mathbf{Z}^n \stackrel{A}{\to} \mathbf{Z}^d)$.

Theorem 1 *Suppose that $v \in (\mathbf{C} \setminus \mathbf{Z}_{<0})^n$ and $Av = \beta$. Then the formal series*

$$\phi_v := \sum_{u \in L} \frac{[v]_{u_-}}{[v + u]_{u_+}} \cdot x^{v+u} \tag{1.1}$$

is well-defined and is a formal solution of $H_A(\beta)$.

As to the proof of this theorem, see [42, Prop. 3.4.1]. We call the formal series the A-hypergeometric series in the falling factorial form.

Let us introduce another expression of the series. We set $\Gamma(u + v + 1) = \prod_{i=1}^n \Gamma(u_i+v_i+1)$ and when $u_i+v_i \in \mathbf{Z}_{<0}$ for an i, we define $1/\Gamma(u+v+1) = 0$. Under this convention, we have $\frac{1}{\Gamma(v+u+1)} = \frac{[v]_{u_-}}{[v+u]_{u_+}} \frac{1}{\Gamma(v+1)}$ for $u \in L$ and $v \in (\mathbf{C} \setminus \mathbf{Z}_{<0})^n$. Define

$$\Phi_v := \sum_{u \in L} \frac{1}{\Gamma(u + v + 1)} x^{v+u}. \tag{1.2}$$

Then, we have $\Phi_v = \frac{1}{\Gamma(v+1)} \phi_v$ when none of v_i is negative integer. We call the formal series the A-hypergeometric series in the gamma function form. Note that when v_i is a negative integer, two series are different. For example, if $v_i = -1$ and $u_i = 1$, then we have $[u_i+v_i]_{u_i} = 0$ and ϕ_v is not well-defined, but $\Gamma(u_i + v_i + 1) = 1$. When we want to express A-hypergeometric series in terms of the Pochhammer symbol or the falling factorial, the formulas

$$\Gamma(\alpha+m)=\Gamma(\alpha)(\alpha)_m,\ \Gamma(\alpha-m+1)=\Gamma(\alpha+1)(-1)^m/(-\alpha)_m$$

are useful.

For a given weight vector $w \in \mathbf{Z}^n$ and $\ell \in I_A$, $\mathrm{in}_w(\ell)$ is the sum of the highest w-order terms in ℓ. The ideal in S_n generated by $\mathrm{in}_w(\ell)$, $\ell \in I_A$ is denoted by $\mathrm{in}_w(I_A)$ and is called the initial ideal of I_A [48]. Let C be the Gröbner cone of I_A for a generic weight vector w. The initial ideal $\mathrm{in}_{w'}(I_A)$ does not change when w' runs over C [48], [42, Chap 2]. For a series f with a support on a translate of the dual cone C^*, for which we may assume $(w, C^*\setminus\{0\}) > 0$, the starting term of f is the sum of the lowest weight terms in f with respect to w. If f is a solution of $\ell \bullet f = 0$, $\ell \in D$, then the starting term of f is a solution of $\mathrm{in}_{(-w,w)}(\ell)$, which is the sum of the highest order terms in ℓ with respect to the weight $(-w, w)$ where $-w$ (resp. w) stands for x (resp. ∂). This observation gives us the following method [42] to find series solutions of $H_A(\beta)$; (1) determine the initial ideal $\mathrm{in}_{(-w,w)}(H_A(\beta))$, (2) solve it to determine the starting terms, (3) extend the starting terms to series solutions.

Theorem 2 *For generic β, the initial ideal $\mathrm{in}_{(-w,w)}(H_A(\beta))$ is generated by $E_i - \beta_i$, $1 \le i \le d$ and $\mathrm{in}_w(I_A)$.*

We note that the proof of [42, Th. 3.1.3] needs to be corrected to utilize the homogenized Weyl algebra. We suppose that I_A is a homogeneous ideal and take a generaic weight vector w such that $\mathrm{in}_w(I_A)$ is a monomial ideal. Let G be the reduced Gröbner basis of I_A with respect to the order $\prec_w$ [48]. We consider the system of differential equations

$$(E_i-\beta_i)\bullet s=0, \quad i=1,\ldots,d, \quad \text{and} \quad \ell\bullet s=0, \quad \ell\in \mathrm{in}_w(G) \tag{1.3}$$

Let v be a solution of algebraic equations

$$Av=\beta, \quad \prod_{i=1}^{n} v_i(v_i-1)\cdots(v_i-e_i+1)=0 \text{ for } \partial^e\in \mathrm{in}_w(G) \tag{1.4}$$

It is called a fake exponent. We note that the fake exponents can be expressed in terms of standard pairs of the monomial ideal $\mathrm{in}_w(I_A)$ [42, 3.2]. When β_i are generic, there are linearly independent $\mathrm{vol}(A)$ solutions of (1.3) of the form $s = x^v = \prod_{i=1}^n x_i^{v_i}$ where v is a fake exponent and they span the solution space over $\mathbf{C}$ when v runs over the fake exponents.

Theorem 3 *[18], [42, Th 3.4.2] If v is a fake exponent and $v \in (\mathbf{C}\setminus\mathbf{Z}_{<0})^n$, then ϕ_v is a formal solution of $H_A(\beta)$ with the support in $v + (C^* \cap L)$.*

Note that Gel'fand, Kaparanov, Zelevinsky constructed series solutions by regular triangulations of A [18]. Our construction differs with their construction, but it is related with the construction via the theorem [48, Th 8.3] "$\sqrt{\mathrm{in}_w(I_A)}$ is the Stanley-Reisner ideal for the regular triangulation by w". The function ϕ_v converges when $(-\log|x_1|, \ldots, -\log|x_n|)$ lies in a translate of the secondary cone attached to the regular triangulation.

For a good class of A-hypergeometric functions, more explicit form of A-hypergeometric series is known as we will describe. For $A = A(p-1, q-1)$, the stair case Groöbner basis in [48, Prop.5.4] gives series solutions. A sequence of indexes $\{(1,1), \ldots, (p,q)\}$ is called a stair if (i,j) is an element of the stair and is not (p,q), then the next element of (i,j) is either $(i+1,j)$ or $(i,j+1)$ (see Table 1.2).

The initial ideal of I_A for the reverse lexicographic order is generated by $\partial_{i\ell}\partial_{jk}$, $1 \le i < j \le p, 1 \le k < \ell \le q$ [48, Prop.5.4]. We can obtain the fake exponents from this initial ideal by solving (1.4). It is known that there is a one-to-one correspondence between the roots of the system of equations and the stairs. For a given stair S, the system has a unique solution such that $v_{ij} = 0$ for $(i,j) \notin S$. In other words, the support of each exponent has the form of the stair for generic β. In the sequel, we use e rather than v to denote exponents. The support of the series solution standing for the exponent e has the form

$$e + L', \quad L' = \sum_{(i,j)\in\overline{\mathrm{supp}(e)}} \mathbf{Z}_{\ge 0} b_e^{(i,j)}$$

where $b_e^{(i,j)}$ is an element of Ker A such that (i,j)-th element of $b_e^{(i,j)}$ is 1 for $(i,j) \in \overline{\mathrm{supp}(e)}$ and (i',j')-th element is 0 for $(i',j') \in \overline{\mathrm{supp}(e)} \setminus \{(i,j)\}$.

Example 2 We put $A = A(1, N-1)$ in this example. Let a, $b_1, \ldots, b_{N-1}$, c be (generic) constants. Put $b_N = a + 1 - c$ and

$$e(k) = \begin{pmatrix} -b_1 & \cdots & -b_{k-1} & -\sum_{j=k}^N b_j + a & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \sum_{j=k+1}^N b_j - a & -b_{k+1} & \cdots & -b_{N-1} & -b_N \end{pmatrix},$$

which is the fake exponent standing for the k-th stair.
Put $m = (m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_N)$, $m_k = -\sum_{j=1}^{k-1} m_j + \sum_{j=k+1}^N m_j$, and $z_j = \frac{x_{2j}x_{1N}}{x_{1j}x_{2N}}$ for $1 \le j \le N$. Note that $z_N = 1$. Define a series $\phi_k(e; z)$ by

$$\sum_{m\in\mathbf{Z}_{\ge 0}^{N-1}} \frac{\prod_{j=1}^{k-1}[e_{1j}]_{m_j} \prod_{j=k+1}^{N}[e_{2j}]_{m_j}}{\prod_{j=1}^{k-1} m_j! \prod_{j=k+1}^{n} m_j!} c_m \prod_{j=1}^{k-1} \left(z_j z_k^{-1}\right)^{m_j} \prod_{j=k+1}^{n} \left(z_k z_j^{-1}\right)^{m_j} \tag{1.5}$$

where $e = e(k)$, $c_m = [e_{1k}]_{m_k}/[e_{2k} + m_k]_{m_k}$ when $m_k > 0$, and $c_m = [e_{2k}]_{-m_k}/[e_{1k} - m_k]_{-m_k}$ when $m_k < 0$, and $c_m = 1$ when $m_k = 0$. For $\beta = (-\sum b_i + c - 1, -a, -b_1, \dots, -b_{N-1}, c - 1 - a)$, the function $x^{e(k)}\phi_k(e(k); z)$, $1 \le k \le N$ is a solution of $H_A(\beta)$ and $x^{e(k)-e(n)}\phi_k(e(k); z)$ is a solution of the Lauricella system $E_D(a, (b), c)$. The series $\phi_N(e(N); z)$ is the Lauricella's F_D. The series ϕ_k's have a common domain of convergence $|z_1| < \cdots < |z_{N-1}| < 1$.

Example 3 The function

$$u_0^{-a} \prod_{j=1}^{m} u_j^{-b_j} \prod_{j=1}^{m} u_{m+j}^{c_j-1} f_A \left(a, b_1, \dots, b_m, c_1, \dots, c_m; \frac{u_{m+1}u_{2m+j}}{u_0 u_1}, \dots, \frac{u_{m+m}u_{2m+m}}{u_0 u_m} \right) \tag{1.6}$$

is a solution of $H_{A(F_A,m)}(\beta)$, $\beta^T = (-a, -b_1, \dots, -b_m, c_1-1, \dots, c_m-1)$ when f_A is a solution of the Lauricella's $E_A(a, (b), (c))$. Any classical solution of $H_{A(F_A,m)}(\beta)$ can be expressed as (1.6).

Example 4 The function

$$u_{m+1}^{-a} u_{-m}^{-b} \prod_{j=1}^{m} u_{-j}^{c_j-1} f_C \left(a, b, c_1, \dots, c_m; \frac{u_1 u_{-1}}{u_{m+1}u_{-(m+1)}}, \dots, \frac{u_m u_{-m}}{u_{m+1}u_{-(m+1)}} \right) \tag{1.7}$$

is a solution of $H_{A(F_C,m)}(\beta)$, $\beta^T = (1-c_1, \dots, 1-c_m, b-a, \sum_{j=1}^m c_j - a - b - m)$ when f_c is a solution of the Lauricella's $E_C(a, b, (c))$. Any classical solution of $H_{A(F_C,m)}(\beta)$ can be expressed as (1.7).

Example 5 Series solutions for $A(2, 2)$ and $\beta^T = (\alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2)$ ($E'(3, 6)$) have attracted special interests [31], [46]. We present a set of series solutions of this system. When we express an exponent as a 3×3 matrix under the double index notation, α_i is the i-th row sum and γ_j is the j-th column sum.

Hypergeometric series associated to the exponent $e(i)$ is written as

$$\phi_{e(i)}(x) = x^{e(i)} \sum_{m \in \mathbf{N}_0^4} \frac{[e(i)]_{u_-}}{[e(i) + u]_{u_+}} x^u, \quad u = \sum_{j=1}^{4} b_{e(i)}^j m_j. \tag{1.8}$$

For other series solutions, see [46] and its references. An interesting series solution of $E'(3, 6)$, which is not obtained with the method in this section, is studied in [33] in terms of arithmetic and geometric means.

In case of non-generic parameters, we have series solutions containing logarithmic functions. We can construct vol(A) linearly independent solutions

stair	e : exponent
$\begin{pmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}$	$e(1) = \begin{pmatrix} \gamma_1 & \gamma_2 & \alpha_1-\gamma_1-\gamma_2 \\ 0 & 0 & \alpha_2 \\ 0 & 0 & \alpha_3 \end{pmatrix}$
$\begin{pmatrix} * & * & 0 \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$	$e(2) = \begin{pmatrix} \gamma_1 & \alpha_1-\gamma_1 & 0 \\ 0 & -\alpha_1+\gamma_1+\gamma_2 & \alpha_1+\alpha_2-\gamma_1-\gamma_2 \\ 0 & 0 & \alpha_3 \end{pmatrix}$
$\begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix}$	$e(3) = \begin{pmatrix} \gamma_1 & \alpha_1-\gamma_1 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & -\alpha_1-\alpha_2+\gamma_1+\gamma_2 & \alpha_1+\alpha_2+\alpha_3-\gamma_1-\gamma_2 \end{pmatrix}$
$\begin{pmatrix} * & 0 & 0 \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$	$e(4) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ -\alpha_1+\gamma_1 & \gamma_2 & \alpha_1+\alpha_2-\gamma_1-\gamma_2 \\ 0 & 0 & \alpha_3 \end{pmatrix}$
$\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ 0 & * & * \end{pmatrix}$	$e(5) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ -\alpha_1+\gamma_1 & \alpha_1+\alpha_2-\gamma_1 & 0 \\ 0 & -\alpha_1-\alpha_2+\gamma_1+\gamma_2 & \alpha_1+\alpha_2+\alpha_3-\gamma_1-\gamma_2 \end{pmatrix}$
$\begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & * & * \end{pmatrix}$	$e(6) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \\ -\alpha_1-\alpha_2+\gamma_1 & \gamma_2 & \alpha_1+\alpha_2+\alpha_3-\gamma_1-\gamma_2 \end{pmatrix}$

Table 1.2 Exponents

when I_A is homogeneous by introducing a purturbation parameter ε in parameters and expand the series solution in terms of ε [42, 3.5, Th 3.5.1]. We will explain the procedure by an example.

Example 6 We consider the case of $\alpha_i = \gamma_i = 1/2$ for $E'(3,6)$ (Table 1.2). The system with this parameter has a special importance in the algebraic geometry ([31], [51]). Let us construct a set of series solutions for this case. The exponenents $e(1)$ and $e(6)$ are not degenerated and give two linearly independent solutions. The exponents $e(i)$, $i = 2, \ldots, 5$ are degenerated: $e(2) = e(3) = e(4) = e(5) = \mathrm{diag}(1/2, 1/2, 1/2)$. We will construct four linearly independent solutions for the degenerated exponent. We set $\alpha_1 = 1/2+3\varepsilon, \alpha_2 = 1/2+2\varepsilon, \alpha_3 = 1/2+\varepsilon, \gamma_1 = 1/2+\varepsilon, \gamma_2 = 1/2+2\varepsilon, \gamma_3 = 1/2+3\varepsilon$. We put $y_i = x^{b^i_{e(2)}}$. Then, we have the following series containing the parameter ε.

$$\phi_{e(2)} = x^{e(2)} f_2(\varepsilon; y_1, y_2, y_3, y_4),$$
$$\phi_{e(3)} = x^{e(2)}(1 - 2\varepsilon \log y_4 + 2\varepsilon^2 (\log y_4)^2 + O(\varepsilon^3)) f_3(\varepsilon; y_1 y_4, y_2, y_4, y_3/y_4),$$
$$\phi_{e(4)} = x^{e(2)}(1 - 2\varepsilon \log y_2 + 2\varepsilon^2 (\log y_2)^2 + O(\varepsilon^3)) f_4(\varepsilon; y_2, y_2 y_2, y_3/y_2, y_4),$$
$$\phi_{e(5)} = x^{e(2)}(1 - 2\varepsilon \log(y_2 y_4) + 2\varepsilon^2 (\log(y_2 y_4))^2 + O(\varepsilon^3)) f_5(\varepsilon; y_2, y_1 y_2 y_4, y_4, y_3/(y_2 y_4)),$$

where $f_i(\varepsilon; z_1, z_2, z_3, z_4) = \sum_{m \in \mathbf{N}_0^4} \frac{[e(i)]_{u_-}}{[e(i)+u]_{u_+}} z^m$, $u = \sum_{j=1}^4 m_j b^j_{e(i)}$. We expand f_i in ε as $f_i^{(0)} + \varepsilon f_i^{(1)} + \varepsilon^2 f_i^{(2)} + O(\varepsilon^2)$. We note that all $\phi_{e(i)}$, $i = 2, 3, 4, 5$

stair	b_e^1	b_e^2	b_e^3	b_e^4
$\begin{pmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$
$\begin{pmatrix} * & * & 0 \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$
$\begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$
$\begin{pmatrix} * & 0 & 0 \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$
$\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ 0 & * & * \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$
$\begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & * & * \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

Table 1.3 Bases of Ker A

gives the same series when $\varepsilon = 0$, which implies $f_i^{(0)}$, $i = 2, 3, 4, 5$ are the same series. Therefore, we have

$$\begin{aligned}
\phi_{e(3)} - \phi_{e(2)} &= \varepsilon(x_{11}x_{22}x_{33})^{1/2}(\underline{-2f_2^{(0)} \log y_4} + f_3^{(1)} - f_2^{(1)}) + O(\varepsilon^2), \\
\phi_{e(4)} - \phi_{e(2)} &= \varepsilon(x_{11}x_{22}x_{33})^{1/2}(\underline{-2f_2^{(0)} \log y_2} + f_4^{(1)} - f_2^{(1)}) + O(\varepsilon^2), \\
\phi_{e(5)} - \phi_{e(2)} &= \varepsilon(x_{11}x_{22}x_{33})^{1/2}(\underline{-2f_2^{(0)} \log(y_2 y_4)} + f_5^{(1)} - f_2^{(1)}) + O(\varepsilon^2).
\end{aligned}$$

The coeffients of ε are solutions. Let us find the fourth solution. We have $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} f_{2345} = 0$, $f_{2345} = (\phi_{e(5)} - \phi_{e(2)}) - (\phi_{e(3)} - \phi_{e(2)}) - (\phi_{e(4)} - \phi_{e(2)})$. Therefore, the series f_{2345} starts with ε^2 and the coefficients ε^2 of f_{2345} is the fourth solution. It is

$$\begin{aligned}
&(x_{11}x_{22}x_{33})^{1/2}(\underline{2(\log y_2)(\log y_4) f_2^{(0)}} - 2\log(y_2 y_4) f_5^{(1)} + f_5^{(2)} \\
&\quad -2\log(y_2) f_3^{(1)} + f_3^{(2)} - 2\log(y_4) f_4^{(1)} + f_4^{(2)} + f_2^{(2)}).
\end{aligned}$$

Example 7 Let $\beta = (1, 2)$ and $A = A(0134)$. We set $w = (0, 1, 2, 0)$. Then, the Gröbner basis of I_A with respect to this order is

$$\underline{\partial_2 \partial_3} - \partial_1 \partial_4, \ \underline{\partial_1 \partial_3^2} - \partial_2^2 \partial_4, \ \underline{\partial_2^3} - \partial_1^2 \partial_3, \ \underline{\partial_3^3} - \partial_2 \partial_4^2.$$

Therefore, fake exponents are $v^{(1)} = (1/2, 0, 0, 1/2)$, $v^{(2)} = (1/4, 1, 0, 1/4)$, $v^{(3)} = (1/4, 0, 1, -1/4)$, $v^{(4)} = (-1, 2, 0, 0)$. $\phi^{(1)}, \phi^{(2)}$ and $\phi^{(3)}$ are convergent series solutions, but $\phi^{(4)} \equiv 0$. By examining $\mathrm{in}_{(-w,w)}(I_A)$, we can find two more solutions: x_2^2/x_1, x_3^2/x_4, [11], [49].

Series solutions with logarithms are constructed for a class of non-generic β's to apply for the mirror symmetry [26], [27], [47]. For non-homogeneous I_A, series solutions are divergent in most cases, but there are a class of series solutions which are convergent. They are studied in [35] and [15]. The Gevrey order of divergent series solutions is studied in [44], [16]. The notion of fully supported series solutions is introduced in [25]. Rational solutions of $H_A(\beta)$ are studied in [12]. Algebraic solutions of it are studied in [7].

1.4 $E(k,n)$

We fix two numbers k and n satisfying $n \geq 2k \geq 4$. Let α_j be generic parameters satisfying $\sum_{j=1}^n \alpha_j = n-k$. The *hypergeometric function* of type $E(k,n)$ is defined by the integral

$$\Psi(\alpha; z) = \int_C \prod_{j=1}^n (\sum_{i=1}^k u_{ij} s_i)^{\alpha_j} ds_2 \cdots ds_k,$$

where we put $s_1 = 1$ and u is a $k \times n$ matrix and C is a bounded $(k-1)$-cell in the hyperplane arrangement defined by $\prod_{j=1}^n \sum_{i=1}^k u_{ij} s_i = 0$ in the $(s_2, \ldots, s_k)$-space [17].

The hypergeometric function of type $E(k,n)$ is quasi-invariant under the action of complex torus $(\mathbf{C}^*)^n$ and the general linear group $GL(k) = GL(k, \mathbf{C})$. In fact, we have, for $h = \mathrm{diag}(h_1, \ldots, h_n) \in (\mathbf{C}^*)^n$ and $g \in GL(k)$,

$$\Psi(\alpha; uh) = \left(\prod_j h_j^{\alpha_j} \right) \Psi(\alpha; u), \quad \Psi(\alpha; gu) = |g|^{-1} \Psi(\alpha; u).$$

It follows from the quasi-invariant property and the integral representation that the function $\Psi(\alpha; u)$ satisfies a system of first order equations and a system of second order equations respectively.

Theorem 4 [17] *The function $f = \Psi(\alpha; u)$ satisfies*

$$\left(\sum_{i=1}^k u_{ip} \frac{\partial}{\partial u_{ip}} - \alpha_p \right) f = 0, \; p = 1, \ldots, n, \quad \left(\sum_{p=1}^n u_{ip} \frac{\partial}{\partial u_{jp}} + \delta_{ij} \right) f = 0, \; i,j = 1, \ldots, k,$$

$$\left(\frac{\partial^2}{\partial u_{ip} \partial u_{jq}} - \frac{\partial^2}{\partial u_{iq} \partial u_{jp}} \right) f = 0, \; i,j = 1, \ldots, k, p, q = 1, \ldots, n.$$

We call this system of equations $E(k,n)$.

When we restrict the hypergeometric system $E(k,n)$ to $u_{ij} = \delta_{ij}$ for

$1 \le i \le k, 1 \le j \le k$, we obtain the A-hypergeometric system associated to $A(k-1, n-k-1)$ and $\beta = (-\alpha_1 - 1, \ldots, -\alpha_k - 1, \alpha_{k+1} - 1, \ldots, \alpha_{n-1} - 1)$. We denoted it by $E'(k, n)$. Here, $u_{i,j+k}$ stands for the variable x_{ij} in Section 1.1.

If $\Psi(\alpha; u)$ is a solution of $E(k, n)$, then $\Psi(\alpha^s; u^s)$, $s \in \mathfrak{S}_n$ is also a solution. This $\mathfrak{S}_n$ symmetry leads us Kummer type relations [50]. The confluent $E(k, n)$ is geometrically studied and a general framework to derive Kummer type relations are given (see [30] and its references).

1.5 Contiguity relations

1.5.1 Contiguity relations

We note the relation in the Weyl algebra D

$$\left(\sum_{j=1}^{n} a_{ij}\theta_j - \beta_i \right) \partial_k = \partial_k \left(\sum_{j=1}^{n} a_{ij}\theta_j - \beta_i - a_{ik}) \right).$$

Since ∂_k commutes with $\Box_u$, we can see that if f is a solution of $H_A(\beta - a_k)$, then $\partial_k \bullet f$ is a solution of $H_A(\beta)$.

We consider the ideal B_k which is the intersection of $\mathbf{C}[s_1, \ldots, s_d]$ and the left ideal generated by ∂_k and $H_A(s)$ in $D[s_1, \ldots, s_d]$. When A is normal and I_A is homogeneous, this ideal can be expressed in terms of primitive support functions.

Theorem 5 *[37] The ideal B_k is the principal ideal generated by*

$$\prod_{\sigma \in S} \prod_{i=0}^{F_\sigma(a_k)-1} (F_\sigma(s) - i),$$

where S is a set of the facets of the convex hull of A for which $F_\sigma(a_k) > 0$ holds.

It follows from the theorem that if $\beta \notin V(B_k)$, then there exists an operator $Q_k \in D$ such that $Q_k \partial_k = 1 \mod H_A(\beta)$. The operators ∂_k and Q_k give contiguity relations for A-hypergeometric series.

The symmetry algebra introduced in [39] gives contiguity relations of A-hypergeometric system in a general framework. The ideal B_k is a special case of the b-ideal introduced in the paper.

1.5.2 Contiguity relations for $E'(k,n)$

We give a contiguity relation for $E'(k,n)$ following [43]. We use the variable u_{ij} instead of x_{ij} as in Section 1.4. Put

$$X_{pa} = -u_{ap} - \sum_{q=k+1}^{n} u_{aq} \sum_{i=1}^{k} u_{ip}\partial_{iq}. \tag{1.9}$$

Let $\varphi(\alpha; u)$ be a solution of the system $E'(k,n)$ with the set of parameters α.

Theorem 6 *[43]. We have* $\partial_{ap}\varphi(\alpha; u) = \varphi(\alpha + 1_a - 1_p; u)$, $X_{pa}\varphi(\alpha; u) = \varphi(\alpha - 1_a + 1_p; u)$ *and* $X_{pa}\partial_{ap} - (\alpha_p - 1)\alpha_a \in H_A(\beta)$

Introducing extra variables to hypergeometric series in several variables was done in the pioneering work of [29] to study contiguity relations. Contiguity relations for the Lauricella functions F_A, F_B, and F_C are derived with this idea and by utilizing the b-ideal B_k for them in [38].

1.5.3 Isomorphism among $M_A(\beta)$'s

We gave contiguity operators ∂_k and Q_k. If they exist, they give an isomorphism $\partial_k : M_A(\beta - a_k) \to M_A(\beta)$.

The question if $M_A(\beta)$ and $M_A(\beta')$ are isomorphic or not as left D-modules is a fundamental question. It was studied in [42, §4.4, §4.5] and a final answer was given in [39]. Let τ be a face of $\mathrm{pos}(A)$. Define

$$E_\tau(\beta) = \{\lambda \in \mathbf{C}(A \cap \tau)/\mathbf{Z}(A \cap \tau) \mid \beta - \lambda \in \mathbf{N}_0 A + \mathbf{Z}(A \cap \tau)\} \tag{1.10}$$

Theorem 7 *[39], [40, Th. 3.4.4] The left D-modules $M_A(\beta)$ and $M_A(\beta')$ are isomorphic if and only if $E_\tau(\beta) = E_\tau(\beta')$ for all faces τ of* $\mathrm{pos}(A)$.

The condition can be rewrited to a condition on the primitive integral supporting function when A is normal.

Theorem 8 *[39, Th 5.2] Assume A is normal and I_A is homogeneous. The left D-module $M_A(\beta)$ is isomorphic to $M_A(\beta')$ if and only if $\beta - \beta' \in \mathbf{Z}A$ and*

$$\{\sigma \mid \sigma \text{ is a facet and } F_\sigma(\beta) \in \mathbf{N}_0\} = \{\sigma \mid \sigma \text{ is a facet and } F_\sigma(\beta') \in \mathbf{N}_0\}. \tag{1.11}$$

1.6 Properties of A-hypergeometric equations

1.6.1 Rank formula and the Euler-Koszul complex

The holonomic rank $H_A(\beta)$ is the dimension of $R/(RH_A(\beta))$ as the vector space over the field of rational functions $\mathbf{C}(x_1,\ldots,x_n)$. Here, R is the ring of differential operators with rational function coefficients. The rank of $H_A(\beta)$ is equal to the normalized volume of A for generic β and we have the inequality $\operatorname{rank} H_A(\beta) \geq \operatorname{vol}(A)$, [2], [18], [42]. More precise discussion requires the Euler-Koszul complex [24], [6].

We assume that A is pointed in the subsection. For $\partial^v \in S_n = \mathbf{C}[\partial_1,\ldots,\partial_n]$, we define the A-multidegree of ∂^v by $-Av \in \mathbf{Z}^d$. We denote it by $\deg(\partial^u)$. Its i-th component is denoted by $\deg_i(\partial^u)$. This multidegree is naturally extended to the Weyl algebra D as $\deg(x^u\partial^v) = Au - Av$. Put $E_i = \sum_{j=1}^n a_{ij}\theta_j$. The multidegree of E_i is $\mathbf{0}$. The identity $\partial^v E_i = E_i\partial^v - \deg_i(\partial^v)\partial^v = (E_i - \deg_i(\partial^v))\,\partial^v$ is fundamental.

Let S_A be the ring $\mathbf{C}[\partial_1,\ldots,\partial_n]/I_A$ which is isomorphic to $\mathbf{C}[t^{a_1},\ldots,t^{a_n}] = \mathbf{C}[\mathbf{N}_0 A]$. We denote $D_n \otimes_{S_n} S_A \simeq D_n/(D_n I_A)$ by D_A. We consider the complex

$$\mathcal{K}_\bullet \;:\; 0 \overset{d_0}{\longleftarrow} D_A^{\binom{n}{0}} \overset{d_1}{\longleftarrow} D_A^{\binom{n}{1}} \overset{d_2}{\longleftarrow} \cdots \overset{d_{n-1}}{\longleftarrow} D_A^{\binom{n}{n-1}} \overset{d_n}{\longleftarrow} D_A^{\binom{n}{n}} \longleftarrow 0.$$

For A-homogeneous $a \otimes b \in D_A$, we define $(E_i - \beta_i)\circ(a\otimes b) = (E_i - \beta_i - \deg_i(a\otimes b))a\otimes b$. We denote the basis of $D_A^{\binom{d}{k}}$ by $e_{i_1,\ldots,i_k}$, $1 \leq i_1 < \cdots < i_k \leq d$. The boundary map d_k is defined by

$$D_A^{\binom{d}{k}} \ni (a\otimes b)e_{i_1\ldots,i_k} \mapsto \sum_{i_j\in\{i_1,\ldots,i_k\}} (E_{i_j}-\beta_{i_j})\circ(a\otimes b)(-1)^{j-1} e_{\{i_1,\ldots,i_k\}\setminus\{i_j\}} \in D_A^{\binom{d}{k-1}}. \tag{1.12}$$

The complex is called the Euler-Koszul complex over D_A.

The Euler-Koszul complex on D_A by $E_i - \beta_i$, $i = 1,\ldots,d$ is well-defined, because we have $(E_i - \beta_i)\circ(a \otimes \Box_u) = (a\Box_u(E_i - \beta_i))\otimes 1 = (a(E_i - \beta_i - \deg_i(\partial^{u_+}))\Box_u)\otimes 1 \equiv 0$. The homology group $\mathcal{H}_i(E-\beta; S_A) = H_i(\ker d_i/\operatorname{Im} d_{i-1})$ of the Euler-Koszul complex has a natural A grading by the A-multidegree. The 0-th homology group is nothing but $M_A(\beta)$. This leads us to more functorial object to study A-hypergeometric system, which is the Euler-Koszul homology for toric modules [24]. We fix $E-\beta$ and replace S_A by (A-)toric modules. We only present an example of toric modules. Let A be $A(0134)$ and $\tilde{A}$ be its saturation. Note that $n = 4$ and the multigrading is defined by A. We may suppose $\tilde{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 & 2 \end{pmatrix}$ and $S_{\tilde{A}} = D_5/I_{\tilde{A}}$. Then, we have a short exact sequence

$$0 \longrightarrow D_4 \otimes_{S_4} S_A \longrightarrow D_4 \otimes_{S_4} S_{\tilde{A}} \longrightarrow D_4 \otimes_{S_4} S_{\tilde{A}}/S_A \longrightarrow 0$$

All modules are A-graded and toric modules. $C = D_4 \otimes S_{\tilde{A}}/S_A$ has the support only at the degree $(1,2)$. We have $\mathcal{H}_0(E-\beta; D_4 \otimes S_{\tilde{A}}) \simeq D_5/x_5D_5 \otimes_{D_5} M_{\tilde{A}}(\beta) \simeq M_{\tilde{A}}(\beta)$ and $H_0(E-\beta; C) = 0$ (resp. $= D_4 \otimes [\partial_5]$) when $\beta \neq (1,2)$ (resp. $\beta = (1,2)$).

Theorem 9 *[24] Put $\mathbf{m} = \langle \partial_1, \ldots, \partial_n \rangle$, which is a maximal ideal in $S_n = \mathbf{C}[\partial_1, \ldots, \partial_n]$.*

1. *If k equals the smallest homological degree i for which $-\beta$ is a quasi degree of $H^i_{\mathbf{m}}(S_A)$, then the Euler-Koszul homology $\mathcal{H}_{d-k}(E-\beta; S_A)$ is non-zero rank and $\mathcal{H}_i = 0$ for $i > d-k$. Here, γ is called the quasi degree when γ is contained in the Zariski closure of the non-zero degrees of the homology group.*
2. *$H^i_{\mathbf{m}}(S_A) = 0$ holds for $0 \leq i < d$, if and only if S_A is Cohen-Macaulay.*
3. *The rank of $H_A(\beta)$ equals to the normalized volume of A if and only if β is not a quasi-degree of $H^i_{\mathbf{m}}(S_A)$.*

Put $\varepsilon_A = \sum a_i$. The degree $-\alpha + \varepsilon_A$ part of the local cohomology group is $H^{n-i}_{\mathbf{m}}(S_A)_{-\alpha+\varepsilon_A} = \mathrm{Hom}_{\mathbf{C}}\left(\mathrm{Ext}^i_{S_n}(S_A, S_n)_\alpha, \mathbf{C}\right)$.

Example 8 We consider the case $A = A(0134)$, $\varepsilon_A = (4,8)^T$. Construct A-graded resolution of R/I_A by Schreyer's method. Then, we have $\mathrm{Ext}^4 = 0$ and $\mathrm{Ext}^3 = \mathbf{C}$ at the degree $(5,10)$, which implies that $H^{4-3}_{\mathbf{m}} \neq 0$ at the degree $-(1,2)$. In fact, the rank of the system is 5 when $\beta = (1,2)$ and it is 4 when $\beta \neq (1,2)$.

1.6.2 Characteristic varieity and principal A-determinant

Let I be a left ideal in D. The initial ideal $\mathrm{in}_{(\mathbf{0},\mathbf{1})}(I)$ is the ideal in $\mathbf{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$ generated by the principal symbols of I. The ideal is called the characteristic ideal of I, and the zero set of the ideal in $\mathbf{C}^{2n}$ is called the characteristic variety of D/I and is denoted by $\mathrm{Ch}(D/I)$. The projection of $\mathrm{Ch}(D/I) \setminus V(\xi_1, \ldots, \xi_n)$ to $\mathbf{C}^n = \{x\}$ is called the singular locus of D/I and is denoted by $\mathrm{Sing}(D/I)$ (see, e.g., [42, p.36]).

Theorem 10 *[18], [20]*

1. *If $H_1(\mathrm{gr}_{(\mathbf{0},\mathbf{1})} \mathcal{K}_\bullet) = 0$, then the characteristic ideal of $H_A(\beta)$ is generated by $Ax\xi$ and $I'_A = I_A|_{\partial \to \xi}$. Here, we denote by $Ax\xi$ the ideal generated by $\sum_{j=1}^n a_{ij}x_j\xi_j$, $(i = 1, \ldots, d)$.*

2. If I_A is Cohen-Macaulay, then the first homology above vanishes.

Characteristic varieties and micro-characteristic varieties of $M_A(\beta)$ are combinatorially studied in [18], [44].

Let E_A be the pricipal A-determinant [21]. The projection of $V(\langle Ax\xi, I'_A\rangle)\setminus V(\xi_1, \ldots, \xi_n)$ to $\mathbf{C}^n$ is expressed as $V(E_A)$.

Theorem 11 *[21, p.300] The principal A-determinant for $A(k, k')$ $(k \leq k')$ is the product of the determiants of all $p\times p$ minors of the matrix (x_{ij}) where $1 \leq p \leq k$.*

Example 9 For $A = A(1, k' - 1)$, we have

$$E_A = \prod_{i=1}^{2}\prod_{j=1}^{k'} x_{ij} \prod_{1\leq j<j'\leq k'} \begin{vmatrix} x_{1j} & x_{1j'} \\ x_{2j} & x_{2j'} \end{vmatrix}.$$

The variety $V(E_A)$ is the singular locus of $H_A(\beta)$.

1.6.3 Reducibility and monodromy groups

We consider the set $\cup_\tau (\mathbf{Z}A + \tau)$ where the union is taken over all linear subspaces τ of $\mathbf{C}^d$ that form a boundary component of $\mathrm{pos}(A)$. The set is called the resonant parameters and is denoted by $\mathrm{Res}(A)$.

Let R be the ring of differential operators with rational function coefficients. We consider the left R-module $\mathbf{C}(x_1, \ldots, x_n)\otimes_{D_n} M_A(\beta) = R/(RH_A(\beta))$. If this module has a non-zero proper R-submodule, it is called reducible.

Theorem 12 *[8] When I_A is homogeneous and A is not a pyramid, $\mathbf{C}(x_1, \ldots, x_n)\otimes M_A(\beta)$ is reducible if and only if $\beta \notin \mathrm{Res}(A)$.*

An analog of this theorem holds without the homogeneous condition. See [45].

The irreducible quotients as D-modules of $M_A(\beta)$ are combinatorially discussed in [41].

The global monodromy groups are calculated for some interesting A's. See [31], [32], [51] for the case of $A(2, 2)$. See [34] for some of 3-dimensional Fano polytopes related to families of $K3$ surfaces. Recently, a general method to compute a subgroup of monodromy groups is proposed [9].

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