4

A-Hypergeometric Functions
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4.1 Introduction

The $A$-hypergeometric differential equations in the present form were introduced by Gel’fand, Zelevinsky, Kapranov [20] about 30 years ago. Series solutions are multivariable hypergeometric series defined by a matrix $A$. Although, there have been analogous approaches before their work, they found that affine toric ideals and their algebraic and combinatorial properties describe solution spaces of the $A$-hypergeometric differential equations, which also opened new research areas in commutative algebra, combinatorics and algebraic statistics. Several textbooks describe some topics of these new research areas, see [28], [54], [60] and their references. The book [47] and its reference give a comprehensive presentation on the $A$-hypergeometric equations at the year 2000, and the study has made a substantial progress after it. This chapter hopes to give a directory for these new advances as well as to describe fundamental facts. Applications of $A$-hypergeometric functions are getting broader. Early applications were mainly for period maps and the algebraic geometry. The interplay with the commutative algebra and combinatorics has been a source of new ideas for both of these and the theory of hypergeometric functions. Recent new applications are for the multivariate analysis in statistics.

This chapter starts with systems of differential equations and examples of matrices $A$ which define $A$-hypergeometric functions. We briefly describe an interplay with combinatorics, Gröbner basis, and software systems. Series solutions are discussed with some important examples. In the next to the last section, we illustrate that contiguity relations, isomorphisms, holonomic ranks, reducibility conditions have simple and beautiful descriptions. Recent new applications to statistics will be briefly discussed in the last section.

4.2 $A$-hypergeometric equations

Let $A$ be a $d \times n$ matrix with integer entries $a_{ij}$. We denote by the point $a_j$ in $\mathbb{Z}^d$ the $j$-th column vector of $A$. We suppose that $a_j$’s generate the lattice $\mathbb{Z}^d$, in other words, we have $\sum_{j=1}^n a_j = \mathbb{Z}^d$. Let $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{C}^d$ be a vector of parameters. The ring of differential operators

$$C(x_1, \ldots, x_n, \partial_1, \ldots, \partial_d), \quad x_jx_j = x_j \partial_j, \quad \partial_j \partial_j = \partial_j \partial_j, \quad \partial_j x_j = x_j \partial_j + \partial_j$$
is denoted by $D$ or by $D_n$. We use the multi-index notations $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$. The action of $x^\alpha \partial^\beta$ to a function $f(x)$ is defined by $x^\alpha \partial^\beta \cdot f(x) = x^\alpha \frac{\partial^\beta f(x)}{\partial x^\beta}$.

**Definition 4.2.1** [20] We call the following system of differential equations an $A$-hypergeometric system or a GKZ hypergeometric system:

$$(E_i - \beta_i) \cdot f = 0, \quad \text{where } E_i - \beta_i = \sum_{j=1}^n \alpha_{ij} x_j \partial_j - \beta_i, \quad (i = 1, \ldots, d)$$

$$\square_u \cdot f = 0, \quad \text{where } \square_u = \prod_{i \in \{i \in [1, \ldots, n] : \mu_i > 0\}} \partial_i^{\alpha_i} - \prod_{i \in \{i \in [1, \ldots, n] : \mu_i < 0\}} \partial_i^{-\alpha_i}$$

with $u \in \mathbb{Z}^n$ running over all $u$ such that $A u = 0, u \neq 0$.

We denote by $I_A$ the ideal in $S_n = C[\partial_1, \ldots, \partial_n]$ generated by $\square_u$ for all $u \in \mathbb{Z}^n$ such that $A u = 0$. This is an affine toric ideal, see [54]. The left ideal in $D$ generated by $E_i - \beta_i$, $i = 1, \ldots, d$ and $I_A$ is denoted by $H_A(\beta)$ and is called the A-hypergeometric ideal. The quotient left D-module $D/H_A(\beta)$ is denoted by $M_A(\beta)$ and called the A-hypergeometric D-module. When the points $\alpha_i$’s lie on a hyperplane which does not pass through the origin, the D-module $M_A(\beta)$ is regular holonomic [32]. Such matrix $A$ is called a configuration matrix.

Several invariants of the D-module can be described in terms of the set of points $\{\alpha_1, \ldots, \alpha_n\}$ as we will see later. We also denote the set of points by A in this chapter; the symbol $A$ stands for a matrix or a set of points. When the meaning of $A$ is clear in the context, we do not say which it means. $N_0 A$ and $Z A$ mean $\sum_{i=1}^n N_0 \alpha_i$ and $\sum_{i=1}^n Z \alpha_i$ respectively.

Although the $A$-hypergeometric system can be defined for any matrix $A$, there are nice classes of matrices $A$ (or sets of points $\alpha_i$) which lead to systems having well-known special functions as solutions. Let us introduce some of them. Take integers $k$ and $k'$ satisfying $1 \leq k \leq k'$. Put $e_1 = (1, 0, \ldots, 0)^T \in \mathbb{Z}^{k+1}$, $e_2 = (0, 1, 0, \ldots, 0)^T \in \mathbb{Z}^{k+1}$, $e_3 = (1, 0, 1, 0, \ldots, 0)^T \in \mathbb{Z}^{k+1}$, ..., and $e_{k+1} = (1, 0, \ldots, 0)^T \in \mathbb{Z}^{k+1}$. Let $A(k, k')$ be a $(k + k' + 1) \times (k + k' + 1)$ matrix of which columns consist of $p e_i \oplus e_j^T$ where $p$ is the projection to the first $k + k' + 1$ coordinates (the projection which removes the last coordinate). $A(1, 1)$, $A(1, 2)$, $A(2, 2)$ are given in Table 4.1.

The columns of $A(k, k')$ generate $\mathbb{Z}^{k+k'+1}$ and they lie on the hyperplane $\sum_{j=1}^{k+1} y_j = 1$ in $\mathbb{R}^{k+k'+1}$. Since the convex hull of $e_1, \ldots, e_{k+1}$ is the simplex $\Delta_k$ and that of $e_1', \ldots, e_{k+1}'$ is the simplex $\Delta_r$, we call this A-hypergeometric system $\Delta_k \times \Delta_r$-hypergeometric system or the hypergeometric system $E'(k + 1, k + k' + 2)$. The latter naming comes from a relation of this system with the hypergeometric system $E(k, n)$ (Section 4.5). For this hypergeometric system, we often denote the variable $x_p$ by $x_j$ where $p = (i - 1)k' + (j - 1) + 1$. This double index notation is convenient. We also regard a vector of length $(k + 1)(k' + 1)$ as a matrix under this double index notation. For example, for a vector $e$, the condition $A(k, k') e = \beta$ means that the row sums and the column sums of $e$ expressed in terms of the $(k + 1) \times (k' + 1)$ matrix are $(\beta_1, \ldots, \beta_{k+1})$ and $(\beta_{k+2}, \ldots, \beta_{k+k'+1})$, $\sum_{i=k+1}^{k+k'+1} \beta_i - \sum_{i=k+1}^{k+k'+1} \beta_i$, respectively.

The ideal $I_A$ for $A = A(k, k')$ is generated by

$$\partial_q \partial_p - \partial_p \partial_q, 1 \leq i \leq j \leq k + 1, 1 \leq p < q \leq k' + 1.$$
4.2 A-hypergeometric equations

\[ A(0,1) = \begin{pmatrix} x_{11} & x_{12} & x_{21} & x_{22} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad A(0,2) = \begin{pmatrix} x_{11} & x_{12} & x_{21} & x_{22} \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \]

\[ A(2,2) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad A(2,3) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \]

\[ A(0|34) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \]

\[ A(0|34) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \]

\[ \begin{array}{c} \theta_1 + \theta_2 - \beta_1 \cdot x^F(z) = (v_{11} + v_{12} - \beta_1)x^F(z) \\ \theta_2 + \theta_2 - \beta_2 \cdot x^F(z) = (v_{21} + v_{22} - \beta_2)x^F(z) \\ \theta_1 + \theta_1 - \beta_3 \cdot x^F(z) = (v_{11} + v_{21} - \beta_3)x^F(z) \end{array} \]

Table 4.1 A

More precisely, it is the reduced Gröbner basis with respect to the graded reverse lexicographic order \( > \) with \( \partial_{1,1} > \partial_{1,2} > \cdots > \partial_{1,k} > \partial_{2,1} > \cdots \) [54, Prop. 5.4]. For any \( A \), generators of \( I_A \) can be obtained by a Gröbner basis computation [54, Alg. 4.5]. Generators of \( I_A \) is called the Markov basis in algebraic statistics. There are theoretical and computational efforts to find explicit Markov bases. We have a database of Markov bases for several matrix \( A \) ([1] or [25]).

The matrix \( A(1,k') \) stands for the Lauricella function \( F_{10} \) of \( k' \) variables (see Example 4.4.4 for the correspondence). In particular, when \( k' = 1 \), it stands for the Gauss hypergeometric function. As we will see later, the correspondence can be described in terms of series solutions or integral representations. In a more sophisticated context, a categorical correspondence is given in [59]. Here, we explain an elementary correspondence in the case of Gauss hypergeometric equation \( (A(1,1) \text{ case}) \) as an introduction. Put \( A = A(1,1) \) and suppose that \( f(z) = x^F(z), x^F = \prod_{j=1}^{2} x_j^{y_j}, z = \frac{x_1 x_2}{x_2 x_0}, \) is a solution of the \( A \)-hypergeometric system \( H_A(\beta) \).

We denote \( x_j \partial_j \) by \( \theta_j \), then we have

\[ (\theta_1 + \theta_2 - \beta_1) \cdot x^F(z) = (v_{11} + v_{12} - \beta_1)x^F(z) \]

\[ (\theta_2 + \theta_2 - \beta_2) \cdot x^F(z) = (v_{21} + v_{22} - \beta_2)x^F(z) \]

\[ (\theta_1 + \theta_1 - \beta_3) \cdot x^F(z) = (v_{11} + v_{21} - \beta_3)x^F(z) \]

by the relation \( \theta_j x^r = x^r(\theta_j + v_j) \) in the ring of differential operators, and \( \theta_i \cdot F(z) = z F'(z), \theta_j \cdot F(z) = -z F'(z), (i \neq j) \). These are equal to 0 by \( (E_i - \beta_i) \cdot f = 0 \), which implies \( AV = \beta \) where \( v = (v_{11}, v_{12}, v_{21}, v_{22})^T \) and \( \beta = (\beta_1, \beta_2, \beta_3)^T \). Since, \( \text{Ker}(A : \mathbb{Z}^{d} \to \mathbb{Z}^{3}) = \mathbb{Z}(1, -1, 1)^T \), we can show that the toric ideal \( I_A \) is generated by \( \partial_2 \partial_2 - \partial_1 \partial_2 \). Note that the basis of the kernel is not enough to generate the toric ideal \( I_A \) in the case of \( A(1,k'), k' > 1 \). Act the operator \( x_1 x_2 (\partial_1 \partial_2 - \partial_1 \partial_2) = \theta_1 \theta_2 - z \theta_1 \theta_2 \) to the function \( x^F(z) \), then we...
have
\[ x^r((-\theta_1 + v_{12})(-\theta_2 + v_{21}) - z(\theta_1 + v_{11})(\theta_2 + v_{22})) \cdot F(z) = 0 \]
where \( \theta_k = z \partial_k \). When \( v_{12} = 0 \) or \( v_{21} = 0 \), this is the Gauss hypergeometric equation. Other \( v_{ij} \)'s are determined by \( Av = \beta \).

Let us give matrices \( A \) for other Lauricella functions (see this volume, Chapter 3 on these functions). Let \( e_0, e_1, \ldots, e_{2m} \) be the standard basis of \( \mathbb{Z}^{2m+1} \). Put \( A = \{e_0, e_1, \ldots, e_{2m}, e_0 + e_1 - e_{m+1}, e_0 + e_2 - e_{m+2}, \ldots, e_0 + e_{m} - e_{2m}\} \). Then \( A \) is a \((2m+1) \times (3m+1)\) matrix, which stands for the Lauricella function \( F_A \) of \( m \) variables [42]. They lie on the hyperplane \( y_0 + y_1 + \cdots + y_{2m} = 1 \) in \( \mathbb{R}^{2m+1} \). We denote the matrix by \( A(F_A, m) \). The associated toric ideal \( I_A \) is generated by \( \partial_0 \partial_{j} - \partial_{m+j} \partial_{2m+j}, \ j = 1, \ldots, m \). Here, we use the variables \( u_0, u_1, \ldots, u_m \) as independent variables instead of \( x_1, \ldots, x_n \). When \( m = 2 \), it is the Appell function \( F_2 \); the matrix is given in Table 4.1.

Let \( e_1, \ldots, e_{m+1}, e_{m+2} \) be the standard basis of \( \mathbb{Z}^{m+2} \). Put \( A = \{e_1 + e_{m+2}, e_2 + e_{m+2}, \ldots, e_{m+1} + e_{m+2}, -e_1 + e_{m+2}, -e_2 + e_{m+2}, \ldots, -e_{m+1} + e_{m+2}\} \). Then \( A \) is an \((m+2) \times (m+1)\) matrix, which stands for the Lauricella function \( F_C \) of \( m \) variables [42]. They lie on the hyperplane \( z_{m+2} = 1 \) in \( \mathbb{R}^{m+2} \). We denote the matrix by \( A(F_C, m) \). Note that the lattice generated by the columns of \( A(F_C, m) \) is a proper sublattice of \( \mathbb{Z}^{m+2} \). Then, we need to regard the sublattice as \( \mathbb{Z}^{m+2} \). The associated toric ideal \( I_A \) is generated by \( \partial_j \partial_{j} - \partial_{m+j} \partial_{2m+j}, \ j = 1, \ldots, m \). Here, we use the variables \( u_1, \ldots, u_m, u_{m+1}, u_{m+2}, \ldots, u_{m+(m+1)} \) as independent variables. When \( m = 2 \), it is the Appell function \( F_4 \); the matrix is given in Table 4.1. The notion of binomial \( D \)-modules is proposed and studied in [15]. Binomial \( D \)-modules are generalizations of \( A \)-hypergeometric equations and they fit to study Appel-Horn equations and their generalizations to several variables in algebraic methods.

\( A \)-hypergeometric systems associated to smooth fano polytopes have importance in studies of period maps for \( K3 \) and Calabi-Yau varieties (see, e.g., [30], [31], [53] and their references). For example, the matrix \( A(P_4) \) [38] appears in this context.

Let us discuss on integral representations of solutions of \( A \)-hypergeometric equations. Suppose that we are given \( n_k \) points \( \alpha_i \in \mathbb{Z}^n \). We divide these points into \( k \) groups and construct \( m \times n_k \) matrices \( A_1 = (\alpha_1, \ldots, \alpha_{n_1}), \ldots, A_k = (\alpha_{n_{k-1}+1}, \ldots, \alpha_{n_k}) \). For each group, define the polynomial \( f_j(x, t) \) by \( f_j^{(n_{j+1})} \delta^{(n_j)} \) where \( \delta^{(n_j)} = \prod_{j=1}^{n_j} f_j \). Note that we use the multi-index notation for \( t = (t_1, \ldots, t_m) \). Take complex numbers \( \alpha_j, \gamma = (\gamma_1, \ldots, \gamma_m) \). We consider the integral
\[ \Phi(\alpha, \gamma; x) = \int_C \prod_{j=1}^{k} f_j(x, t)^{\alpha_j} t^\gamma dt_1 \cdots dt_m, \]
where \( C \) is any twisted \( m \)-cycle defined for \( \prod_{j=1}^{k} f_j(x, t)^{\alpha_j} \). Define \((k + m) \times n_k \) matrix \( A(A_1, \ldots, A_k) \) by the Cayley matrix in Figure 4.1. The function \( \Phi(\alpha, \gamma; x) \) satisfies the \( A \)-hypergeometric system \( H_A(\beta) \) for \( A = A(A_1, \ldots, A_m) \) and \( \beta = (\alpha_1, \ldots, \alpha_k, -\gamma_1 - 1, \ldots, -\gamma_m - 1)^T \). When \( f_j \) are linear with respect to the variable \( t \), we call the function \( \Phi \) hypergeometric function for hyperplane arrangements. Note that when \( A_1 = \ldots = A_k = A_k \), in other words, all \( A_i \)'s are equal to \( k \times (k+1) \) matrix \( E_k \oplus 0 \) in Figure 4.2, we have \( A(A_1, \ldots, A_k) = A(k, k') \).
4.3 Some definitions from combinatorics, polytopes and Gröbner basis

As to studies on these hypergeometric functions in terms of twisted cohomology groups, see [5], [6], [3], [40].

When the toric ideal \( I_\Lambda \) is not a homogeneous ideal (the case that \( \alpha_i \)'s do not lie on an affine hyperplane), the integral

\[
\Phi(\gamma; x) = \int_C \exp \left( \sum_{i=1}^{n} x_i t_i^{\alpha_i} \right) y^\gamma dt_1 \cdots dt_d
\]

satisfies \( H_\Lambda(\beta) \) with \( \beta = (-\gamma_1 - 1, \ldots, -\gamma_d - 1)^T \) for any rapid decay \( d \)-cycle under some conditions [17].

4.3 Some definitions from combinatorics, polytopes and Gröbner basis

The matrix \( A \) is said to be pointed when \( a_1, \ldots, a_d \) lie in a single open half-space. For example, \( A = (-1, 1) \) is not pointed and all \( A \)'s in Table 4.1 are pointed. The set of points \( A \) is called normal, when \( A \) satisfies \( (\sum R_{\sigma_0} a_i) \cap \mathbb{Z}^d = \sum \mathbb{Z} a_i \).

For a facet \( \sigma \) of the poset \( A = R_{\sigma_0} A \), \( F_\sigma \) is a linear function on \( RA = \mathbb{R}^d \) uniquely determined by the conditions:
1. \( F_\sigma(ZA) = Z \), 2. \( F_\sigma(a_i) \geq 0 \) for all \( i = 1, \ldots, n \), 3. \( F_\sigma(a_i) = 0 \) for all \( a_i \in \sigma \).

We call \( F_\sigma \) the primitive integral support function of \( \sigma \).

For \( \Delta_\Lambda \times \Delta_\Lambda \), embedded in \( \mathbb{R}^{k+1} \times \mathbb{R}^{k+1} = \{(x_1, \ldots, x_{k+1}; y_1, \ldots, y_{k+1})\} \), the support functions are \( x_i \) and \( y_j \). When we project the points to \( \mathbb{R}^{k+1} \times \mathbb{R}^{k} \), the primitive integral support functions are \( x_i \) (\( i = 1, \ldots, k + 1 \)), and \( y_j \) (\( j = 1, \ldots, k' \)), and \( 1 - \sum_{j=1}^{k'} y_j \).
The supporting functions for \( A(F, x^m) \) are \( s_j, s_j + s_{m+j}, 1 \leq j \leq m \) and \( s_0 + \sum_{j \in J} s_{m+j} \), \( J \subseteq [1, m] \) where \( \{s_i\} \) is the dual basis of \( \{e_i\} \). Those for \( A(F, x^m) \) are \((1/2)(s_{m+2} + \sum_{j \in J} s_j - \sum_{j \notin J} s_j)\), \( J \subseteq [1, m + 1] \) \cite{42}.

Let \( Z_\Lambda \) be the lattice generated by the columns of \( \Lambda \). Let us set the volume of the convex hull \( U \) of the lattice base and the origin to 1. The volume of polytopes in \( RA \) normalized with the \( U \) is called the normalized volume. The normalized volume of the convex hull of \( A \) and the origin is denoted by \( \text{vol}(A) \). The normalized volume of \( A(k, k') \) is known to be equal to \((\binom{k+k'}{k})\).

For given \( A \), it can be evaluated by geometry software systems like polymake, or by computer algebra systems which use a formula \( \deg(I_\Lambda) = \text{vol}(A) \).

An interplay of theory and computation has been indispensable in the study of \( A \)-hypergeometric systems. A lot of algorithms and software systems have been developed to study it and its related areas. The text book \cite{28} gives a comprehensive introduction to them including mathematical software systems. Here, we give a few examples.

**Example 4.3.1** Macaulay2 \cite{24} commands to evaluate the volume (the degree) of \( A(0134) \). Here, \( o_5 \) is \( I_\Lambda \).

```
loadPackage "FourTiTwo"
M=matrix "1,1,1,1; 0,1,3,4"
l=toricGroebner(M,R)
o5 = ideal (b3 - a2*c, b*c - a*d, -a*c2 + b2*d, c3 - b*d2)
degree(I)
o6 = 4
```

For a given weight vector \( w \in \mathbb{R}^d \) (weights below), consider points \( \{(w_i, w_j)\} \) in \( \mathbb{R}^{d+1} \) and the convex hull of them. The projection of the convex hull to the first \( d \) coordinates naturally induces a triangulation of the set of points \( A \) for a generic weight \( w \), which is called a regular triangulation \cite{23, 54, Chapter 8, cite{5.5.2}dojo}. We compute a regular triangulation of \( \Delta_1 \times \Delta_2 \) for \( w = (4, 2, 0, 10, 8, 6) \) by the computer algebra system Macaulay2

```
11 : loadPackage "FourTiTwo"
12 : M=matrix "1,1,0,0,0,0; 0,1,0,1,0,0; 0,1,0,0,0,0"
13 : R=QQ[x11,x12,x13,x21,x22,x23, MonomialOrder=>{Weights=>{4,2,0,10,8,6}}]
14 : l=toricGroebner(M,R)
o4 = ideal (x13^2*x11 - x11^2*x23, x12^2*x21 - x11^2*x22, x13^2*x12 - x12^2*x23)
15 : J=leadTerm(I)
o5 = ideal (x13, x12, ideal (x13, x12))
o6 = 4
```

By taking the complements of the indices of each associated primes, we get a regular triangulation \((11, 12, 13, 23), (11, 21, 22, 23), (11, 12, 22, 23) \). We note that computer experiments have played important roles in studies of \( A \)-hypergeometric systems.

### 4.4 A-hypergeometric series

Let us introduce \( A \)-hypergeometric series following \cite{20} and \cite{47, 3.4}. Let \( v = (v_1, \ldots, v_d) \) be a vector in \( \mathbb{C}^d \) and \( u = (u_1, \ldots, u_d) \) a vector in \( \mathbb{Z}^d \). We decompose \( u \) into positive and negative parts, \( u = u_+ - u_- \), where \( u_+ \) and \( u_- \) are non-negative vectors with disjoint support. Consider
4.4 A-hypergeometric series

the following two scalars in $C$, which can be expressed by falling factorials:

$$[v]_{u_v} = \prod_{i=0}^{v} \prod_{j=1}^{u_i} (v_i - j + 1),$$

$$[u + v]_{u_v} = \prod_{i=0}^{v} \prod_{j=1}^{u_i} (u_i + v_i - j + 1) = \prod_{i=0}^{v} \prod_{j=1}^{u_i} (v_i + j).$$

For example, when $v = (v_1, v_2, 0, v_4)$ and $u = (-2, 2, 2, -2)$, we have $\frac{[v]_{u_v}}{[u + v]_{u_v}} = v(v_i + (u_i + v_i - 1)).$

Note that when $v \in (C \setminus \mathbb{Z}_0)^2$, we have $[u + v]_{u_v} \neq 0$. We set $L = \text{Ker}(\mathcal{Z}^u \xrightarrow{A} \mathbb{Z}^d)$.

**Theorem 4.4.1** Suppose that $v \in (C \setminus \mathbb{Z}_0)^2$ and $A v = \beta$. Then the formal series

$$\Phi_v := \sum_{u \in L} \frac{[v]_{u_v}}{[v + u]_{u_v}} \cdot x^{vu}$$

is well-defined and is a formal solution of $H_A(\beta)$.

As to the proof of this theorem, see [47, Prop. 3.4.1]. We call the formal series the A-hypergeometric series in the falling factorial form.

Let us introduce another expression of the series. We set $\Gamma(u + v + 1) = \prod_{i=1}^{v+1} \Gamma(u_i + v_i + 1)$ and when $u_i + v_i \in \mathbb{Z}_0$ for an $i$, we define $1/\Gamma(u_i + v_i + 1) = 0$. Under this convention, we have $\frac{1}{\Gamma(v_i + u_i + 1)} = \frac{1}{\Gamma(v_i + u_i)} \cdot \frac{1}{v_i + u_i + 1}$ for $v, u \in L$ and $v \in (C \setminus \mathbb{Z}_0)^2$ (use $\Gamma(x + i) = \Gamma(x)(x)_i$, $\Gamma(x + i + 1) = \Gamma(x + i)(-1)^i/(x)_i$). Define

$$\Phi_v := \sum_{u \in L} \frac{1}{\Gamma(u + v + 1)} x^{vu}.$$  

(4.4.2)

Then, we have $\Phi_v = \frac{1}{|v|^2} \Phi_v$ when none of $v_i$ is negative integer. We call the formal series the A-hypergeometric series in the gamma function form. Note that when $v_i$ is a negative integer, two series are different. For example, if $v_i = -1$ and $u_i = 1$, then we have $[u_i + v_i]_{u_v} = 0$ and $\Phi_v$ is not well-defined, but $\Gamma(u_i + v_i + 1) = 1$. When $v = (1, -2, 3, 0)$ and $L = \mathbb{Z}(1, -1, -1, 1)$, $\Phi_v$ is a non-zero polynomial, but $\Phi_v$ is identically 0.

For a given weight vector $w \in \mathbb{Z}^d$ and $\ell \in I_k$, $\text{in}_w(\ell)$ is the sum of the highest $w$-order terms in $\ell$. The ideal in $S_\ell$ generated by $\text{in}_w(\ell)$, $\ell \in I_k$ is denoted by $\text{in}_w(I_k)$ and is called the initial ideal of $I_k$ [54]. Let $C$ be the Gröbner cone of $I_k$ for a generic weight vector $w$. Two weight vectors $w$ and $w'$ are equivalent with respect to the ideal $I$ when $\text{in}_w(I) = \text{in}_{w'}(I)$. Fix $w$. The closure of the equivalent class of the weight vector $w$ is called the Gröbner cone for $w$ [54, Chapter 1], [47, 2.1], [28, 5.3.2]. The initial ideal $\text{in}_w(I_k)$ does not change by definition when $w'$ runs over $C$ [54], [47, Chap. 2]. For a series $f$ with a support on a translate of the dual cone $C^*$, for which we may assume $(w, C^* \setminus \{0\}) > 0$, the starting term of $f$ is the sum of the lowest $w$ weight terms in $f$ with respect to $w$. If $f$ is a solution of $\ell \bullet f = 0$, $\ell \in D$, then the starting term of $f$ is a solution of $\text{in}_{-w} \circ \ell(\ell)$, which is the sum of the highest order terms in $\ell$ with respect to the weight $(-w, w)$ where $-w$ (resp. $w$) stands for $x$ (resp. $\partial$). This observation gives us the following method [47, Chapter 2] to find series solutions of $H_A(\beta)$: (1) determine the initial ideal $\text{in}_{-w} \circ H_A(\beta)$, (2) solve it to determine the starting terms, (3) extend the starting terms to series solutions.
Theorem 4.4.2 For generic $\beta$, the initial ideal $\text{in}_{\prec w}(\mathcal{H}_A(\beta))$ is generated by $E_i - \beta_i$, $1 \leq i \leq d$ and $\text{in}_w(I_A)$.

We note that the proof of [47, Th. 3.1.3] needs to be corrected to utilize the homogenized Weyl algebra. We suppose that $I_A$ is a homogeneous ideal and take a generic weight vector $w$ such that $\text{in}_w(I_A)$ is a monomial ideal. Let $G$ be the reduced Gröbner basis of $I_A$ with respect to the order $\prec_w$ [54]. We consider the system of differential equations

$$
(E_i - \beta_i) \ast s = 0, \quad i = 1, \ldots, d, \quad \text{and} \quad \ell \ast s = 0, \quad \ell \in \text{in}_w(G) \quad (4.4.3)
$$

Let $\nu$ be a solution of algebraic equations

$$
Av = \beta, \quad \prod_{i=1}^n v_i(v_i - 1) \cdots (v_i - e_i + 1) = 0 \quad \text{for} \quad \partial^\nu \in \text{in}_w(G) \quad (4.4.4)
$$

It is called a fake exponent. We note that the fake exponents can be expressed in terms of standard pairs of the monomial ideal $\text{in}_w(I_A)$ [47, 3.2]. When $\beta$ are generic, there are linearly independent vol($A$) solutions of (4.4.3) of the form $s = x^\nu = \prod_{i=1}^n v_i^\nu_i$ where $\nu$ is a fake exponent and they span the solution space over $\mathbb{C}$ when $\nu$ runs over the fake exponents.

Theorem 4.4.3 [20], [47, Th 3.4.2] If $\nu$ is a fake exponent and $\nu \in (\mathbb{C} \setminus \mathbb{Z})^d$, then $\Phi_\nu$ is a formal solution of $\mathcal{H}_A(\beta)$ with the support in $\nu + (C^r \cap L)$.

Note that Gel’fand, Kapranov, Zelevinsky constructed series solutions by regular triangulations of $A$ [20]. Our construction differs with their construction, but it is related with the construction via the theorem [54, Th 8.3] “ $\mathcal{H}_A(I_A)$ is the Stanley-Reisner ideal for the regular triangulation by $w$”. The function $\Phi_\nu$ converges when $(-\log |x_1|, \ldots, -\log |x_n|)$ lies in a translate of the secondary cone attached to the regular triangulation.

For a good class of $A$-hypergeometric functions, more explicit form of $A$-hypergeometric series is known as we will describe. For $A = A(p, q - 1)$, the stair case Gröbner basis in [54, Prop.5.4] gives series solutions. A sequence of indexes $(1, 1, \ldots, (p, q))$ is called a stair if $(i, j)$ is an element of the stair and is not $(p, q)$, then the next element of $(i, j)$ is either $(i + 1, j)$ or $(i, j + 1)$ (see Table 4.2).

The initial ideal of $I_A$ for the reverse lexicographic order is generated by $\partial_{ij}$, $1 \leq i < j \leq p$, $1 \leq k < \ell \leq q$ [54, Prop.5.4]. We can obtain the fake exponents from this initial ideal by solving (4.4.4). It is known that there is a one-to-one correspondence between the roots of the system of equations and the stairs. For a given stair $S$, the system has a unique solution such that $v_{ij} = 0$ for $(i, j) \notin S$. In other words, the support of each exponent has the form of the stair for generic $\beta$. In the sequel, we use $e$ rather than $\nu$ to denote exponents. The support of the series solution standing for the exponent $e$ has the form

$$

 e + L', \quad L' = \sum_{(i,j) \in \text{supp}(e)} Z_{ij} b_{e,i,j} \quad \text{where} \quad b_{e,i,j} \text{ is an element of Ker } A \text{ such that } (i, j)\text{-th element of } b_{e,i,j} \text{ is } 1 \text{ for } (i,j) \in \text{supp}(e) \text{ and } (i', j')\text{-th element is } 0 \text{ for } (i', j') \in \text{supp}(e) \setminus \{(i,j)\}.
$$


Let us see some $A$’s of which series solutions can be written in terms of solutions of Lauricella systems (this volume, Chapter 3, section 3.4).

**Example 4.4.4** We put $A = A(1, N - 1)$ in this example. Let $\alpha, b_1, \ldots, b_{N-1}, c$ be (generic) constants. Put $b_N = \alpha + 1 - c$ and

$$e(k) = \begin{pmatrix} -b_1 & \cdots & -b_{k-1} & -\sum_{j=1}^{k-1} b_j + a & 0 & \cdots & 0 \\
0 & \cdots & 0 & -b_{k+1} & \cdots & -b_N \
\end{pmatrix},$$

which is the fake exponent standing for the $k$-th stair. Put $m = (m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_N)$, $m_k = -\sum_{j=1}^{k-1} m_j + \sum_{j=k+1}^{N} m_j$, and $z_j = \frac{z_{j+1}}{z_j^{k+1}}$ for $1 \leq j \leq N$. Note that $z_N = 1$. Define a series $\phi(e;z)$ by

$$\sum_{m \in \mathbb{Z}_{m_1}^{m_N}} \frac{\prod_{j=1}^{k-1} [e_{j,m_j}] \prod_{j=k+1}^{n} [e_{j,m_j}]}{\prod_{j=1}^{k-1} m_j! \prod_{j=k+1}^{n} m_j!} c_m \prod_{j=1}^{k-1} (z_j z_k^{-1})^{m_j} \prod_{j=k+1}^{n} (z_j z_k^{-1})^{m_j}$$

(4.4.5)

where $e = e(k)$, $c_m = [e_{1,k}] m_k / [e_{2,k} - m_k] m_k$ when $m_k > 0$, and $c_m = [e_{2,k}] - m_k / [e_{1,k} - m_k] - m_k$ when $m_k < 0$, and $c_m = 0$ when $m_k = 0$. For $\beta = (\sum b_i + c - 1, -\alpha, -b_1, \ldots, -b_{N-1}, c - 1 - \alpha)$, the function $x^{(k)} \phi_e(e; k)$, $1 \leq k \leq N$ is a solution of $H_A(\beta)$ and $x^{(k)} \phi_e(e; k)$ is a solution of the Lauricella system $E_D(\alpha, (b, c))$. The series $\phi_e(e(N); z)$ is the Lauricella’s $F_D$. The series $\phi_e$’s have a common domain of convergence $|z_1| < \cdots < |z_N| < 1$.

**Example 4.4.5** The function

$$u_0 \prod_{j=1}^{m} u_j b_j \prod_{j=1}^{m} u_j c_j - 1, \ldots, c_m \prod_{j=1}^{m} u_j a_j, \ldots, u_{m+1} u_{2m+1} \prod_{j=1}^{m} u_j a_j$$

(4.4.6)

is a solution of $H_{A(1, m)}(\beta)$, $\beta^T = (-\alpha, -b_1, \ldots, -b_{m-1}, c_{m-1}, c_{m-2}, \ldots, c_1 - 1, c_m - 1)$ when $f_A$ is a solution of the Lauricella’s $E_A(\alpha, (b, c))$. Any classical solution of $H_{A(1, m)}(\beta)$ can be expressed as (4.4.6).

**Example 4.4.6** The function

$$u_{m+1} u_{m+2} \cdots u_{m+n} \prod_{j=1}^{n} f_c \left( a, b, c_1, \ldots, c_m, \frac{u_{j+1} u_{j+2} \cdots u_{m+n}}{u_{j+1} a_j} \right)$$

(4.4.7)

is a solution of $H_{B(1, m)}(\beta)$, $\beta^T = (1 - c_1, \ldots, 1 - c_m, -\alpha, -b_1, \ldots, -b_{m-1}, c_m - 1, -b_m)$ when $f_c$ is a solution of the Lauricella’s $E_B(\alpha, (b, c))$. Any classical solution of $H_{B(1, m)}(\beta)$ can be expressed as (4.4.7).

**Example 4.4.7** Series solutions for $A(2, 2)$ and $\beta^T = (a_1, a_2, a_3, c_1, c_2) \ (E(3, 6))$ have attracted special interests [35], [52]. We present a set of series solutions of this system. When we express an exponent as a $3 \times 3$ matrix under the double index notation, $a_i$ is the $i$-th row sum and $c_j$ is the $j$-th column sum.

Hypergeometric series associated to the exponent $e(i)\alpha$ is written as

$$\phi_{e(i)}(x) = x^{(i)} \sum_{m \in \mathbb{N}^m} \frac{[e(i)]_m}{[e(i) + a]_m} x^m, \quad u = \sum_{j=1}^{4} h_{e(i)}^j m_j,$$

(4.4.8)
Table 4.2 Exponents

<table>
<thead>
<tr>
<th>stair</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$<em>$ $</em>$ $*$</td>
<td>(−1 0 1)</td>
<td>(−1 0 1)</td>
<td>(−1 0 1)</td>
<td>(−1 0 1)</td>
</tr>
<tr>
<td>0 0 0</td>
<td>0 1 −1</td>
<td>0 0 0</td>
<td>0 0 0</td>
<td>0 0 0</td>
</tr>
</tbody>
</table>

Table 4.3 Bases of Ker A

<table>
<thead>
<tr>
<th>stair</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$<em>$ $</em>$ $*$</td>
<td>(−1 1 0)</td>
<td>(−1 0 1)</td>
<td>(−1 0 1)</td>
<td>(−1 0 1)</td>
</tr>
<tr>
<td>0 0 0</td>
<td>0 1 −1</td>
<td>0 0 0</td>
<td>0 0 0</td>
<td>0 0 0</td>
</tr>
</tbody>
</table>

For other series solutions, see [52] and its references. An interesting series solution of $E'(3, 6)$, which is not obtained with the method in this section, is studied in [37] in terms of arithmetic and geometric means.
In case of non-generic parameters, we have series solutions containing logarithmic functions. We can construct $\text{vol}(A)$ linearly independent solutions when $I_A$ is homogeneous by introducing a perturbation parameter $\varepsilon$ in parameters and expand the series solution in terms of $\varepsilon$ [47, 3.5, Th 3.5.1]. We will explain the procedure by an example.

**Example 4.48** We consider the case of $a_1 = \gamma_1 = 1/2$ for $E'(3,6)$ (Table 4.2). The system with this parameter has a special importance in the algebraic geometry ([35], [58]). Let us construct a set of series solutions for this case. The exponents $e(1)$ and $e(6)$ are not degenerated and give two linearly independent solutions. The exponents $e(i)$, $i = 2, \ldots, 5$ are degenerated: $e(2) = e(3) = e(4) = e(5) = \text{diag}(1/2, 1/2, 1/2)$. We will construct four linearly independent solutions for the degenerated exponent. We set $a_1 = 1/2 + 3\varepsilon$, $a_2 = 1/2 + 2\varepsilon$, $a_3 = 1/2 + \varepsilon$, $\gamma_1 = 1/2 + \varepsilon$, $\gamma_2 = 1/2 + 2\varepsilon$, $\gamma_3 = 1/2 + 3\varepsilon$. We put $\gamma_i = x^2 \phi_i$. Then, we have the following series containing the parameter $\varepsilon$.

\[
\phi_{e(2)} = x^{e(2)} f(e(2), y_2, y_3, y_4),
\phi_{e(3)} = x^{e(3)}(1 - 2\varepsilon \log y_2 + 2\varepsilon(\log y_2)^2 + \mathcal{O}(\varepsilon^3)) f(e(3), y_2, y_3, y_5),
\phi_{e(4)} = x^{e(4)}(1 - 2\varepsilon \log y_3 + 2\varepsilon(\log y_3)^2 + \mathcal{O}(\varepsilon^3)) f(e(4), y_2, y_3, y_4),
\phi_{e(5)} = x^{e(5)}(1 - 2\varepsilon \log y_4 + 2\varepsilon(\log y_4)^2 + \mathcal{O}(\varepsilon^3)) f(e(5), y_2, y_3, y_4, y_5).
\]

where $f_i(e; z_1, z_2, z_3, z_4) = \sum_{m=0}^{\infty} \frac{\phi_k}{m!} z_1^m$, $u = \sum_{i=1}^{4} m_i$. We expand $f_i$ in $\varepsilon$ as $f_i^{(0)} + \varepsilon f_i^{(1)} + \varepsilon^2 f_i^{(2)} + \mathcal{O}(\varepsilon^3)$. We note that all $\phi_k$, $i = 2, 3, 4, 5$ gives the same series when $\varepsilon = 0$, which implies $f_i^{(0)}$, $i = 2, 3, 4, 5$ are the same series. Therefore, we have

\[
\phi_{e(i)} - \phi_{e(i)}^{(0)} = (\phi_{e(i)}^{(0)} - \phi_{e(i)}^{(0)}) - (\phi_{e(i)} - \phi_{e(i)}^{(0)}) = \mathcal{O}(\varepsilon^2).
\]

The coefficients of $\varepsilon$ are solutions. Let us find the fourth solution. We have $\lim_{\varepsilon \to 0} \frac{1}{2} f_{2345} = 0$, $f_{2345} = (\phi_{e(3)} - \phi_{e(2)}) - (\phi_{e(2)} - \phi_{e(1)}) - (\phi_{e(1)} - \phi_{e(0)})$. Therefore, the series $f_{2345}$ starts with $\varepsilon^2$ and the coefficients $\varepsilon^2$ of $f_{2345}$ is the fourth solution. It is

\[
\sum_{i=1}^{4} m_i = \sum_{i=1}^{4} m_i f_i^{(0)} + \sum_{i=1}^{4} m_i f_i^{(1)} + \sum_{i=1}^{4} m_i f_i^{(2)} - 2\varepsilon \log y_2 f_i^{(1)} + f_i^{(3)} - f_i^{(2)} - f_i^{(3)} - f_i^{(4)}.
\]

**Example 4.49** Let $\beta = (1, 2)$ and $A = A(0134)$. We set $w = (0, 1, 2, 0)$. Then, the Gröbner basis of $I_A$ with respect to this order is

\[
\partial_2 \partial_3 - \partial_1 \partial_1, \partial_1 \partial_2, \partial_1 \partial_3 - \partial_2 \partial_4, \partial_2 \partial_2, \partial_2 \partial_3 - \partial_3 \partial_4.
\]

Therefore, fake exponents are $\nu^{(1)} = (1/2, 0, 0, 1/2), \nu^{(2)} = (1/4, 1, 0, 1/4), \nu^{(3)} = (1/4, 0, 1, -1/4), \nu^{(4)} = (-1, 2, 0, 0)$. $\phi^{(1)}, \phi^{(2)}$ and $\phi^{(3)}$ are convergent series solutions, but $\phi^{(4)} \equiv 0$. By examining $\ln_{w,z}(I_A)$, we can find two more solutions: $x^2/\partial x_1, x^2/\partial x_4$, [12], [55].

Series solutions with logarithms are constructed for a class of non-generic $\beta$’s to apply for the mirror symmetry [30], [31], [53]. For non-homogeneous $I_A$, series solutions are divergent in most cases, but there are a class of series solutions which are convergent. They are studied in [39] and [16]. The Gevrey order of divergent series solutions is studied in [50], [18]. The notion of fully supported series solutions is introduced in [27]. Rational solutions of $H_A(\beta)$ are studied in [13]. Algebraic solutions of it are studied in [8].
4.5 $E(k, n)$

We fix two numbers $k$ and $n$ satisfying $n \geq 2k \geq 4$. Let $\alpha_j$ be generic parameters satisfying $\sum_{j=1}^n \alpha_j = n - k$. The hypergeometric function of type $E(k, n)$ or the Aomoto-Gel’fand hypergeometric function is defined by the integral

$$\Psi(\alpha; z) = \int_C \prod_{j=1}^n \left( \sum_{i=1}^k u_i s_i \right)^{\alpha_j} ds_1 \cdots ds_k,$$

where we put $s_1 = 1$ and $u$ is a $k \times n$ matrix and $C$ is a bounded $(k - 1)$-cell in the hyperplane arrangement defined by $\prod_{j=1}^n \sum_{i=1}^k u_i s_i = 0$ in the $(s_2, \ldots, s_k)$-space [19].

The hypergeometric function of type $E(k, n)$ is quasi-invariant under the action of complex torus $(\mathbb{C}^*)^k$ and the general linear group $GL(k) = GL(k, \mathbb{C})$. In fact, we have, for $h = \text{diag}(h_1, \ldots, h_n) \in (\mathbb{C}^*)^k$ and $g \in GL(k)$,

$$\Psi(\alpha; uh) = \left( \prod_{j} h_j^{\alpha_j} \right) \Psi(\alpha; u), \quad \Psi(\alpha; gu) = |g|^{-1} \Psi(\alpha; u).$$

It follows from the quasi-invariant property and the integral representation that the function $\Psi(\alpha; u)$ satisfies a system of first order equations and a system of second order equations respectively.

**Theorem 4.5.1** [19] *The function $f = \Psi(\alpha; u)$ satisfies*

$$\sum_{j=1}^k \mu_{ij} \frac{\partial}{\partial u_{ij}} - \alpha_j f = 0, \quad p = 1, \ldots, n, \quad \sum_{j=1}^k \nu_{ij} \frac{\partial}{\partial u_{ij}} + \delta_{ij} f = 0, \quad i, j = 1, \ldots, k,$$

$$\left( \frac{\partial^2}{\partial u_{ij} \partial u_{kl}} - \frac{\partial^2}{\partial u_{il} \partial u_{kj}} \right) f = 0, \quad i, j = 1, \ldots, k, p, q = 1, \ldots, n.$$

We call this system of equations $E(k, n)$.

When we restrict the hypergeometric system $E(k, n)$ to $u_{ij} = \delta_{ij}$ for $1 \leq i \leq k, 1 \leq j \leq k$, we obtain the $A$-hypergeometric system associated to $A(k - 1, n - k - 1)$ and $\beta = (\alpha_1 - 1, \ldots, -\alpha_k - 1, \alpha_{k+1} - 1, \ldots, \alpha_{n-1} - 1)$. We denoted it by $E'(k, n)$. Here, $u_{ij}, s_{ij}$ stands for the variable $x_{ij}$ in Section 4.2.

If $\Psi(\alpha; u)$ is a solution of $E(k, n)$, then $\Psi(\alpha^s; u^s), s \in \mathbb{Z}_n$ is also a solution. This $\mathbb{Z}_n$ symmetry leads us Kummer type relations [56]. The confluent $E(k, n)$ is geometrically studied and a general framework to derive Kummer type relations are given (see [34] and its references).

4.6 Contiguity relations

4.6.1 Contiguity relations

We note the relation in the Weyl algebra $D$

$$\left( \sum_{j=1}^n \alpha_j \theta_j - \beta \right) \partial_k = \partial_k \left( \sum_{j=1}^n \alpha_j \theta_j - \beta - \alpha_k \right).$$
4.6 Contiguity relations

Since \( \partial_k \) commutes with \( \mathfrak{a}_i \), we can see that if \( f \) is a solution of \( H_A(\beta - \mathfrak{a}_k) \), then \( \partial_k \circ f \) is a solution of \( H_A(\beta) \).

We consider the ideal \( B_k \) which is the intersection of \( C[s_1, \ldots, s_d] \) and the left ideal generated by \( \partial_k \) and \( H_A(s) \) in \( D[s_1, \ldots, s_d] \). When \( A \) is normal and \( I_A \) is homogeneous, this ideal can be expressed in terms of primitive support functions.

**Theorem 4.6.1** [41] The ideal \( B_k \) is the principal ideal generated by

\[
\prod_{\sigma \in \mathfrak{a}} \prod_{i=0}^{r_{\sigma}(\mathfrak{a}_k)^{-1}} (F_{\sigma}(s) - i),
\]

where \( S \) is a set of the facets of the convex hull of \( A \) for which \( F_\sigma(\mathfrak{a}_k) > 0 \) holds.

It follows from the theorem above that if \( \beta \notin V(B_k) \), then there exists an operator \( Q_k \in D \) such that \( Q_k \partial_k = 1 \mod H_A(\beta) \). The operators \( \partial_k \) and \( Q_k \) give contiguity relations for \( A \)-hypergeometric series.

The symmetry algebra introduced in [43] gives contiguity relations of \( A \)-hypergeometric system in a general framework. The ideal \( B_k \) is a special case of the \( b \)-ideal introduced in the paper.

### 4.6.2 Contiguity relations for \( E'(k, n) \)

We give a contiguity relation for \( E'(k, n) \) following [49]. We use the variable \( u_{ij} \) instead of \( x_{ij} \) as in Section 4.5. Put

\[
X_{\rho a} = -u_{ap} - \sum_{q=1}^{n} u_{aq} \sum_{i=1}^{k} u_{ip} \partial_{\rho q}.
\]

(4.6.9)

Let \( \varphi(\alpha; u) \) be a solution of the system \( E'(k, n) \) with the set of parameters \( \alpha \).

**Theorem 4.6.2** [49]. We have \( \partial_{\rho p} \varphi(\alpha; u) = \varphi(\alpha + 1_{a} - 1_{p}; u) \), \( X_{\rho a} \varphi(\alpha; u) = \varphi(\alpha - 1_{a} + 1_{p}; u) \) and \( X_{\rho a} \partial_{\rho p} - (\alpha_{p} - 1) \alpha_{a} \in H_A(\beta) \).

Introducing extra variables to hypergeometric series in several variables was done in the pioneering work of [33] to study contiguity relations. Contiguity relations for the Lauricella functions \( F_A \), \( F_B \), and \( F_C \) are derived with this idea and by utilizing the \( b \)-ideal \( B_k \) for them in [42]. See also the section 3.5 (of this volume, Chapter 3) as to some explicit contiguity relations of Lauricella functions.

### 4.6.3 Isomorphism among \( M_A(\beta) \)'s

We gave contiguity operators \( \partial_k \) and \( Q_k \). If they exist, they give an isomorphism \( \partial_k : M_A(\beta - \mathfrak{a}_k) \rightarrow M_A(\beta) \).

The question if \( M_A(\beta) \) and \( M_A(\beta') \) are isomorphic or not as left \( D \)-modules is a fundamental
question. It was studied in [47, §4.4, §4.5] and a final answer was given in [43]. Let \( \tau \) be a face of \( \text{pos}(A) \). Define

\[
E_\tau(\beta) = \{ \lambda \in C(A \cap \tau) / Z(A \cap \tau) | \beta - \lambda \in N_0 A + Z(A \cap \tau) \}
\]

(4.6.10)

**Theorem 4.6.3** [43, §4.4] [44, Th. 3.4.4] The left \( D \)-modules \( M_A(\beta) \) and \( M_A(\beta') \) are isomorphic if and only if \( E_\tau(\beta) = E_\tau(\beta') \) for all faces \( \tau \) of \( \text{pos}(A) \).

The condition can be rewritten to a condition on the primitive integral supporting function when \( A \) is normal.

**Theorem 4.6.4** [43, Th. 5.2] Assume \( A \) is normal and \( I_A \) is homogeneous. The left \( D \)-module \( M_A(\beta) \) is isomorphic to \( M_A(\beta') \) if and only if \( \beta - \beta' \in ZA \) and

\[
[\sigma] \cap \tau \text{ is a facet and } F_\sigma(\beta) \in N_0 = [\sigma] \cap \tau \text{ is a facet and } F_\sigma(\beta') \in N_0.
\]

(4.6.11)

### 4.7 Properties of \( A \)-hypergeometric equations

#### 4.7.1 Rank formula and the Euler-Koszul complex

The holonomic rank \( H_A(\beta) \) is the dimension of \( R/(RH_A(\beta)) \) as the vector space over the field of rational functions \( C(x_1, \ldots, x_n) \). Here, \( R \) is the ring of differential operators with rational function coefficients. The rank of \( H_A(\beta) \) is equal to the normalized volume of \( A \) for generic \( \beta \) and we have the inequality rank \( H_A(\beta) \geq \text{vol}(A) \). [2], [20], [47]. More precise discussion requires the Euler-Koszul complex [26], [7].

We assume that \( A \) is pointed in the subsection. For \( \partial^\alpha \in S_\alpha = C[\partial_1, \ldots, \partial_n] \), we define the \( A \)-multidegree of \( \partial^\alpha \) by \( -Av \in Z^d \). We denote it by \( \text{deg}(\partial^\alpha) \). Its \( i \)-th component is denoted by \( \text{deg}_i(\partial^\alpha) \). This multidegree is naturally extended to the Weyl algebra \( D \) as \( \text{deg}(\partial^\alpha \partial^\beta) = Ah - Av \). Put \( E_i = \sum_{j=1}^n a_{ij} \partial_j \). The multidegree of \( E_i \) is \( 0 \). The identity \( \text{deg}(\partial^\alpha) \partial^\beta = (E_i - \text{deg}_i(\partial^\alpha)) \partial^\beta \) is fundamental.

Let \( S_A = C[\partial_1, \ldots, \partial_n] / I_A \) which is isomorphic to \( C[\partial^1, \ldots, \partial^n] = C[N_0 A] \). We denote \( D_n \otimes_{S_\alpha} S_A \cong D_{\alpha} / (D_{\alpha} I_A) \) by \( D_A \). We consider the complex

\[
\begin{array}{ccccccc}
K^*_A & : & 0 & \longrightarrow & D_A^{(0)}(\partial^\alpha) & \overset{d_1}{\longrightarrow} & D_A^{(1)}(\partial^\alpha) & \overset{d_2}{\longrightarrow} & \cdots & \overset{d_{n-1}}{\longrightarrow} & D_A^{(n-1)}(\partial^\alpha) & \overset{d_n}{\longrightarrow} & \longrightarrow & 0
\end{array}
\]

For \( A \)-homogeneous \( \alpha \otimes b \in D_A \), we define \( (E_i - \beta_i) (\alpha \otimes b) = (E_i - \beta_i - \text{deg}_i(\alpha \otimes b) \alpha \otimes b \). We denote the basis of \( D_{\alpha}^{(i)} \) by \( e_{i_1, \ldots, i_d} \), \( 1 \leq i_1 < \cdots < i_d \leq d \). The boundary map \( d_i \) is defined by

\[
D_A^{(i)} \ni (\alpha \otimes b) e_{i_1, \ldots, i_d} \mapsto \sum_{i_{j_1}, \ldots, i_{j_d}} (E_{i_j} - \beta_{i_j}) (\alpha \otimes b) (-1)^{j_1} e_{i_{j_1}, \ldots, i_{j_d}} \in D_A^{(i-1)}(\partial^\alpha).
\]

(4.7.12)

The complex is called the Euler-Koszul complex over \( D_A \).

The Euler-Koszul complex on \( D_A \) by \( E_i - \beta_i, i = 1, \ldots, d \) is well-defined, because we have \( (E_i - \beta_i) (\alpha \otimes b) = (\alpha \otimes (E_i - \beta_i)) \otimes 1 = (\alpha(E_i - \beta_i - \text{deg}(\partial^\beta)) \otimes 1 \equiv 0 \). The homology group \( \mathcal{H}_i(E - \beta, S_A) = H_i(ker d_i / \text{Im} d_{i-1}) \) of the Euler-Koszul complex has a natural \( A \) grading by the \( A \)-multidegree. The 0-th homology group is nothing but \( M_A(\beta) \). This leads us to more
functorial object to study $A$-hypergeometric system, which is the Euler-Koszul homology for toric modules [26]. We fix $E = \beta$ and replace $S_A$ by $(A_1)$-toric modules. We only present an example of toric modules. Let $A = (0134)$ and $\bar{A}$ be its saturation. Note that $n = 4$ and the multigrading is defined by $A$. We may suppose $\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix}$ and $S_{\bar{A}} = D_8/I_{\bar{A}}$.

Then, we have a short exact sequence

$$0 \rightarrow D_{\bar{A}} \otimes S_A \rightarrow D_4 \otimes_{S_{\bar{A}}} S_A \rightarrow D_4 \otimes S_A/S_{\bar{A}} \rightarrow 0$$

All modules are $A$-graded and toric modules. $C = D_{\bar{A}} \otimes_{S_{\bar{A}}} S_A$ has the support only at the degree $(1, 2)$. We have $H_0(E - \beta; D_{\bar{A}} \otimes S_A) \cong D_5/s_5D_5\otimes_{0, A} M_A(\beta) \cong M_A(\beta)$ and $H_0(E - \beta; C) = 0$ (resp. $= D_4 \otimes [\partial_{\bar{A}}]$) when $\beta \neq (1, 2)$ (resp. $\beta = (1, 2)$).

**Theorem 4.7.1** [26] Put $m = \langle \partial_1, \ldots, \partial_d \rangle$, which is a maximal ideal in $S_n = C[\partial_1, \ldots, \partial_n]$.

1. If $k$ equals the smallest homological degree $i$ for which $-\beta$ is a quasi degree of $H^i_m(S_{\bar{A}})$, then the Euler-Koszul homology $H^i_{\bar{A}}(E - \beta; S_{\bar{A}})$ is non-zero rank and $H_i = 0$ for $i > d - k$.

   Here, $\gamma$ is called the quasi degree when $\gamma$ is contained in the Zariski closure of the non-zero degrees of the homology group.

2. $H^i_m(S_{\bar{A}}) = 0$ holds for $0 \leq i < d$, if and only if $S_{\bar{A}}$ is Cohen-Macaulay.

3. The rank of $H^i_m(S_{\bar{A}})$ equals to the normalized volume of $A$ if and only if $\beta$ is not a quasi-degree of $H^i_m(S_{\bar{A}})$.

Put $e_A = \sum e_i$. The degree $-\alpha + e_A$ part of the local cohomology group is $H^i_m(S_{\bar{A}})_{-\alpha + e_A} = \text{Hom}_C(\text{Ext}^i_{\bar{A}}(S_A, S_{\bar{A}}), C)$.

**Example 4.7.2** We consider the case $A = (0134)$, $e_A = (4, 8)^T$. Construct $A$-graded resolution of $R/I_A$ by Schreyer's method. Then, we have $\text{Ext}^4 = 0$ and $\text{Ext}^3 = C$ at the degree $(5, 10)$, which implies that $H^4_{m} \neq 0$ at the degree $-(1, 2)$. In fact, the rank of the system is 5 when $\beta = (1, 2)$ and it is 4 when $\beta \neq (1, 2)$.

### 4.7.2 Characteristic variety and principal $A$-determinant

Let $I$ be a left ideal in $D$. The initial ideal in $\mathfrak{p}_{(0,1)}(I)$ is the ideal in $C[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$ generated by the principal symbols of $I$. The ideal is called the characteristic ideal of $I$, and the zero set of the ideal in $C^\infty$ is called the characteristic variety of $D/I$ and is denoted by $\text{Ch}(D/I)$. The projection of $\text{Ch}(D/I) \setminus V(\xi_1, \ldots, \xi_n)$ to $C^n = [x]$ is called the singular locus of $D/I$ and is denoted by $\text{Sing}(D/I)$ (see, e.g., [47, p.36]).

**Theorem 4.7.3** [20], [22]

1. If $H_1(\mathfrak{p}_{(0,1)}(K_\alpha)) = 0$, then the characteristic ideal of $H_1(\beta)$ is generated by $A_1x_i$ and $I_A = I_{A(0, \ldots, 0)}$. Here, we denote by $A_1x_i$ the ideal generated by $\sum^n_{j=1} a_{ij} x_j, (i = 1, \ldots, d)$.

2. If $I_A$ is Cohen-Macaulay, then the first homology above vanishes.
Characteristic varieties and micro-characteristic varieties of $M_{\lambda}(\beta)$ are combinatorially studied in [20], [50].

Let $E_{A}$ be the principal $A$-determinant [23]. The projection of $V((Ax_{i}, I_{A}')) \setminus V(\xi_{1}, \ldots, \xi_{n})$ to $C^{n}$ is expressed as $V(E_{A})$.

**Theorem 4.7.4** [23, p.300] The principal $A$-determinant for $A(k, k')$ ($k \leq k'$) is the product of the determinants of all $p \times p$ minors of the matrix $(x_{ij})$ where $1 \leq p \leq k$.

**Example 4.7.5** For $A = A(1, k' - 1)$, we have

$$E_{A} = \prod_{i=1}^{2} \prod_{j=1}^{k'} x_{ij} \prod_{1 \leq j < k', 1 \leq j' \leq k'} \begin{vmatrix} x_{ij} & x_{i j'} \\ x_{2j} & x_{2 j'} \end{vmatrix}.$$  

The variety $V(E_{A})$ is the singular locus of $H_{A}(\beta)$.

### 4.7.3 Reducibility and monodromy groups

We consider the set $U_{\tau}(ZA + \tau)$ where the union is taken over all linear subspaces $\tau$ of $C^{d}$ that form a boundary component of $pos(A)$. The set is called the resonant parameters and is denoted by $Res(A)$.

Let $R$ be the ring of differential operators with rational function coefficients. We consider the left $R$-module $C(x_{1}, \ldots, x_{n}) \otimes_{C} M_{\lambda}(\beta) = R/(R_{H}(\beta))$. If this module has a non-zero proper $R$-submodule, it is called reducible.

**Theorem 4.7.6** [9] When $I_{A}$ is homogeneous and $A$ is not a pyramid, $C(x_{1}, \ldots, x_{n}) \otimes M_{\lambda}(\beta)$ is reducible if and only if $\beta \notin Res(A)$.

An analog of this theorem holds without the homogeneous condition. See [51]. The irreducible quotients as $D$-modules of $M_{\lambda}(\beta)$ are combinatorially discussed in [45].

Connection formulas are studied for $A(1, n)$ by restrictions [48]. The global monodromy groups are calculated for some interesting $A$'s. See [35], [36], [58] for the case of $A(2, 2)$. See [38] for some of 3-dimensional Fano polytopes related to families of $K3$ surfaces. The monodromy at infinities is discussed; see [4] and its references. Recently, a general method to compute a subgroup of monodromy groups is proposed [10].

### 4.8 $A$-hypergeometric polynomials and statistics

We denote by $N$ the set of the non-negative integers. We assume $A$ is a configuration matrix. The $A$-hypergeometric polynomial [46] for $A$ and $\beta \in \mathbb{N}^{d}$ is defined by

$$Z(\beta; p) = \sum_{\mu \in \mathbb{N}^{d}} \frac{p^{\mu}}{\mu!},$$

where $p^{\mu} = \prod_{i=1}^{n} p_{i}^{\mu_{i}}$ and $\mu! = \prod_{i=1}^{n} \mu_{i}!$. Set $p_{\xi} = \exp \xi_{i}$ and let $\exp \xi$ denote the vector $(\exp \xi_{1}, \ldots, \exp \xi_{n})$. We fix $\beta \neq 0$ such that $\beta \in NA = \sum_{i=1}^{n} N \alpha_{i}$. Let $U \in \mathbb{N}^{d}$ be a random
variable of the \((A, \beta)\) hypergeometric distribution with the parameter \(p \in \mathbb{R}_{>0}^n\) (or \(\xi \in \mathbb{R}^n\)), which is defined by the probability that \(U\) takes the value \(u\)

\[
P(U = u|Au = \beta) = \frac{p(\xi)^p}{u!Z(\beta; p(\xi))} = \frac{\exp(u \cdot \xi)}{u!Z(\beta; p(\xi))}, \quad u \cdot \xi = \sum_{i=1}^n u_i \xi_i.
\]

(4.8.14)

It is the conditional distribution of \(u\) given by \(\beta = Au\) under the Poisson distribution

\[
P(U = u) = \frac{p^u}{u!} \exp(-1 \cdot p), \quad 1 = (1, \ldots, 1).
\]

(4.8.15)

The polynomial \(Z\) is the normalizing constant or the partition function of the \((A, \beta)\) hypergeometric distribution. The \((A(p - 1, q - 1), \beta)\) hypergeometric distribution has been called the generalized hypergeometric distribution for \(p \times q\) contingency tables with the marginal sum \(\beta\) in statistics [28, 4.1].

Let \(\tilde{A}\) be an \(n \times (n - d)\) matrix with integer entries satisfying the conditions \(A\tilde{A} = 0\) and that the rank of \(\tilde{A}\) as a \(\mathbb{Q}\)-matrix is \(n - d\). We denote by \(a_i\) the \(i\)-th column vector of \(\tilde{A}\). An asymptotic study of this probability distribution gives the following theorem.

**Theorem 4.8.1** [57] We fix \(p \in \mathbb{R}_{>0}^n\) and \(\beta\) and suppose that \(\beta \in \mathbb{N}A \cap \text{int}(\mathbb{R}_{>0}^nA)\). There exists a unique \(m \in \mathbb{R}_{>0}^n\) such that \(Am = \beta\), \(m^\alpha = p^\beta\). When \(k \to +\infty\), we have

\[
Z(\alpha; p) \sim \left(\prod_i p_{i}^{m_i}\right)^{\frac{1}{k}} \frac{(2 \pi k)^{d-d}}{\Gamma(km + 1) \det(\tilde{A}M^{-1}\tilde{A}^T)^{1/2}}.
\]

where \(M = \text{diag}(m)\).

Conversely, studies on \(A\)-hypergeometric equations can be applied to statistics ([29], [57]).

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