4

A-Hypergeometric Functions

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4.1 Introduction

The A-hypergeometric differential equations in the present form were introduced by Gel'fand, Zelevinsky, Kapranov [21] about 30 years ago. Series solutions are multivariable hypergeometric series defined by a matrix A. Although, there have been analogous approaches before their work, they found that affine toric ideals and their algebraic and combinatorial properties describe solution spaces of the A-hypergeometric differential equations, which also opened new research areas in commutative algebra, combinatorics, polyhedral geometry, and algebraic statistics. Several text books describe some topics of these new research areas, see [30], [56], [62] and their references. The book [49] and its reference give a comprehensive presentation on the A-hypergeometric equations at the year 2000, and the study has made a substantial progress after it. This chapter hopes to give a directory for these new advances as well as to describe fundamental facts. Applications of A-hypergeometric functions are getting broader. Early applications were mainly for period maps and the algebraic geometry. An interplay with the commutative algebra and combinatorics has been a source of new ideas for both of these and the theory of hypergeometric functions. Recent new applications are for the multivariate analysis in statistics.

This chapter starts with systems of differential equations and examples of matrices A which define A-hypergeometric functions. We briefly describe an interplay with combinatorics, Gröbner basis, and software systems. Series solutions are discussed with some important examples. In Sections 4.6 and 4.7, we illustrate that contiguity relations, isomorphisms, holonomic ranks, reducibility conditions have simple descriptions in terms of algebra and combinatorics. Recent new applications to statistics will be briefly discussed in the last section.

4.2 *A*-hypergeometric equations

Let A be a $d \times n$ matrix with integer entries a_{ij}. We denote by the point a_j in $\mathbf{Z}^d$ the j-th column vector of A. We suppose that a_j's generate the lattice $\mathbf{Z}^d$, in other words, we have $\sum_{j=1}^n \mathbf{Z} a_j = \mathbf{Z}^d$. Let $\beta = (\beta_1, \ldots, \beta_d)^T \in \mathbf{C}^d$ be a vector of parameters. The ring of differential operators

$$\mathbf{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle, \ x_j x_j = x_j x_i, \partial_i \partial_j = \partial_j \partial_i, \partial_i x_j = x_j \partial_i + \delta_{ij}$$

is denoted by D or by D_n. We use the multi-index notations $x^p = x_1^{p_1} \cdots x_n^{p_n}$ and $\partial^q = \partial_1^{q_1} \cdots \partial_n^{q_n}$. The action of $x^p \partial^q$ to a function $f(x)$ is defined by $x^p \partial^q \bullet f(x) = x^p \frac{\partial^{|q|} f(x)}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}}$.

Definition 4.2.1 [21] We call the following system of differential equations an *A-hypergeometric system* or a GKZ hypergeometric system:

$$(E_i - \beta_i) \bullet f = 0, \quad \text{where } E_i - \beta_i = \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i, \qquad (i = 1, \dots, d)$$
$$\Box_u \bullet f = 0, \quad \text{where } \Box_u = \prod_{\{i \mid 1 \le i \le n, u_i > 0\}} \partial_i^{u_i} - \prod_{\{j \mid 1 \le j \le n, u_j < 0\}} \partial_j^{-u_j}$$

with $u \in \mathbf{Z}^n$ running over all u such that $Au = 0, u \neq 0$.

We denote by I_A the ideal in $S_n = \mathbf{C}[\partial_1, \dots, \partial_n]$ generated by $\Box_u$ for all $u \in \mathbf{Z}^n$ such that $Au = 0$. This is an *affine toric ideal*, see [56]. The left ideal in D generated by $E_i - \beta_i$, $i = 1, \dots, d$ and I_A is denoted by $H_A(\beta)$ and is called the A-hypergeometric ideal. The quotient left D-module $D/H_A(\beta)$ is denoted by $M_A(\beta)$ and called the A-hypergeometric D-module. When the points a_i's lie on a hyperplane which does not pass through the origin, the D-module $M_A(\beta)$ is regular holonomic [34]. Such matrix A is called a *configuration matrix*.

Several invariants of the D-module can be described in terms of the set of points $\{a_1, \dots, a_n\}$ as we will see later. We also denote the set of points by A in this chapter; the symbol A stands for a matrix or a set of points. When the meaning of A is clear in the context, we do not say which it means. $\mathbf{N}_0 A$ and $\mathbf{Z}A$ mean $\sum_{i=1}^n \mathbf{N}_0 a_i$ and $\sum_{i=1}^n \mathbf{Z} a_i$ respectively where $\mathbf{N}_0$ is the set of non-negative integers, which is also denoted by $\mathbf{Z}_{\ge 0}$.

Although the A-hypergeometric system can be defined for any matrix A, there are nice classes of matrices A (or sets of points a_i) which lead to systems having well-known special functions as solutions. Let us introduce some of them. Take integers k and k' satisfying $1 \le k \le k'$. Put $e_1 = (1, 0, \dots, 0)^T \in \mathbf{Z}^{k+1}$, $e_2 = (0, 1, 0, \dots, 0)^T \in \mathbf{Z}^{k+1}$, …, and $e'_1 = (1, 0, \dots, 0)^T \in \mathbf{Z}^{k'+1}$, $e'_2 = (0, 1, 0, \dots, 0)^T \in \mathbf{Z}^{k'+1}$, …. Let $A(k, k')$ be a $(k + k' + 1) \times (k + 1)(k' + 1)$ matrix of which columns consist of $p(e_i \oplus e'_j)$ where p is the projection to the first $k + k' + 1$ coordinates (the projection which removes the last coordinate). $A(1, 1)$, $A(1, 2)$, $A(2, 2)$ are given in Table 4.1.

The columns of $A(k, k')$ generate $\mathbf{Z}^{k+k'+1}$ and they lie on the hyperplane $\sum_{j=1}^{k+1} y_j = 1$ in $\mathbf{R}^{k+k'+1} = \{(y_1, \dots, y_{k+k'+1})\}$. Since the convex hull of $e_1, \dots, e_{k+1}$ is the simplex Δ_k and that of $e'_1, \dots, e'_{k'+1}$ is the simplex $\Delta_{k'}$, we call this A-hypergeometric system $\Delta_k \times \Delta_{k'}$-hypergeometric system or the hypergeometric system $E'(k + 1, k + k' + 2)$. The latter naming comes from a relation of this system with the hypergeometric system $E(k, n)$ (Section 4.5). For this hypergeometric system, we often denote the variable x_p by x_{ij} where $p = (i - 1)k' + (j - 1) + 1$, $1 \le i \le k$, $1 \le j \le k'$. This double index notation is convenient. We also regard a vector of length $(k + 1)(k' + 1)$ as a matrix under this double index notation. For example, for a vector e, the condition $A(k, k')e = \beta$ means that the row sums and the column sums of e expressed in terms of the $(k+1)\times(k'+1)$ matrix are $(\beta_1, \dots, \beta_{k+1})$ and $(\beta_{k+2}, \dots, \beta_{k+k'+1}, \sum_{i=1}^{k+1} \beta_i - \sum_{j=k+2}^{k+k'+1} \beta_j)$ respectively.

$$A(1,1) = \begin{array}{c} \begin{array}{cccc} x_{11} & x_{12} & x_{21} & x_{22} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}, \quad A(1,2) = \begin{array}{c} \begin{array}{cccccc} x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$A(2,2) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A(F_A,2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}, \quad A(F_C,2) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A(0134) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}, \quad A_s = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad A(P_4) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix}$$

Table 4.1 A

The ideal I_A for $A = A(k, k')$ is generated by

$$\partial_{iq}\partial_{jp} - \partial_{ip}\partial_{jq}, 1 \leq i < j \leq k+1, 1 \leq p < q \leq k'+1.$$

More precisely, it is the reduced Gröbner basis with respect to the graded reverse lexicographic order $>$ with $\partial_{1,1} > \partial_{1,2} > \cdots > \partial_{1,k} > \partial_{2,1} > \cdots$ [56, Prop 5.4]. For any A, generators of I_A can be obtained by a Gröbner basis computation [56, Alg. 4.5]. Generators of I_A is called the Markov basis in algebraic statistics. There are theoretical and computational efforts to find explicit Markov basis. We have a database of Markov bases for several matrix A ([1] or [26]).

The matrix $A(1, k')$ stands for the Lauricella function F_D of k' variables (see Example 4.4.4 for the correspondence). In particular, when $k' = 1$, it stands for the Gauss hypergeometric function. As we will see later, the correspondence can be described in terms of series solutions or integral representations. In a more sophisticated context, a categorical correspondence is given in [61]. Here, we explain an elementary correspondence in the case of Gauss hypergeometric equation ($A(1,1)$ case) as an introduction. Put $A = A(1,1)$ and suppose that $f(x) = x^v F(z)$, $x^v = \prod_{i,j=1}^{2} x_{ij}^{v_{ij}}$, $z = \frac{x_{11}x_{22}}{x_{12}x_{21}}$, is a solution of the A-hypergeometric system $H_A(\beta)$. We denote $x_{ij}\partial_{ij}$ by θ_{ij}. Then, we have

$$(\theta_{11} + \theta_{12} - \beta_1) \bullet x^v F(z) = (v_{11} + v_{12} - \beta_1) x^v F(z)$$
$$(\theta_{21} + \theta_{22} - \beta_2) \bullet x^v F(z) = (v_{21} + v_{22} - \beta_2) x^v F(z)$$
$$(\theta_{11} + \theta_{21} - \beta_3) \bullet x^v F(z) = (v_{11} + v_{21} - \beta_3) x^v F(z)$$

by the relation $\theta_{ij} x^v = x^v(\theta_{ij} + v_{ij})$ in the ring of differential operators, and $\theta_{ii} \bullet F(z) = zF'(z)$, $\theta_{ij} \bullet F(z) = -zF'(z)$, $(i \neq j)$. These are equal to 0 by $(E_i - \beta_i) \bullet f = 0$, which implies $Av = \beta$ where $v = (v_{11}, v_{12}, v_{21}, v_{22})^T$ and $\beta = (\beta_1, \beta_2, \beta_3)^T$. Since, $\mathrm{Ker}\,(A : \mathbf{Z}^4 \to \mathbf{Z}^3) = \mathbf{Z}(1, -1, -1, 1)^T$, we can show that the toric ideal I_A is generated by $\partial_{12}\partial_{21} - \partial_{11}\partial_{22}$. Note that the basis of the kernel is not enough to generate the toric ideal I_A in the case of $A(1,k')$, $k' > 1$. Act the operator $x_{11}x_{22}(\partial_{11}\partial_{22} - \partial_{12}\partial_{21}) = \theta_{11}\theta_{22} - z\theta_{12}\theta_{21}$ to the function $x^v F(z)$, then we have

$$x^v\left((\theta_z + v_{11})(\theta_z + v_{22}) - z(\theta_z - v_{12})(\theta_z - v_{21})\right) \bullet F(z) = 0$$

where $\theta_z = z\partial_z$. When $v_{11} = 0$ or $v_{22} = 0$, this is the Gauss hypergeometric equation. Other v_{ij}'s are determined by $Av = \beta$.

Let us give matrices A for other Lauricella functions (see this volume, Chapter 3 on these functions). Let $e_0, e_1, \ldots, e_{2m}$ be the standard basis of $\mathbf{Z}^{2m+1}$. Put $A = \{e_0, e_1, \ldots, e_{2m}, e_0 + e_1 - e_{m+1}, e_0 + e_2 - e_{m+2}, \ldots, e_0 + e_m - e_{2m}\}$. Then A is a $(2m+1)\times(3m+1)$ matrix, which stands for the Lauricella function F_A of m variables [44]. They lie on the hyperplane $y_0+y_1+\cdots+y_{2m} = 1$ in $\mathbf{R}^{2m+1} = \{(y_0, \ldots, y_{2m})\}$. We denote the matrix by $A(F_A, m)$. The associated toric ideal I_A is generated by $\partial_0\partial_j - \partial_{m+j}\partial_{2m+j}$, $j = 1, \ldots, m$. Here, we use the variables $u_0, u_1, \ldots, u_{3m}$ as independent variables instead of $x_1, \ldots, x_n$. When $m = 2$, it is the Appell function F_2; the matrix is given in Table 4.1.

Let $e_1, \ldots, e_{m+1}, e_{m+2}$ be the standard basis of $\mathbf{Z}^{m+2}$. Put $A = \{e_1 + e_{m+2}, e_2 + e_{m+2}, \ldots, e_{m+1} + e_{m+2}, -e_1 + e_{m+2}, -e_2 + e_{m+2}, \ldots, -e_{m+1} + e_{m+2}\}$. Then A is an $(m+2)\times 2(m+1)$ matrix, which stands for the Lauricella function F_C of m variables [44]. They lie on the hyperplane $z_{m+2} = 1$ in $\mathbf{R}^{m+2}$. We denote the matrix by $A(F_C, m)$. Note that the lattice generated by the columns of $A(F_C, m)$ is a proper sublattice of $\mathbf{Z}^{m+2}$. Then, we need to regard the sublattice as $\mathbf{Z}^{m+2}$. The associated toric ideal I_A is generated by $\partial_j\partial_{-j} - \partial_{m+1}\partial_{-(m+1)}$, $j = 1, \ldots, m$. Here, we use the variables $u_1, \ldots, u_{m+1}, u_{-1}, \ldots, u_{-(m+1)}$ as independent variables. When $m = 2$, it is the Appell function F_4; the matrix is given in Table 4.1. The notion of binomial D-modules is proposed and studied in [16]. Binomial D-modules are generalizations of A-hypergeometric equations and they fit to study Appell-Horn equations and their generalizations to several variables in algebraic methods.

A-hypergeometric systems associated to smooth fano polytopes have importance in studies of period maps for $K3$ and Calabi-Yau varieties (see, e.g., [7], [32], [33], [55] and their references). For example, the matrix $A(P_4)$ [40] appears in this context.

Let us discuss on integral representations of solutions of A-hypergeometric equations. Suppose that we are given n_k points $a_i \in \mathbf{Z}^m$. We divide these points into k groups and construct $m \times n_i$ matrices $A_1 = (a_1, \ldots, a_{n_1})$, $\ldots$, $A_k = (a_{n_{k-1}+1}, \ldots, a_{n_k})$. For each group, define the polynomial $f_j(x,t) = \sum_{i=n_{j-1}+1}^{n_j} x_i t^{a_i}$ where $t^{a_i} = \prod_{j=1}^{m} t_j^{(a_i)_j}$. Note that we use the multi-index notation for $t = (t_1, \ldots, t_m)$. Take complex numbers α_j, $\gamma = (\gamma_1, \ldots, \gamma_m)$. We consider the integral

$$\Phi(\alpha, \gamma; x) \quad = \quad \int_C \prod_{j=1}^{k} f_j(x,t)^{\alpha_j} t^\gamma dt_1 \cdots dt_m,$$

where C is any twisted m-cycle defined for $\prod_{j=1}^{k} f_j(x,t)^{\alpha_j} t^\gamma$. Define $(k + m) \times n_k$ matrix $A(A_1, \ldots, A_k)$, which is called the *Caylay matrix*,as in Figure 4.1. The function $\Phi(\alpha, \gamma; x)$ satisfies the A-hypergeometric system $H_A(\beta)$ for $A = A(A_1, \ldots, A_m)$ and $\beta = (\alpha_1, \ldots, \alpha_k, -\gamma_1 -$

$$\begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & & \cdot & \cdots & \cdot \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 1 \\ a_1 & \cdots & a_{n_1} & a_{n_1+1} & \cdots & a_{n_2} & & a_{n_{k-1}+1} & \cdots & a_{n_k} \end{pmatrix}$$

Figure 4.1 Caylay matrix $A(A_1, \ldots, A_k)$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Figure 4.2 $E_{k'} \oplus 0$

$1, \ldots, -\gamma_m - 1)^T$. When f_j are linear with respect to the variable t, we call the function Φ hypergeometric function for hyperplane arrangements. Note that when $A_1 = \ldots = A_k = \Delta_{k'}$, in other words, all A_i's are equal to $k' \times (k' + 1)$ matrix $E_{k'} \oplus 0$ in Figure 4.2, we have $A(A_1, \ldots, A_k) = A(k, k')$. As to studies on these hypergeometric functions in terms of twisted cohomology groups, see [5], [6], [3], [42].

When the toric ideal I_A is not a homogeneous ideal (the case that a_i's do not lie on an affine hyperplane), the integral

$$\Phi(\gamma; x) \quad = \quad \int_C \exp\left(\sum_{i=1}^{n} x_i t^{a_i}\right) t^\gamma dt_1 \cdots dt_d$$

satisfies $H_A(\beta)$ with $\beta = (-\gamma_1 - 1, \ldots, -\gamma_d - 1)^T$ for any rapid decay d-cycle under some conditions [18].

4.3 Combinatorics, polytopes and Gröbner basis

The matrix A is said to be pointed when $a_1, \ldots, a_n$ lie in a single open half-space. For example, $A = (-1, 1)$ is not pointed and all A's in Table 4.1 are pointed. The set of points A is called normal, when A satisfies $(\sum \mathbf{R}_{\geq 0} a_k) \cap \mathbf{Z}^n = \sum \mathbf{Z}_{\geq 0} a_k$.

For a facet σ of the cone $\mathrm{pos}(A) = \mathbf{R}_{\geq 0} A$, F_σ is a linear function on $\mathbf{R}A = \mathbf{R}^d$ uniquely determined by the conditions:

1. $F_\sigma(\mathbf{Z}A) = \mathbf{Z}$, 2. $F_\sigma(a_i) \geq 0$ for all $i = 1, \ldots, n$, 3. $F_\sigma(a_i) = 0$ for all $a_i \in \sigma$.

We call F_σ the *primitive integral support function* of σ.

For $\Delta_k \times \Delta_{k'}$ embedded in $\mathbf{R}^{k+1} \times \mathbf{R}^{k'+1} = \{(x_1, \dots, x_{k+1}; y_1, \dots, y_{k'+1})\}$, the support functions are x_i and y_j. When we project the points to $\mathbf{R}^{k+1} \times \mathbf{R}^{k'}$, the primitive integral support functions are x_i $(i = 1, \dots, k+1)$, and y_j $(j = 1, \dots, k')$, and $1 - \sum_{j=1}^{k'} y_j$.

The supporting functions for $A(F_A, m)$ are s_j, $s_j + s_{m+j}$, $1 \leq j \leq m$ and $s_0 + \sum_{j \in J} s_{m+j}$, $J \subseteq [1, m]$ where $\{s_i\}$ is the dual basis of $\{e_i\}$. Those for $A(F_C, m)$ are $(1/2)(s_{m+2} + \sum_{j \in J} s_j - \sum_{j \notin J} s_j)$, $J \subseteq [1, m+1]$ [44].

Let $\mathbf{Z}A$ be the lattice generated by the columns of A. Let us set the volume of the convex hull U of the lattice base and the origin to 1. The volume of polytopes in $\mathbf{R}A$ normalized with the U is called the *normalized volume*. The normalized volume of the convex hull of A and the origin is denoted by $\mathrm{vol}(A)$. The normalized volume of $A(k, k')$ is known to be equal to $\binom{k+k'}{k}$. For given A, it can be evaluated by geometry software systems like polymake, or by computer algebra systems which use a formula $\mathrm{degree}(I_A) = \mathrm{vol}(A)$.

An interplay of theory and computation has been indispensable in the study of A-hypergeometric systems. A lot of algorithms and software systems have been developed to study it and its related areas. The text book [30] gives an introduction to them including mathematical software systems. Here, we give a few examples.

Example 4.3.1 Macaulay2 [25] commands to evaluate the volume (the degree) of $A(0134)$. Here, `o5` is I_A.

```
loadPackage "FourTiTwo"
M=matrix "1,1,1,1; 0,1,3,4"
R=QQ[a..d]
I=toricGroebner(M,R)
   o5 = ideal (b^3  - a^2*c, b*c - a*d, - a*c^2  + b^2*d, c^3  - b*d^2 )
degree(I)
   o6 = 4
```

For a given weight vector $w \in \mathbf{R}^n$ (`Weights` below), consider points $\{(a_i, w_i)\}$ in $\mathbf{R}^{d+1}$ and the convex hull of them. The projection of the convex hull to the first d coordinates naturally induces a triangulation of the set of points A for a generic weight w, which is called a regular triangulation [24], [56, Chapter 8], [30, 5.5.2]. We compute a regular triangulation of $\Delta_1 \times \Delta_2$ for $w = (4, 2, 0, 10, 8, 6)$ by the computer algebra system Macaulay2

```
i1 : loadPackage "FourTiTwo"
i2 : M=matrix "1,1,1,0,0,0; 0,0,0,1,1,1; 1,0,0,1,0,0; 0,1,0,0,1,0"
i3 : R=QQ[x11,x12,x13,x21,x22,x23, MonomialOrder=>{Weights=>{4,2,0,10,8,6}}]
i4 : I=toricGroebner(M,R)
  o4 = ideal (x13*x21 - x11*x23, x12*x21 - x11*x22, x13*x22 - x12*x23)
i5 : J=leadTerm(I)
  o8 = | x13x22 x13x21 x12x21 |
i6 : associatedPrimes(ideal(J))
  o12 = {ideal (x22, x21), ideal (x13, x12), ideal (x13, x21)}
```

By taking the complements of the indices of each associated primes, we get a regular triangulation $(11, 12, 13, 23)$, $(11, 21, 22, 23)$, $(11, 12, 22, 23)$.

4.4 A-hypergeometric series

Let us introduce A-hypergeometric series following [21] and [49, 3.4]. Let $v = (v_1, \dots, v_n)^T$ be a vector in $\mathbf{C}^n$ and $u = (u_1, \dots, u_n)^T$ a vector in $\mathbf{Z}^n$. We omit the sign of the transpose ^{T} to change a row vector to the column vector in the sequel as long as no confusion arises. We decompose u into positive and negative parts, $u = u_+ - u_-$, where u_+ and u_- are non-negative vectors with disjoint support. Consider the following two scalars in $\mathbf{C}$, which can be expressed by falling factorials:

$$[v]_{u_-} = \prod_{i:u_i<0} \prod_{j=1}^{-u_i} (v_i - j + 1),$$
$$[u+v]_{u_+} = \prod_{i:u_i>0} \prod_{j=1}^{u_i} (u_i + v_i - j + 1) = \prod_{i:u_i>0} \prod_{j=1}^{u_i} (v_i + j).$$

For example, when $v = (v_1, v_2, 0, v_4)$ and $u = (-2, 2, 2, -2)$, we have $\frac{[v]_{u_-}}{[v+u]_{u_+}} = \frac{v_1(v_1-1)v_4(v_4-1)}{(v_2+2)(v_2+1)2!}$. Note that when $v \in (\mathbf{C} \setminus \mathbf{Z}_{<0})^n$, we have $[u+v]_{u_+} \neq 0$. We set $L = \mathrm{Ker}(\mathbf{Z}^n \overset{A}{\to} \mathbf{Z}^d)$.

Theorem 4.4.1 *Suppose that* $v \in (\mathbf{C} \setminus \mathbf{Z}_{<0})^n$ *and* $Av = \beta$. *Then the formal series*

$$\phi_v := \sum_{u \in L} \frac{[v]_{u_-}}{[v+u]_{u_+}} \cdot x^{v+u} \tag{4.4.1}$$

is well-defined and is a formal solution of $H_A(\beta)$.

As to the proof of this theorem, see [49, Prop. 3.4.1]. We call the formal series the A-hypergeometric series in the falling factorial form.

Let us introduce another expression of the series. We set $\Gamma(u + v + 1) = \prod_{i=1}^{n} \Gamma(u_i + v_i + 1)$ and when $u_i + v_i \in \mathbf{Z}_{<0}$ for an i, we define $1/\Gamma(u + v + 1) = 0$. Under this convention, we have $\frac{1}{\Gamma(v+u+1)} = \frac{[v]_{u_-}}{[v+u]_{u_+}} \frac{1}{\Gamma(v+1)}$ for $u \in L$ and $v \in (\mathbf{C} \setminus \mathbf{Z}_{<0})^n$ (use $\Gamma(\alpha + m) = \Gamma(\alpha)(\alpha)_m$, $\Gamma(\alpha - m + 1) = \Gamma(\alpha + 1)(-1)^m/(-\alpha)_m$). Define

$$\Phi_v := \sum_{u \in L} \frac{1}{\Gamma(u+v+1)} x^{v+u}. \tag{4.4.2}$$

Then, we have $\Phi_v = \frac{1}{\Gamma(v+1)}\phi_v$ when none of v_i is negative integer. We call the formal series the A-hypergeometric series in the gamma function form. Note that when v_i is a negative integer, two series are different. For example, if $v_i = -1$ and $u_i = 1$, then we have $[u_i + v_i]_{u_i} = 0$ and ϕ_v is not well-defined, but $\Gamma(u_i + v_i + 1) = 1$. When $v = (1, -2, 3, 0)$ and $L = \mathbf{Z}(1, -1, -1, 1)$, ϕ_v is a non-zero polynomial, but Φ_v is identically 0.

For a given weight vector $w \in \mathbf{Z}^n$ and $\ell \in I_A$, $\mathrm{in}_w(\ell)$ is the sum of the highest w-order terms in ℓ. The ideal in S_n generated by $\mathrm{in}_w(\ell)$, $\ell \in I_A$ is denoted by $\mathrm{in}_w(I_A)$ and is called the initial ideal of I_A [56]. Two weight vectors w and w' are equivalent with respect to the ideal I when $\mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$. Fix w. The closure of the equivalent class of the weight vector w is called the Gröbner cone for w [56, Chapter 1], [49, 2.1], [30, 5.3.2]. Let C be the Gröbner cone of I_A for a generic weight vector w. The initial ideal $\mathrm{in}_{w'}(I_A)$ does not change by definition when w' runs over the relative interior of C [56], [49, Chap 2]. For a series f with a support on a translate of the dual cone C^*, for which we may assume $(w, C^* \setminus \{0\}) > 0$, the starting term of f is the sum of the lowest weight terms in f with respect to w. If f is a solution of $\ell \bullet f = 0$, $\ell \in D$, then the starting term of f is a solution of $\mathrm{in}_{(-w,w)}(\ell)$, which is the sum of the highest order terms in ℓ with respect to the weight $(-w, w)$ where $-w$ (resp. w) stands for x (resp. ∂). This observation gives us the following method [49, Chapter 2] to find series solutions of $H_A(\beta)$; (1) determine the initial ideal $\mathrm{in}_{(-w,w)}(H_A(\beta))$, (2) solve it to determine the starting terms, (3) extend the starting terms to series solutions.

Theorem 4.4.2 *For generic β, the initial ideal* $\mathrm{in}_{(-w,w)}(H_A(\beta))$ *is generated by $E_i - \beta_i$, $1 \leq i \leq d$ and* $\mathrm{in}_w(I_A)$.

We note that the proof of [49, Th. 3.1.3] needs to be corrected to utilize the homogenized Weyl algebra. We suppose that I_A is a homogeneous ideal and take a generic weight vector w such that $\mathrm{in}_w(I_A)$ is a monomial ideal. Let G be the reduced Gröbner basis of I_A with respect to the order $<_w$ [56]. We consider the system of differential equations

$$(E_i - \beta_i) \bullet s = 0, \quad i = 1, \ldots, d, \quad \text{and} \quad \ell \bullet s = 0, \quad \ell \in \mathrm{in}_w(G) \tag{4.4.3}$$

Let v be a solution of algebraic equations

$$Av = \beta, \quad \prod_{i=1}^{n} v_i(v_i - 1) \cdots (v_i - e_i + 1) = 0 \text{ for } \partial^e \in \mathrm{in}_w(G) \tag{4.4.4}$$

It is called a fake exponent. We note that the fake exponents can be expressed in terms of standard pairs of the monomial ideal $\mathrm{in}_w(I_A)$ [49, 3.2]. When β_i are generic, there are linearly independent $\mathrm{vol}(A)$ solutions of (4.4.3) of the form $s = x^v = \prod_{i=1}^n x_i^{v_i}$ where v is a fake exponent and they span the solution space over $\mathbf{C}$ when v runs over the fake exponents.

Theorem 4.4.3 *[21], [49, Th 3.4.2] If v is a fake exponent and $v \in (\mathbf{C} \setminus \mathbf{Z}_{<0})^n$, then ϕ_v is a formal solution of $H_A(\beta)$ with the support in $v + (C^* \cap L)$.*

Note that Gel'fand, Kapranov, Zelevinsky constructed series solutions by regular triangulations of A [21]. Our construction differs with their construction, but it is related with the construction via the theorem [56, Th 8.3] "$\sqrt{\mathrm{in}_w(I_A)}$ is the Stanley-Reisner ideal for the regular triangulation by w". The function ϕ_v converges when $(-\log|x_1|, \ldots, -\log|x_n|)$ lies in a translate of the secondary cone attached to the regular triangulation.

For a good class of A-hypergeometric functions, more explicit form of A-hypergeometric series is known as we will describe. For $A = A(p-1, q-1)$, the stair case Gröbner basis in [56, Prop.5.4] gives series solutions. A sequence of indexes $\{(1,1), \ldots, (p,q)\}$ is called a stair if (i,j) is an element of the stair and is not (p,q), then the next element of (i,j) is either $(i+1, j)$ or $(i, j+1)$ (see Table 4.2).

The initial ideal of I_A for the reverse lexicographic order is generated by $\partial_{i\ell}\partial_{jk}$, $1 \leq i < j \leq p, 1 \leq k < \ell \leq q$ [56, Prop.5.4]. We can obtain the fake exponents from this initial ideal by solving (4.4.4). It is known that there is a one-to-one correspondence between the roots of the system of equations and the stairs. For a given stair S, the system has a unique solution such that $v_{ij} = 0$ for $(i,j) \notin S$. In other words, the support of each exponent has the form of the stair for generic β. In the sequel, we use e rather than v to denote exponents. The support of the series solution standing for the exponent e has the form

$$e + L', \quad L' = \sum_{(i,j) \in \mathrm{supp}(e)} \mathbf{Z}_{\geq 0} b_e^{(i,j)}$$

where $b_e^{(i,j)}$ is an element of Ker A such that (i, j)-th element of $b_e^{(i,j)}$ is 1 for $(i, j) \in \overline{\mathrm{supp}(e)}$ and (i', j')-th element is 0 for $(i', j') \in \overline{\mathrm{supp}(e)} \setminus \{(i, j)\}$. Here, $\overline{S}$ denotes the complement of the set S.

Let us see some A's of which series solutions can be written in terms of solutions of Lauricella systems (this volume, Chapter 3, Section 3.4).

Example 4.4.4 We put $A = A(1, N-1)$ in this example. Let a, $b_1, \dots, b_{N-1}$, c be (generic) constants. Put $b_N = a + 1 - c$ and

$$e(k) = \begin{pmatrix} -b_1 & \cdots & -b_{k-1} & -\sum_{j=k}^N b_j + a & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \sum_{j=k+1}^N b_j - a & -b_{k+1} & \cdots & -b_{N-1} & -b_N \end{pmatrix},$$

which is the fake exponent standing for the k-th stair.
Put $m = (m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_N)$, $m_k = -\sum_{j=1}^{k-1} m_j + \sum_{j=k+1}^N m_j$, and $z_j = \frac{x_{2j}x_{1N}}{x_{1j}x_{2N}}$ for $1 \leq j \leq N$. Note that $z_N = 1$. Define a series $\phi_k(e; z)$ by

$$\sum_{m \in \mathbf{Z}_{\geq 0}^{N-1}} \frac{\prod_{j=1}^{k-1}[e_{1j}]_{m_j} \prod_{j=k+1}^N [e_{2j}]_{m_j}}{\prod_{j=1}^{k-1} m_j! \prod_{j=k+1}^N m_j!} c_m \prod_{j=1}^{k-1} \left(z_j z_k^{-1}\right)^{m_j} \prod_{j=k+1}^{n} \left(z_k z_j^{-1}\right)^{m_j} \tag{4.4.5}$$

where $e = e(k)$, $c_m = [e_{1k}]_{m_k}/[e_{2k} + m_k]_{m_k}$ when $m_k > 0$, and $c_m = [e_{2k}]_{-m_k}/[e_{1k} - m_k]_{-m_k}$ when $m_k < 0$, and $c_m = 1$ when $m_k = 0$. For $\beta = (-\sum b_i + c - 1, -a, -b_1, \dots, -b_{N-1}, c - 1 - a)$, the function $x^{e(k)}\phi_k(e(k); z)$, $1 \leq k \leq N$ is a solution of $H_A(\beta)$ and $x^{e(k)-e(N)}\phi_k(e(k); z)$ is a solution of the Lauricella system $E_D(a, (b), c)$. The series $\phi_N(e(N); z)$ is the Lauricella's F_D. The series ϕ_k's have a common domain of convergence $|z_1| < \cdots < |z_{N-1}| < 1$.

Example 4.4.5 The function

$$u_0^{-a} \prod_{j=1}^m u_j^{-b_j} \prod_{j=1}^m u_{m+j}^{c_j - 1} f_A \left(a, b_1, \dots, b_m, c_1, \dots, c_m; \frac{u_{m+1}u_{2m+j}}{u_0 u_1}, \dots, \frac{u_{m+m}u_{2m+m}}{u_0 u_m} \right) \tag{4.4.6}$$

is a solution of $H_{A(F_A,m)}(\beta)$, $\beta^T = (-a, -b_1, \dots, -b_m, c_1 - 1, \dots, c_m - 1)$ when f_A is a solution of the Lauricella's $E_A(a, (b), (c))$. Any classical solution of $H_{A(F_A,m)}(\beta)$ can be expressed as (4.4.6).

Example 4.4.6 The function

$$u_{m+1}^{-a} u_{-m}^{-b} \prod_{j=1}^m u_{-j}^{c_j - 1} f_C \left(a, b, c_1, \dots, c_m; \frac{u_1 u_{-1}}{u_{m+1}u_{-(m+1)}}, \dots, \frac{u_m u_{-m}}{u_{m+1}u_{-(m+1)}} \right) \tag{4.4.7}$$

is a solution of $H_{A(F_C,m)}(\beta)$, $\beta^T = (1 - c_1, \dots, 1 - c_m, b - a, \sum_{j=1}^m c_j - a - b - m)$ when f_c is a solution of the Lauricella's $E_C(a, b, (c))$. Any classical solution of $H_{A(F_C,m)}(\beta)$ can be expressed as (4.4.7).

Example 4.4.7 Series solutions for $A(2, 2)$ and $\beta^T = (\alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2)$ ($E'(3, 6)$) have attracted special interests [37], [54]. We present a set of series solutions of this system. When we express an exponent as a 3×3 matrix under the double index notation, α_i is the i-th row sum and γ_j is the j-th column sum.

stair	e : exponent
$\begin{pmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}$	$e(1) = \begin{pmatrix} \gamma_1 & \gamma_2 & \alpha_1 - \gamma_1 - \gamma_2 \\ 0 & 0 & \alpha_2 \\ 0 & 0 & \alpha_3 \end{pmatrix}$
$\begin{pmatrix} * & * & 0 \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$	$e(2) = \begin{pmatrix} \gamma_1 & \alpha_1 - \gamma_1 & 0 \\ 0 & -\alpha_1 + \gamma_1 + \gamma_2 & \alpha_1 + \alpha_2 - \gamma_1 - \gamma_2 \\ 0 & 0 & \alpha_3 \end{pmatrix}$
$\begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix}$	$e(3) = \begin{pmatrix} \gamma_1 & \alpha_1 - \gamma_1 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & -\alpha_1 - \alpha_2 + \gamma_1 + \gamma_2 & \alpha_1 + \alpha_2 + \alpha_3 - \gamma_1 - \gamma_2 \end{pmatrix}$
$\begin{pmatrix} * & 0 & 0 \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$	$e(4) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ -\alpha_1 + \gamma_1 & \gamma_2 & \alpha_1 + \alpha_2 - \gamma_1 - \gamma_2 \\ 0 & 0 & \alpha_3 \end{pmatrix}$
$\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ 0 & * & * \end{pmatrix}$	$e(5) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ -\alpha_1 + \gamma_1 & \alpha_1 + \alpha_2 - \gamma_1 & 0 \\ 0 & -\alpha_1 - \alpha_2 + \gamma_1 + \gamma_2 & \alpha_1 + \alpha_2 + \alpha_3 - \gamma_1 - \gamma_2 \end{pmatrix}$
$\begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & * & * \end{pmatrix}$	$e(6) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \\ -\alpha_1 - \alpha_2 + \gamma_1 & \gamma_2 & \alpha_1 + \alpha_2 + \alpha_3 - \gamma_1 - \gamma_2 \end{pmatrix}$

Table 4.2 Exponents

stair	b_e^1	b_e^2	b_e^3	b_e^4
$\begin{pmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$
$\begin{pmatrix} * & * & 0 \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$
$\begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$
$\begin{pmatrix} * & 0 & 0 \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$
$\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ 0 & * & * \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$
$\begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & * & * \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

Table 4.3 Bases of Ker A

Hypergeometric series associated to the exponent $e(i)$ is written as

$$\phi_{e(i)}(x) = x^{e(i)} \sum_{m \in \mathbf{N}_0^4} \frac{[e(i)]_{u_-}}{[e(i)+u]_{u_+}} x^u, \quad u = \sum_{j=1}^{4} b_{e(i)}^j m_j. \tag{4.4.8}$$

For other series solutions, see [54] and its references. An interesting series solution of $E'(3,6)$, which is not obtained with the method in this section, is studied in [39] in terms of arithmetic and geometric means.

In case of non-generic parameters, we have series solutions containing logarithmic functions. We can construct vol(A) linearly independent solutions when I_A is homogeneous by introducing a perturbation parameter ε in parameters and expand the series solution in terms of ε [49, 3.5, Th 3.5.1]. We will explain the procedure by an example.

Example 4.4.8 We consider the case of $\alpha_i = \gamma_i = 1/2$ for $E'(3,6)$ (Table 4.2). The system with this parameter has a special importance in the algebraic geometry ([37], [60]). Let us construct a set of series solutions for this case. The exponents $e(1)$ and $e(6)$ are not degenerated and give two linearly independent solutions. The exponents $e(i)$, $i = 2, \ldots, 5$ are degenerated: $e(2) = e(3) = e(4) = e(5) = \mathrm{diag}(1/2, 1/2, 1/2)$. We will construct four linearly independent solutions for the degenerated exponent. We set $\alpha_1 = 1/2+3\varepsilon, \alpha_2 = 1/2+2\varepsilon, \alpha_3 = 1/2+\varepsilon, \gamma_1 = 1/2 + \varepsilon, \gamma_2 = 1/2 + 2\varepsilon, \gamma_3 = 1/2 + 3\varepsilon$. We put $y_i = x^{b^i_{e(2)}}$. Then, we have the following series containing the parameter ε.

$$\begin{aligned}
\phi_{e(2)} &= x^{e(2)} f_2(\varepsilon; y_1, y_2, y_3, y_4),\\
\phi_{e(3)} &= x^{e(2)}(1 - 2\varepsilon \log y_4 + 2\varepsilon^2(\log y_4)^2 + O(\varepsilon^3)) f_3(\varepsilon; y_1 y_4, y_2, y_4, y_3/y_4),\\
\phi_{e(4)} &= x^{e(2)}(1 - 2\varepsilon \log y_2 + 2\varepsilon^2(\log y_2)^2 + O(\varepsilon^3)) f_4(\varepsilon; y_2, y_2 y_2, y_3/y_2, y_4),\\
\phi_{e(5)} &= x^{e(2)}(1 - 2\varepsilon \log(y_2 y_4) + 2\varepsilon^2(\log(y_2 y_4))^2 + O(\varepsilon^3)) f_5(\varepsilon; y_2, y_1 y_2 y_4, y_4, y_3/(y_2 y_4)),
\end{aligned}$$

where $f_i(\varepsilon; z_1, z_2, z_3, z_4) = \sum_{m \in \mathbf{N}_0^4} \frac{[e(i)]_{u_-}}{[e(i)+u]_{u_+}} z^m$, $u = \sum_{j=1}^4 m_j b^j_{e(i)}$. We expand f_i in ε as $f_i^{(0)} + \varepsilon f_i^{(1)} + \varepsilon^2 f_i^{(2)} + O(\varepsilon^2)$. We note that all $\phi_{e(i)}$, $i = 2, 3, 4, 5$ gives the same series when $\varepsilon = 0$, which implies $f_i^{(0)}$, $i = 2, 3, 4, 5$ are the same series. Therefore, we have

$$\begin{aligned}
\phi_{e(3)} - \phi_{e(2)} &= \varepsilon(x_{11}x_{22}x_{33})^{1/2}(\underline{-2f_2^{(0)} \log y_4} + f_3^{(1)} - f_2^{(1)}) + O(\varepsilon^2),\\
\phi_{e(4)} - \phi_{e(2)} &= \varepsilon(x_{11}x_{22}x_{33})^{1/2}(\underline{-2f_2^{(0)} \log y_2} + f_4^{(1)} - f_2^{(1)}) + O(\varepsilon^2),\\
\phi_{e(5)} - \phi_{e(2)} &= \varepsilon(x_{11}x_{22}x_{33})^{1/2}(\underline{-2f_2^{(0)} \log(y_2 y_4)} + f_5^{(1)} - f_2^{(1)}) + O(\varepsilon^2).
\end{aligned}$$

The coefficients of ε are solutions. Let us find the fourth solution. We have $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} f_{2345} = 0$, $f_{2345} = (\phi_{e(5)} - \phi_{e(2)}) - (\phi_{e(3)} - \phi_{e(2)}) - (\phi_{e(4)} - \phi_{e(2)})$. Therefore, the series f_{2345} starts with ε^2 and the coefficients ε^2 of f_{2345} is the fourth solution. It is

$$\begin{aligned}
&(x_{11}x_{22}x_{33})^{1/2}(\underline{2(\log y_2)(\log y_4) f_2^{(0)}} - 2\log(y_2 y_4) f_5^{(1)} + f_5^{(2)}\\
&\quad -2\log(y_2) f_3^{(1)} + f_3^{(2)} - 2\log(y_4) f_4^{(1)} + f_4^{(2)} + f_2^{(2)}).
\end{aligned}$$

Example 4.4.9 Let $\beta = (1,2)$ and $A = A(0134)$. We set $w = (0,1,2,0)$. Then, the Gröbner basis of I_A with respect to this order is

$$\underline{\partial_2\partial_3} - \partial_1\partial_4,\ \underline{\partial_1\partial_3^2} - \partial_2^2\partial_4,\ \underline{\partial_2^3} - \partial_1^2\partial_3,\ \underline{\partial_3^3} - \partial_2\partial_4^2.$$

Therefore, fake exponents are
$v^{(1)} = (1/2, 0, 0, 1/2)$, $v^{(2)} = (1/4, 1, 0, 1/4)$, $v^{(3)} = (1/4, 0, 1, -1/4)$, $v^{(4)} = (-1, 2, 0, 0)$. $\phi^{(1)}, \phi^{(2)}$ and $\phi^{(3)}$ are convergent series solutions, but $\phi^{(4)} \equiv 0$. By examining $\mathrm{in}_{(-w,w)}(I_A)$, we can find two more solutions: x_2^2/x_1, x_3^2/x_4, [13], [57].

Series solutions with logarithms are constructed for a class of non-generic β's to apply for the mirror symmetry [32], [33], [55]. For non-homogeneous I_A, series solutions are divergent in most cases, but there are a class of series solutions which are convergent. They are studied in [41] and [17]. The Gevrey order of divergent series solutions is studied in [52], [19]. The notion of fully supported series solutions is introduced in [28]. Rational solutions of $H_A(\beta)$ are studied in [14]. Algebraic solutions of it are studied in [9].

4.5 $E(k,n)$

We fix two numbers k and n satisfying $n \geq 2k \geq 4$. Let α_j be generic parameters satisfying $\sum_{j=1}^n \alpha_j = n - k$. The *hypergeometric function* of type $E(k,n)$ or the Aomoto-Gel'fand hypergeometric function is defined by the integral

$$\Psi(\alpha; u) = \int_C \prod_{j=1}^n (\sum_{i=1}^k u_{ij} s_i)^{\alpha_j} ds_2 \cdots ds_k,$$

where we put $s_1 = 1$ and u is a $k \times n$ matrix and C is a bounded $(k-1)$-cell in the hyperplane arrangement defined by $\prod_{j=1}^n \sum_{i=1}^k u_{ij} s_i = 0$ in the $(s_2, \dots, s_k)$-space [20].

The hypergeometric function of type $E(k,n)$ is quasi-invariant under the action of complex torus $(\mathbf{C}^*)^n$ and the general linear group $GL(k) = GL(k, \mathbf{C})$. In fact, we have, for $h = \mathrm{diag}(h_1, \dots, h_n) \in (\mathbf{C}^*)^n$ and $g \in GL(k)$,

$$\Psi(\alpha; uh) = \left(\prod_j h_j^{\alpha_j} \right) \Psi(\alpha; u), \quad \Psi(\alpha; gu) = |g|^{-1} \Psi(\alpha; u).$$

It follows from the quasi-invariant property and the integral representation that the function $\Psi(\alpha; u)$ satisfies a system of first order equations and a system of second order equations respectively.

Theorem 4.5.1 [20] *The function* $f = \Psi(\alpha; u)$ *satisfies*

$$\left(\sum_{i=1}^k u_{ip} \frac{\partial}{\partial u_{ip}} - \alpha_p \right) f = 0, \ p = 1, \dots, n, \quad \left(\sum_{p=1}^n u_{ip} \frac{\partial}{\partial u_{jp}} + \delta_{ij} \right) f = 0, \ i, j = 1, \dots, k,$$

$$\left(\frac{\partial^2}{\partial u_{ip} \partial u_{jq}} - \frac{\partial^2}{\partial u_{iq} \partial u_{jp}} \right) f = 0, \ i, j = 1, \dots, k, p, q = 1, \dots, n.$$

We call this system of equations $E(k,n)$.

When we restrict the hypergeometric system $E(k,n)$ to $u_{ij} = \delta_{ij}$ for $1 \leq i \leq k, 1 \leq j \leq k$, we obtain the A-hypergeometric system associated to $A(k-1, n-k-1)$ and $\beta = (-\alpha_1 - 1, \dots, -\alpha_k - 1, \alpha_{k+1} - 1, \dots, \alpha_{n-1} - 1)$. We denoted it by $E'(k,n)$. Here, $u_{i,j+k}$ stands for the variable x_{ij} in Section 4.2.

If $\Psi(\alpha; u)$ is a solution of $E(k,n)$, then $\Psi(\alpha^s; u^s)$, $s \in \mathfrak{S}_n$ is also a solution. This $\mathfrak{S}_n$ symmetry leads us Kummer type relations [58]. The confluent $E(k,n)$ is geometrically studied and a general framework to derive Kummer type relations are given (see [36] and its references).

4.6 Contiguity relations

4.6.1 Contiguity relations

We note the relation in the Weyl algebra D

$$\left(\sum_{j=1}^{n} a_{ij}\theta_j - \beta_i\right)\partial_k = \partial_k\left(\sum_{j=1}^{n} a_{ij}\theta_j - \beta_i - a_{ik}\right).$$

Since ∂_k commutes with $\Box_u$, we can see that if f is a solution of $H_A(\beta - a_k)$, then $\partial_k \bullet f$ is a solution of $H_A(\beta)$.

We consider the ideal B_k which is the intersection of $\mathbf{C}[s_1, \ldots, s_d]$ and the left ideal generated by ∂_k and $H_A(s)$ in $D[s_1, \ldots, s_d]$. When A is normal and I_A is homogeneous, this ideal can be expressed in terms of primitive support functions.

Theorem 4.6.1 *[43] The ideal B_k is the principal ideal generated by*

$$\prod_{\sigma\in S} \prod_{i=0}^{F_\sigma(a_k)-1} (F_\sigma(s) - i),$$

where S is a set of the facets of the convex hull of A for which $F_\sigma(a_k) > 0$ holds.

It follows from the theorem above that if $\beta \notin V(B_k)$, then there exists an operator $Q_k \in D$ such that $Q_k\partial_k = 1 \mod H_A(\beta)$. The operators ∂_k and Q_k give contiguity relations for A-hypergeometric series.

The symmetry algebra introduced in [45] gives contiguity relations of A-hypergeometric system in a general framework. The ideal B_k is a special case of the b-ideal introduced in the paper.

4.6.2 Contiguity relations for $E'(k, n)$

We give a contiguity relation for $E'(k, n)$ following [51]. We use the variable u_{ij} instead of x_{ij} as in Section 4.5. Put

$$X_{pa} = -u_{ap} - \sum_{q=k+1}^{n} u_{aq} \sum_{i=1}^{k} u_{ip}\partial_{iq}. \tag{4.6.9}$$

Let $\varphi(\alpha; u)$ be a solution of the system $E'(k, n)$ with the set of parameters α and 1_a the a-th unit vector in $\mathbf{Z}^n$.

Theorem 4.6.2 *[51]. We have $\partial_{ap}\varphi(\alpha; u) = \varphi(\alpha + 1_a - 1_p; u)$, $X_{pa}\varphi(\alpha; u) = \varphi(\alpha - 1_a + 1_p; u)$ and $X_{pa}\partial_{ap} - (\alpha_p - 1)\alpha_a \in H_A(\beta)$*

Introducing extra variables to hypergeometric series in several variables was done in the pioneering work of [35] to study contiguity relations. Contiguity relations for the Lauricella functions F_A, F_B, and F_C are derived with this idea and by utilizing the b-ideal B_k for them in [44]. See also Section 3.5 (of this volume, Chapter 3) as to some explicit contiguity relations of Lauricella functions.

4.6.3 Isomorphism among $M_A(\beta)$'s

We gave contiguity operators ∂_k and Q_k. If they exist, they give an isomorphism $\partial_k : M_A(\beta - a_k) \to M_A(\beta)$.

The question if $M_A(\beta)$ and $M_A(\beta')$ are isomorphic or not as left D-modules is a fundamental question. It was studied in [49, §4.4, §4.5] and a final answer was given in [45]. Let τ be a face of $\mathrm{pos}(A)$. Define

$$E_\tau(\beta) = \{\lambda \in \mathbf{C}(A \cap \tau)/\mathbf{Z}(A \cap \tau) \,|\, \beta - \lambda \in \mathbf{N}_0 A + \mathbf{Z}(A \cap \tau)\} \tag{4.6.10}$$

Theorem 4.6.3 *[45], [46, Th. 3.4.4] The left D-modules $M_A(\beta)$ and $M_A(\beta')$ are isomorphic if and only if $E_\tau(\beta) = E_\tau(\beta')$ for all faces τ of* $\mathrm{pos}(A)$.

The condition can be rewrited to a condition on the primitive integral supporting function when A is normal.

Theorem 4.6.4 *[45, Th 5.2] Assume A is normal and I_A is homogeneous. The left D-module $M_A(\beta)$ is isomorphic to $M_A(\beta')$ if and only if $\beta - \beta' \in \mathbf{Z}A$ and*

$$\{\sigma \,|\, \sigma \text{ is a facet and } F_\sigma(\beta) \in \mathbf{N}_0\} = \{\sigma \,|\, \sigma \text{ is a facet and } F_\sigma(\beta') \in \mathbf{N}_0\}. \tag{4.6.11}$$

4.7 Properties of A-hypergeometric equations

4.7.1 Rank formula and the Euler-Koszul complex

The holonomic rank $H_A(\beta)$ is the dimension of $R/(RH_A(\beta))$ as the vector space over the field of rational functions $\mathbf{C}(x_1, \dots, x_n)$. Here, R is the ring of differential operators with rational function coefficients. The rank of $H_A(\beta)$ is equal to the normalized volume of A for generic β and we have the inequality $\mathrm{rank}\, H_A(\beta) \geq \mathrm{vol}(A)$, [2], [21], [49]. More precise discussion requires the Euler-Koszul complex [27], [8].

We assume that A is pointed in the subsection. For $\partial^v \in S_n = \mathbf{C}[\partial_1, \dots, \partial_n]$, we define the A-multidegree of ∂^v by $-Av \in \mathbf{Z}^d$. We denote it by $\deg(\partial^u)$. Its i-th component is denoted by $\deg_i(\partial^u)$. This multidegree is naturally extended to the Weyl algebra D as $\deg(x^u \partial^v) = Au - Av$. Put $E_i = \sum_{j=1}^n a_{ij}\theta_j$. The multidegree of E_i is $\mathbf{0}$. The identity $\partial^v E_i = E_i \partial^v - \deg_i(\partial^v)\partial^v = (E_i - \deg_i(\partial^v))\,\partial^v$ is fundamental.

Let S_A be the ring $\mathbf{C}[\partial_1, \dots, \partial_n]/I_A$ which is isomorphic to $\mathbf{C}[t^{a_1}, \dots, t^{a_n}] = \mathbf{C}[\mathbf{N}_0 A]$. We denote $D_n \otimes_{S_n} S_A \simeq D_n/(D_n I_A)$ by D_A. We consider the complex

$$\mathcal{K}_\bullet \,:\, 0 \xleftarrow{d_0} D_A^{\binom{n}{0}} \xleftarrow{d_1} D_A^{\binom{n}{1}} \xleftarrow{d_2} \cdots \xleftarrow{d_{n-1}} D_A^{\binom{n}{n-1}} \xleftarrow{d_n} D_A^{\binom{n}{n}} \longleftarrow 0.$$

For A-homogeneous $a \otimes b \in D_A$, we define $(E_i - \beta_i) \circ (a \otimes b) = (E_i - \beta_i - \deg_i(a \otimes b))a \otimes b$. We denote the basis of $D_A^{\binom{d}{k}}$ by $e_{i_1, \dots, i_k}$, $1 \leq i_1 < \cdots < i_k \leq d$. The boundary map d_k is defined by

$$D_A^{\binom{d}{k}} \ni (a \otimes b)e_{i_1 \dots, i_k} \mapsto \sum_{i_j \in \{i_1, \dots, i_k\}} (E_{i_j} - \beta_{i_j}) \circ (a \otimes b)(-1)^{j-1} e_{\{i_1, \dots, i_k\} \setminus \{i_j\}} \in D_A^{\binom{d}{k-1}}. \tag{4.7.12}$$

The complex is called the Euler-Koszul complex over D_A.

The Euler-Koszul complex on D_A by $E_i - \beta_i$, $i = 1, \ldots, d$ is well-defined, because we have $(E_i - \beta_i) \circ (a \otimes \Box_u) = (a\Box_u(E_i - \beta_i)) \otimes 1 = (a(E_i - \beta_i - \deg_i(\partial^{u_+}))\Box_u) \otimes 1 \equiv 0$. The homology group $\mathcal{H}_i(E-\beta; S_A) = H_i(\ker d_i / \operatorname{Im} d_{i-1})$ of the Euler-Koszul complex has a natural A grading by the A-multidegree. The 0-th homology group is nothing but $M_A(\beta)$. This leads us to more functorial object to study A-hypergeometric system, which is the Euler-Koszul homology for toric modules [27]. We fix $E - \beta$ and replace S_A by (A-)toric modules. We only present an example of toric modules. Let A be $A(0134)$ and $\tilde{A}$ be its saturation. Note that $n = 4$ and the multigrading is defined by A. We may suppose $\tilde{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 & 2 \end{pmatrix}$ and $S_{\tilde{A}} = D_5 / I_{\tilde{A}}$. Then, we have a short exact sequence

$$0 \longrightarrow D_4 \otimes_{S_4} S_A \longrightarrow D_4 \otimes_{S_4} S_{\tilde{A}} \longrightarrow D_4 \otimes_{S_4} S_{\tilde{A}} / S_A \longrightarrow 0$$

All modules are A-graded and toric modules. $C = D_4 \otimes S_{\tilde{A}} / S_A$ has the support only at the degree $(1, 2)$. We have $\mathcal{H}_0(E-\beta; D_4 \otimes S_{\tilde{A}}) \simeq D_5 / x_5 D_5 \otimes_{D_5} M_{\tilde{A}}(\beta) \simeq M_{\tilde{A}}(\beta)$ and $H_0(E-\beta; C) = 0$ (resp. $= D_4 \otimes [\partial_5]$) when $\beta \neq (1, 2)$ (resp. $\beta = (1, 2)$).

Theorem 4.7.1 *[27] Put* $\mathbf{m} = \langle \partial_1, \ldots, \partial_n \rangle$*, which is a maximal ideal in* $S_n = \mathbf{C}[\partial_1, \ldots, \partial_n]$.

1. *If* k *equals the smallest homological degree* i *for which* $-\beta$ *is a quasi degree of* $H^i_{\mathbf{m}}(S_A)$*, then the Euler-Koszul homology* $\mathcal{H}_{d-k}(E - \beta; S_A)$ *is non-zero rank and* $\mathcal{H}_i = 0$ *for* $i > d - k$*. Here,* γ *is called the quasi degree when* γ *is contained in the Zariski closure of the non-zero degrees of the homology group.*
2. $H^i_{\mathbf{m}}(S_A) = 0$ *holds for* $0 \leq i < d$*, if and only if* S_A *is Cohen-Macaulay.*
3. *The rank of* $H_A(\beta)$ *equals to the normalized volume of* A *if and only if* β *is not a quasi-degree of* $H^i_{\mathbf{m}}(S_A)$.

Put $\varepsilon_A = \sum a_i$. The degree $-\alpha + \varepsilon_A$ part of the local cohomology group is $H^{n-i}_{\mathbf{m}}(S_A)_{-\alpha+\varepsilon_A} = \operatorname{Hom}_{\mathbf{C}}\left(\operatorname{Ext}^i_{S_n}(S_A, S_n)_\alpha, \mathbf{C}\right)$.

Example 4.7.2 We consider the case $A = A(0134)$, $\varepsilon_A = (4, 8)^T$. Construct A-graded resolution of R/I_A by Schreyer's method. Then, we have $\operatorname{Ext}^4 = 0$ and $\operatorname{Ext}^3 = \mathbf{C}$ at the degree $(5, 10)$, which implies that $H^{4-3}_{\mathbf{m}} \neq 0$ at the degree $-(1, 2)$. In fact, the rank of the system is 5 when $\beta = (1, 2)$ and it is 4 when $\beta \neq (1, 2)$.

4.7.2 Characteristic variety and principal *A*-determinant

Let I be a left ideal in D. The initial ideal $\operatorname{in}_{(\mathbf{0},\mathbf{1})}(I)$ is the ideal in $\mathbf{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$ generated by the principal symbols of I. The ideal is called the characteristic ideal of I, and the zero set of the ideal in $\mathbf{C}^{2n}$ is called the characteristic variety of D/I and is denoted by $\operatorname{Ch}(D/I)$. The projection of $\operatorname{Ch}(D/I) \setminus V(\xi_1, \ldots, \xi_n)$ to $\mathbf{C}^n = \{x\}$ is called the singular locus of D/I and is denoted by $\operatorname{Sing}(D/I)$ (see, e.g., [49, p.36]).

Theorem 4.7.3 *[21], [23]*

1. *If $H_1(\mathrm{gr}_{(\mathbf{0},\mathbf{1})}\mathcal{K}_\bullet) = 0$, then the characteristic ideal of $H_A(\beta)$ is generated by $Ax\xi$ and $I'_A = I_A|_{\partial\to\xi}$. Here, we denote by $Ax\xi$ the ideal generated by $\sum_{j=1}^n a_{ij}x_j\xi_j$, $(i = 1,\dots,d)$.*
2. *If I_A is Cohen-Macaulay, then the first homology above vanishes.*

Characteristic varieties and micro-characteristic varieties of $M_A(\beta)$ are combinatorially studied in [21], [52].

Let E_A be the principal A-determinant [24]. The projection of $V(\langle Ax\xi, I'_A\rangle) \setminus V(\xi_1,\dots,\xi_n)$ to $\mathbf{C}^n$ is expressed as $V(E_A)$.

Theorem 4.7.4 *[24, p.300] The principal A-determinant for $A(k,k')$ $(k \le k')$ is the product of the determinants of all $p \times p$ minors of the matrix (x_{ij}) where $1 \le p \le k$.*

Example 4.7.5 For $A = A(1,k'-1)$, we have

$$E_A = \prod_{i=1}^{2}\prod_{j=1}^{k'} x_{ij} \prod_{1\le j<j'\le k'} \begin{vmatrix} x_{1j} & x_{1j'} \\ x_{2j} & x_{2j'} \end{vmatrix}.$$

The variety $V(E_A)$ is the singular locus of $H_A(\beta)$.

4.7.3 Reducibility and monodromy groups

We consider the set $\cup_\tau(\mathbf{Z}A + \tau)$ where the union is taken over all linear subspaces τ of $\mathbf{C}^d$ that form a boundary component of pos(A). The set is called the resonant parameters and is denoted by Res(A).

Let R be the ring of differential operators with rational function coefficients. We consider the left R-module $\mathbf{C}(x_1,\dots,x_n)\otimes_{D_n} M_A(\beta) = R/(RH_A(\beta))$. If this module has a non-zero proper R-submodule, it is called reducible.

Theorem 4.7.6 *[10] When I_A is homogeneous and A is not a pyramid, $\mathbf{C}(x_1,\dots,x_n)\otimes M_A(\beta)$ is reducible if and only if $\beta \notin \mathrm{Res}(A)$.*

An analog of this theorem holds without the homogeneous condition. See [53]. The irreducible quotients as D-modules of $M_A(\beta)$ are combinatorially discussed in [47].

The reducibility of a Lauricella system can be described by the reducibility of the corresponding A-hypergeometric system. See [29, §6] as to the case of Lauricella's F_C. A systematic approach to study reducibilities for Appell-Horn or Mellin type systems is given in [61].

Connection formulas are studied for $A(1,n)$ by restrictions [50]. The global monodromy groups are calculated for some interesting A's. See [37], [38], [60] for the case of $A(2,2)$. See [40] for some of 3-dimensional Fano polytopes related to families of $K3$ surfaces. See this volumne, Chapter 3, Section 3.7 for monodromy groups for Lauricella functions. The monodromy at infinities is discussed; see [4] and its references. Recently, a general method to compute a subgroup of monodromy groups is proposed [11].

4.8 A-hypergeometric polynomials and statistics

We assume A is a configuration matrix. The A-hypergeometric polynomial [48] for the configuration A and the parameter vector $\beta \in \mathbf{N}_0^d$ is defined by

$$Z(\beta; p) = \sum_{Au=\beta, u\in\mathbf{N}_0^n} \frac{p^u}{u!}, \tag{4.8.13}$$

where $p^u = \prod_{i=1}^n p_i^{u_i}$ and $u! = \prod_{i=1}^n u_i!$. It is a solution of the A-hypergeometric system $H_A(\beta)$. Set $p_i = \exp \xi_i$ and let $\exp \xi$ denote the vector $(\exp \xi_1, \ldots, \exp \xi_n)$. We fix $\beta \neq 0$ such that $\beta \in \mathbf{N}_0 A = \sum_{i=1}^n \mathbf{N}_0 a_i$. Let $U \in \mathbf{N}_0^n$ be a random variable of the (A,β) *hypergeometric distribution with the parameter* $p \in \mathbf{R}_{>0}^n$ (or $\xi \in \mathbf{R}^n$), which is defined by the probability that U takes the value u

$$P(U = u \,|\, Au = \beta) = \frac{p(\xi)^u}{u! Z(\beta; p(\xi))} = \frac{\exp(u \cdot \xi)}{u! Z(\beta; p(\xi))}, \quad u \cdot \xi = \sum_{i=1}^n u_i \xi_i. \tag{4.8.14}$$

It is the conditional distribution of u given by $\beta = Au$ under the Poisson distribution

$$P(U = u) = \frac{p^u}{u!} \exp(-\mathbf{1} \cdot p), \quad \mathbf{1} = (1, \ldots, 1). \tag{4.8.15}$$

The polynomial Z is the *normalizing constant* or the *partition function* of the (A,β) hypergeometric distribution. The $(A(k,k'),\beta)$ hypergeometric distribution has been called the generalized hypergeometric distribution for $(k+1) \times (k'+1)$ contingency tables with the marginal sum β in statistics [30, 4.1].

Let $\overline{A}$ be an $n \times (n-d)$ matrix with integer entries satisfying the conditions $A\overline{A} = 0$ and that the rank of $\overline{A}$ as a $\mathbf{Q}$-matrix is $n-d$. We denote by $\overline{a}_i$ the i-th column vector of $\overline{A}$. An asymptotic study of the probability distribution (4.8.14) gives the following theorem, which gives an approximate value when the parameter vector of the A-hypergeometric polynomial becomes large as $\kappa\beta$, $(\kappa \to +\infty)$.

Theorem 4.8.1 *[59] We fix* $p \in \mathbf{R}_{>0}^n$ *and* $\beta \in \mathbf{N}_0 A \cap \mathrm{int}(\mathbf{R}_{\geq 0} A)$. *There exists a unique* $m \in \mathbf{R}_{>0}^n$ *such that* $Am = \beta$, $m^{\overline{a}_i} = p^{\overline{a}_i}$. *When* $\kappa \to +\infty$, *we have*

$$Z(\kappa\beta; p) \sim \frac{\left(\prod p_i^{m_i}\right)^{\kappa}}{\Gamma(\kappa m + 1)} \frac{(2\pi\kappa)^{n-d}}{\det(\overline{A} M^{-1} \overline{A}^T)^{1/2}},$$

where $M = \mathrm{diag}(m)$.

Conversely, applications of A-hypergeometric equations to statistics are given in [31], [59].

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