Pfaffian Systems of A-Hypergeometric Equations I: Bases of Twisted Cohomology Groups

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Abstract: We consider bases of Pfaffian systems for A-hypergeometric systems. These are given by Gröbner deformations, they also provide bases for twisted cohomology groups. For a hypergeometric system associated with a class of order polytopes, these bases have a combinatorial description. The size of the bases associated with a subclass of the order polytopes has a growth rate of polynomial order.

1 Introduction

The gamma distribution in statistics is a probability distribution on $t \in (0, +\infty)$ with two parameters $\gamma > 0$ (shape) and $x > 0$ (rate). The probability density function is written as $\exp(-xt)t^{\gamma-1}/\Phi(\gamma; x)$ where the normalizing constant Φ can be expressed in terms of the gamma function as

$$\Phi(\gamma; x) = \int_0^{+\infty} \exp(-xt)t^{\gamma-1}dt = x^{-\gamma}\Gamma(\gamma).$$

Numerical evaluation of the Gamma function is an important problem to apply the Gamma distribution to problems in statistics. In [12], [22], a new method to evaluate numerically normalizing constants for a class of unnormalized distributions was proposed. It is the holonomic gradient method (HGM). The key step of this method is to construct a Pfaffian system of differential or difference equations associated to the normalizing constant.

In a series of papers, we are going to study numerical evaluations of A-hypergeometric functions regarded as a generalization of the gamma and the beta distributions by the HGM, which leads us interesting mathematical problems. We will discuss on one of them, which is a method to construct bases of Pfaffian systems associated to A-hypergeometric functions.

Let $g(x,t) = \sum_{a \in \mathcal{A}} x_a t^a$, $t^a = t_1^{a_1} \cdots t_d^{a_d}$ be a generic sparse polynomial in $t = (t_1, \ldots, t_d)$ with the support on a finite set of points $\mathcal{A} \subset \mathbf{Z}^d$. The

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coefficients x_a, $a \in \mathcal{A}$ are denoted by x_i, $i = 1, \ldots, n$. The function defined by the integral

$$\Phi(x) = \int_{C_1} g(x,t)^\alpha t^\gamma dt, \quad \text{or } \Phi(x) = \int_{C_2} \exp(g(x,t)) t^\gamma dt, \ dt = dt_1 \cdots dt_d$$

over a cycle C_i in the t-space is called an *A-hypergeometric function* of x with parameters $\alpha \in \mathbf{C}$, $\gamma_i \in \mathbf{C}$ [1], [9], [11]. It is known that the A-hypergeometric function satisfies a system of linear partial differential equations in x, which is called the *A-hypergeometric system*. The A-hypergeometric system is a holonomic system, and the operators of the system generate a zero-dimensional ideal in the ring of differential operators with rational function coefficients (see, e.g., [13, Chapter 6]). A-hypergeometric systems have been studied for the past 25 years (see, e.g., [10], [11], [26]), and they have applications in many fields.

The function $g(x,t)^\alpha t^\gamma / \Phi(x)$ or $\exp(g(x,t)) t^\gamma / \Phi(x)$ can be regarded as a probability distribution function on C_i with parameters x, α, γ satisfying certain conditions. This distribution, which we will call the *A-distribution*, is a generalization of the beta distribution or the gamma distribution. In this context, the function $\Phi(x)$ is called the normalizing constant of the A-distribution. In [12], [13], [22], some new statistical methods were proposed. These were the holonomic gradient method (HGM) and the holonomic gradient descent (HGD). The HGM is a method for numerically evaluating the normalizing constant, which is a function of the parameters x, for a given unnormalized probability distribution, and the HGD uses the HGM to obtain the maximum likelihood estimate. The key step for both of these methods is to construct a Pfaffian system associated with the normalizing constant. The size of the Pfaffian system determines the complexity of the HGM and the HGD (see, e.g., [19]). The HGM and HGD lead us to the following fundamental goals for applying A-hypergeometric systems to statistics.

1. Find an efficient method for constructing a Pfaffian system associated with a given A-hypergeometric system.

2. Find a subclass of A-hypergeometric systems for which the associated Pfaffian systems are of moderate size.

Let us turn into more precise statements of the mathematical problem which will be discussed in this paper. Let F be a vector-valued function in $x_1, \ldots, x_n$. We suppose the length of F is r and that F is a column vector. Let $Q_i(x)$, $i = 1, \ldots, n$, be $r \times r$ matrices satisfying

$$\frac{\partial Q_i}{\partial x_j} + Q_i Q_j = \frac{\partial Q_j}{\partial x_i} + Q_j Q_i$$

for all $i \neq j$. The system of linear differential equations

$$\frac{\partial F}{\partial x_i} = Q_i(x) F, \ i = 1, \ldots, n$$

is called a *Pfaffian system.* We also call the system of linear differential operators $\frac{\partial}{\partial x_i} - Q_i$ a Pfaffian system. The number r is called the *size* or the *rank* of the Pfaffian system. For a given zero-dimensional left ideal in the ring of differential operators with rational function coefficients, it is well known that an associated Pfaffian system can be obtained using a Gröbner basis method and some computer algebra systems can perform this translation (see, e.g., [22, Appendix]). However, in general, this computation is difficult, and we wish to provide an efficient method for translating the A-hypergeometric system into a Pfaffian system.

Twisted cohomology groups can be used as a geometric method for finding a Pfaffian system associated with a given definite integral that contains parameters (see, e.g., the book by Aomoto-Kita [3, Chapter 3, §8]). This approach is as follows: (1) obtain a basis for a twisted cohomology group, and (2) calculate the Pfaffian system associated with that basis. We will use this approach to obtain a Pfaffian system, and in this paper, we consider the step (1).

Gel'fand, Kapranov, and Zelevinsky expressed A-hypergeometric functions with regular singularities as pairings of twisted cycles and twisted cocycles [11]. Esterov and Takeuchi expressed confluent A-hypergeometric functions as pairings of rapid-decay twisted cycles and twisted cocycles [9]. The cohomology groups that come from geometry and are associated with A-hypergeometric systems were discussed by Adolphson and Sperber [2]. The next step is to obtain explicit bases for these twisted cohomology groups. Orlik and Terao provided the βnbc bases for the twisted cohomology groups associated with hyperplane arrangements [25] (see also Remark 3). Aomoto, Kita, Orlik, and Terao [4] provided a basis for a class of confluent hypergeometric integrals. In this paper, we will give a computational method for determining the bases of the twisted cohomology groups associated with generic sparse polynomials or any A-hypergeometric system, and we will also give a combinatorial method for a class of generic sparse polynomials.

Let R_n be the ring of differential operators with rational function coefficients in $K(x)$ of n-variables $x = (x_1, \ldots, x_n)$ where K is a field of characteristic 0. The first step in finding a Pfaffian system associated with a zero-dimensional left ideal I in R_n is to obtain a basis for R_n/I as a $K(x)$-vector space. It is well known that a basis can be obtained by computing a Gröbner basis of I in R_n.

Definition 1 Suppose that a left ideal I of R_n is zero-dimensional. Let $\{u_1, \ldots, u_r\}$ be a basis of R_n/I as a $K(x)$ vector space and $Q_i(x)$ be $r \times r$ matrix satisfying $\frac{\partial}{\partial x_i} U \equiv Q_i U \mod I$, $U = (u_1, \ldots, u_r)^T$. The system of linear differential operators $\partial/\partial x_i - Q_i$, $i = 1, \ldots, n$ is called a *Pfaffian system for the basis* $\{u_i\}$ and the basis is called the *basis of the Pfaffian system.*

In Theorem 1, we show that standard monomials for an ideal in a polynomial ring give bases and provide an algorithm that is more efficient than computing the Gröbner basis of I itself. In Theorems 2 and 3, we show that this gives a basis of the twisted cohomology group.

Our theorems are not only useful for computations, but they also pose interesting theoretical problems in commutative algebra and combinatorics. We study the hypergeometric system associated with a class of order polytopes (see, e.g., [15]). We prove that bases of Pfaffian systems or twisted cohomology groups have combinatorial descriptions (Theorems 4 and 7). The size of the Pfaffian system associated with a subclass of the order polytopes has a growth rate of polynomial order (Theorem 6). We expect that our results will yield a new class of exponential probability distributions for which we can efficiently apply the holonomic gradient method (HGM) and the holonomic gradient descent (HGD). Construction algorithms for Pfaffian systems, utilizing the results of this paper and examples of numerical evaluations, will be discussed in next papers.

2 Bases for the Pfaffian System

We denote by $A = (a_{ij})$ a $d \times n$-matrix whose elements are integers. We suppose that the set of the column vectors of A spans $\mathbf{Z}^d$. Let $s_1, \ldots, s_d$ be indeterminates. Let $D(s)$ be the Weyl algebra

$$D(s) = \mathbf{C}(s_1, \ldots, s_d)\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle,\ \partial_i = \partial/\partial x_i$$

over the field $\mathbf{C}(s) = \mathbf{C}(s_1, \ldots, s_d)$.

Definition 2 ([10], [26, p.105])

1. The ideal in $\mathbf{C}[\partial_1, \ldots, \partial_n]$ generated by

$$\prod_{i=1}^{n} \partial_i^{\mu_i} - \prod_{j=1}^{n} \partial_j^{\nu_j} \tag{1}$$

(with $\mu, \nu \in \mathbf{N}_0^n$ running over all μ, ν such that $A\mu = A\nu$).

   is called the *affine toric ideal* and is denoted by I_A.

2. The left ideal in $D(s)$ generated by the elements of I_A and

$$E_i - s_i := \sum_{j=1}^{n} a_{ij} x_j \partial_j - s_i, \qquad (i = 1, \ldots, d) \tag{2}$$

   is called the *A-hypergeometric ideal or system with indefinite parameters* $\{s_i\}$. The left ideal is denoted by $H_A(s)$.

For complex parameters β_i, the system of linear differential equations $(E_i - \beta_i)f = 0$ $(i = 1, \ldots, d)$, $(\partial^\mu - \partial^\nu)f = 0$ $(A\mu = A\nu)$ is called the A-hypergeometric system of differential equations or just the A-hypergeometric system.

Let R_n be the ring of differential operators with rational function coefficients

$$R_n = \mathbf{C}(s, x)\langle \partial_1, \ldots, \partial_n \rangle. \tag{3}$$

We are interested in bases of $R_n/(R_n H_A(s))$ as a vector space over the field $\mathbf{C}(s, x) = \mathbf{C}(s_1, \ldots, s_d, x_1, \ldots, x_n)$. Any basis of the vector space yields an associated Pfaffian system or an integrable connection associated with $H_A(s)$. Let $u_1, \ldots, u_r$ be a basis of $R_n/(R_n H_A(s))$. For u_j, there exist rational functions $p_{ij}^k \in \mathbf{C}(s,x)$ such that $\partial_i u_j \equiv \sum_{k=1}^r p_{ij}^k u_k \bmod R_n H_A(s)$. The action of a differential operator u on a function F is denoted by $u \bullet F$. The system of differential equations $\partial_i \bullet F = (p_{ij}^k \,|\, 1 \le j,k \le r)F$, where F is a vector valued function of size r, is called a Pfaffian system, and $\{u_i\}$ is the basis of the Pfaffian system (Definition 1).

Let w be a vector in $\mathbf{Z}^n$. For an ideal I in $\mathbf{C}[\partial_1, \ldots, \partial_n]$, we denote by $\mathrm{in}_w(I)$ the ideal generated by $\mathrm{in}_w(\ell)$, $\ell \in I$, which is the sum of the highest w-order terms in ℓ. The ideal is called the *initial ideal* of I for the weight vector w. For a left ideal J in $D(s)$ and $u, v \in \mathbf{Z}^n$ such that $u+v \ge 0$, the initial ideal of J, which is denoted by $\mathrm{in}_{(u,v)}(J)$, is analogously defined (see, e.g., [26, p.4]). It is known that $\mathrm{in}_{(-w,w)}(H_A(s))$ is generated by $\mathrm{in}_w(I_A)$ and $E_i - s_i$, $i = 1, \ldots, d$ [26, Th. 3.1.3].

Bases of $R_n/(R_n H_A(s))$ can be described by simpler quotients as in the following theorem.

Theorem 1 *Let $w \in \mathbf{Z}^n$ be a generic weight vector for the affine toric ideal I_A such that $\deg \mathrm{in}_w(I_A) = \deg I_A$. Let $u_1, \ldots, u_r$ be a monomial basis of $R_n/(R_n J)$, where the left ideal J is generated by $\mathrm{in}_w(I_A)$ and $E_i - s_i$, $i = 1, \ldots, d$ in R_n. Then, the set $\{u_1, \ldots, u_r\}$ is a basis of the vector space $R_n/(R_n H_A(s))$.*

Proof. We denote by r the normalized volume of A. Since the s_i are indeterminate, the holonomic ranks of J and $H_A(s)$ are r by Adolphson's theorem (see, e.g., [1], [26]). In other words, we have $\dim_{\mathbf{C}(s,x)} R_n/(R_n H_A(s)) = r$ and $\dim_{\mathbf{C}(s,x)} R_n/J = r$.

We may assume that the u_i are expressed as monomials in terms of Euler operators $\theta_j = x_j \partial_j$. When we regard J as a system of linear differential equations, it has r linearly independent solutions of the form x^ρ, where $\rho \in \mathbf{C}(s)^n$. We denote them by $g_i = x^{\rho(i)}$, $i = 1, \ldots, r$. Since the g_i are linearly independent solutions, the Wronskian determinant $\det(u_i \bullet g_j)$ is not identically equal to 0. The solution g_j can be extended to a solution f_j of $H_A(s)$ such that g_j is the leading monomial of f_j with respect to the weight vector w (see, e.g., [23], [26, Chapters 2 and 3]). The series f_j is expressed as $f_j = g_j \sum_{\ell \in M_j} C_\ell x^\ell$, $C_0 = 1$, where M_j denotes the set of lattice points in a cone and C_ℓ is a constant belonging to $\mathbf{C}(s)$. The series converges in the space of convergent power series $g_j \cdot \mathcal{O}(U)\{M_j\}$, where U is an open set in the s-space and $\mathcal{O}(U)$ is the space of holomorphic functions on U [23]. We replace x_i by $x_i t^{w_i}$ for all i in f_j and denote by xt^w the vector $(x_1 t^{w_1}, \ldots, x_n t^{w_n})$. From the construction algorithm of f_j, we may assume that $f_j(xt^w) = g_j(xt^w)(1 + O(t))$ when $t \to 0$ as a function of t when x is fixed and s lies in U.

Let us prove $W = \det(u_i \bullet f_j) \not\equiv 0$. We denote by $u_i(\rho(j))$ the constant $(u_i \bullet x^{\rho(j)})/x^{\rho(j)}$. Under this notation, we have $x^{-\rho(j)} u_i \bullet g_j = u_i(\rho(j))$ and

$$(x^{-\rho(j)}(u_i \bullet f_j))(xt^w) = \sum_{\ell \in M_j} u_i(\rho(j) + \ell) C_\ell x^\ell t^{\ell w}. \tag{4}$$

Note that $\ell w > 0$ for $\ell \neq 0$ and $\ell \in M_j$. Therefore, we have

$$\det(x^{-\rho(j)} u_i \bullet f_j)(xt^w) = \det(x^{-\rho(j)} u_i \bullet g_j)(xt^w) + O(t) \tag{5}$$

from (4) when x is fixed and $t \to 0$. This implies that the Wronskian determinant $\det(u_i \bullet f_j) = \left(\prod_j x^{\rho(j)}\right) \det(x^{-\rho(j)} u_i \bullet f_j)$ is not identically equal to 0. Therefore the u_i are linearly independent in $R_n/(R_n H_A(s))$. Q.E.D.

Let M be a monomial ideal in $\mathbf{C}[\partial]$. When M is generated by $\{\partial^\alpha\}$, the distraction $\widetilde{M} \subset \mathbf{C}[\theta]$ is generated by $\prod_{i=1}^n \theta_i(\theta_i - 1) \cdots (\theta_i - \alpha_i + 1)$, where $\theta_i = x_i \partial_i$ [26, p.68]. Let $M = \mathrm{in}_w(I_A)$. Then, the ideal J in Theorem 1 is generated by $\widetilde{M}$ and $\sum_{j=1}^n a_{ij}\theta_j - s_i$, $i = 1, \ldots, d$ [26, Sec. 2.3, Prop. 3.1.5]. This leads us to the following corollary.

Corollary 1 *Retain the assumptions of Theorem 1. Let $\tilde{J}$ be the ideal generated by $\widetilde{M}$ and $\sum a_{ij}\theta_j - s_i$, $i = 1, \ldots, d$ in the polynomial ring $\mathbf{C}(s)[\theta]$ over the field $\mathbf{C}(s)$. The set of the monomial basis of $\mathbf{C}(s)[\theta]/\tilde{J}$ gives a basis of $R_n/R_n H_A(s)$ by the replacement $\theta_i = x_i \partial_i$.*

Remark 1 Theorem 1 and Corollary 1 give an efficient method to find bases for Pfaffian systems. Let w be a vector in $\mathbf{Z}^n$. It is shown in [26, p.6 and Th. 3.1.3] that the set of $<_w$-Gröbner basis of I_A and $E_i - s_i$'s is a $<_{(-w,w)}$-Gröbner basis of $H_A(s)$ in the Weyl algebra $D(s)$. Let $<_1$ be a term order in the ring of differential operators with rational function coefficients R_n and we extend it to an order $<$ on $D(s)$ by a block ordering as $x^\alpha \partial^\beta < x^{\alpha'} \partial^{\beta'}$ if and only if $\partial^\beta <_1 \partial^{\beta'}$ or ($\beta = \beta'$ and $x^\alpha <_2 x^{\alpha'}$) where $<_2$ is a term order in $\mathbf{C}(s)[x]$. It is known that any $<$-Gröbner basis in $D(s)$ is a $<_1$-Gröbner basis in R_n (see, e.g., [13, Th 6.9.3]). This implies that Gröbner bases by block orders in the Weyl algebra can be regarded as Gröbner bases in R_n. Although the $<_{(-w,w)}$-Gröbner basis of $H_A(s)$ has a simple form, Gröbner bases for block orders are not as simple as it in general. For example, set $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. The set $G = \{\partial_1\partial_3 - \partial_2^2, E_1 - s_1 := x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - s_1, E_2 - s_2 := x_2\partial_2 + 2x_3\partial_3 - s_2\}$ is a $<_{(-w,w)}$-Gröbner basis for any $w \in \mathbf{Z}^3$. On the other hand, the Gröbner fan of $H_A(s)$ in the homogenized Weyl algebra [26, Sec. 2.1] when s is specialized to $(1/2, 1/3)$ [1] consists of 26 maximal dimensional cones and only one of them stands for the Gröbner basis of the 3 elements and other cones stand for more complicated Gröbner bases. For example, when $<_1$ is the graded lexicographic order, the Gröbner basis is the union of G and

$$\begin{aligned}\{ \quad & -2x_1\partial_2^2 - 2x_2\partial_2\partial_3 - 2x_3\partial_3^2 - \partial_3, \\ & -12x_1x_3\partial_2\partial_3 - 6x_2x_3\partial_3^2 - 4x_1\partial_2 - 7x_2\partial_3, \\ & g_6 := 6x_2^2x_3\partial_3^2 - 24x_1x_3^2\partial_3^2 + 4x_1x_2\partial_2 + 7x_2^2\partial_3 - 20x_1x_3\partial_3\}\end{aligned}$$

[1] Computing the Gröbner fan with indeterminates s_i is not easy on our software "kan/sm1 gfan.sm1 package (http://www.openxm.org)". Then, we specialize s_i's to numbers.

The corresponding Gröbner basis in R_3 is $\{E_1 - s_1, E_2 - s_2, g_6\}$. The operator g_6 does not have a simple form even in this case. The set of the *standard monomials* for a Gröbner basis is the set of the monomials which are irreducible by the division in terms of the Gröbner basis. The theory of Gröbner basis tells that the set of the standard monomials gives a basis of the quotient ring by the ideal standing for the Gröbner basis as a vector space over the coefficient field. Theorem 1 and Corollary 1 claim that a basis of $R_n/R_n H_A(s)$ can be obtained by a computation of a Gröbner basis in the polynomial ring and we do not need a computation of a Gröbner basis in R_n. We can expect the former method is more efficient than the latter method of computing a Gröbner basis in R_n to obtain a basis of the quotient space. In fact, our computational experiments in [16] and [24, Sec. 6] support this expectation.

3 Bases of Twisted Cohomology Groups

Let $A_1 = (a_1, \ldots, a_{n_1}), \ldots, A_k = (a_{n_{k-1}+1}, \ldots, a_{n_k})$, $a_i \in \mathbf{Z}^\delta$. To each matrix A_j, we associate a generic sparse polynomial in t

$$f_j(x,t) = \sum_{i=n_{j-1}+1}^{n_j} x_i t^{a_i}, \tag{6}$$

where $t^b = \prod_{i=1}^{\delta} t_i^{b_i}$. For parameters $\alpha_1, \ldots, \alpha_k$ and $\gamma_1, \ldots, \gamma_\delta$, we consider the integral

$$\Phi(\alpha, \gamma; x) = \int_C P(x,t) dt_1 \cdots dt_\delta, \ P(x,t) = \prod_{j=1}^{k} f_j(x,t)^{\alpha_j} t^\gamma \tag{7}$$

for a suitable twisted cycle C [11, 2.2]. The function Φ is satisfied by the A-hypergeometric system for

$$A = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & & \cdot & \cdots & \cdot \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 1 \\ a_1 & \cdots & a_{n_1} & a_{n_1+1} & \cdots & a_{n_2} & & a_{n_{k-1}+1} & \cdots & a_{n_k} \end{pmatrix} \tag{8}$$

and $\beta = (\alpha_1, \ldots, \alpha_k, -\gamma_1 - 1, \ldots, -\gamma_\delta - 1)^T$, where we assume that the rank of A is maximal [11]. Set $P' = P|_{\alpha=\gamma=1}$. Let $n = \sum_{i=1}^k n_i$, and define a projection p by

$$p \ : \ \mathbf{C}^{\delta+n} \setminus V(P') \ni (x,t) \mapsto x \in \mathbf{C}^n.$$

We regard P as a function in $t = (t_1, \ldots, t_\delta)$ with the parameter vector x. Define the connection ∇ with rational function coefficients by

$$\nabla = d_t + \sum_{j=1}^{\delta} \left(\frac{\partial P}{\partial t_j} / P \right) dt_j \tag{9}$$

where d_t is the exterior derivative with respect to the variables $t_1, \ldots, t_\delta$.

We fix a parameter vector x. Then, $p^{-1}(x)$ is a complement in $\mathbf{C}^\delta$ of the algebraic variety $V(P')$ where P' is regarded as a polynomial in t. The (rational) twisted cohomology group $H^\delta(p^{-1}(x), \nabla)$ is defined by

$$\frac{\mathbf{C}[t_1, \ldots, t_\delta, 1/P'] dt_1 \wedge \cdots \wedge dt_\delta}{\nabla \left(\sum_{i=1}^{\delta} \mathbf{C}[t_1, \ldots, t_\delta, 1/P'] \wedge_{j \neq i} dt_j \right)} \tag{10}$$

which is a $\mathbf{C}$-vector space. When we say a basis of the twisted cohomology group, it means that a basis of the cohomology group as the $\mathbf{C}$ vector space.

Theorem 2 *Assume that the matrix A is expressed as (8). Let α, γ be generic complex parameters, and let $\{u_1, \ldots, u_r\}$ be a basis as given in Theorem 1. Then the set of rational expressions $\{(u_i \bullet P)/P\}$ is a basis of the twisted cohomology group $H^\delta(p^{-1}(x), \nabla)$ when x lies outside of an analytic set.*

We mean by "generic complex parameters" parameters such that a set of logarithm free solutions [26, Sec. 3.4] spans the solution space when we specialize s to the parameter vector β in the proof of Theorem 1.

Proof. It follows from the local triviality theorem [29, 5.1 Corollaire] that the projection p is a locally trivial map on a Zariski open subset U of $\mathbf{C}^n$. Take a point x_0 in U. Then the inverse image $p^{-1}(U')$ of a small neighborhood U' of x_0 is isomorphic, as a smooth manifold, to the direct product of $p^{-1}(x_0) \times U'$. Therefore, we can form a basis from the twisted homology group $H_\delta(p^{-1}(x), \mathcal{P}_x)$, $x \in U'$ of the form $\sum c_i \Delta_i \otimes P$, where c_i is a constant that does not depend on x and Δ_i is a smooth simplex that does not depend on x. Here, $\mathcal{P}_x$ is the local system defined by P at x. Note that $\mathcal{P}_x$ and $\mathcal{P}_{x'}$ are isomorphic for any $x, x' \in U'$.

Let $\{u_1, \ldots, u_r\}$ be a basis of $R_n/(R_n H_A(s))$ as given in Theorem 1. For generic parameters α and γ and a twisted cycle $C_j = \sum_k c_{jk} \Delta_{jk} \otimes P$, where c_{jk} and Δ_{jk} does not depend on the parameter x, the integral

$$u_i \bullet \Phi(\alpha, \gamma; x) = \sum c_{jk} \int_{\Delta_{jk}} u_i \bullet P dt = \sum c_{jk} \int_{\Delta_{jk}} \frac{u_i \bullet P}{P} P dt$$

can be regarded as a pairing $\langle \varphi_i, C_j \rangle$ of the twisted cocycle $\varphi_i = \frac{u_i \bullet P}{P} dt \in H^\delta(p^{-1}(x), \nabla)$ and the twisted cycle C_j. Since the matrix-valued function $(\langle \varphi_i, C_j \rangle)$ is a fundamental set of solutions of the Pfaffian system for the A-hypergeometric system $H_A(\beta)$, its determinant does not vanish away from an analytic set. This implies that the pairing of $H^\delta(p^{-1}(x), \nabla) \times H_\delta(p^{-1}(x), \mathcal{P}_x)$ is perfect away from the analytic set in the x space. Thus, the set $u_i \bullet P/P$ is a basis of the twisted cohomology group. Q.E.D.

We consider general A-hypergeometric systems without assuming the special form of A in (8). We put

$$g(x,t) = \sum_{i=1}^{n} x_i t^{a_i}, \quad t^{a_i} = \prod_{j=1}^{d} t_j^{(a_i)_j}$$

where a_i's are column vectors of the matrix $A = (a_{ij})$ and $(a_i)_j = a_{ji}$ denotes the j-th element of the vector a_i. Define a projection p by

$$p \;:\; (\mathbf{C}^*)^{d+n} \ni (x,t) \mapsto x \in \mathbf{C}^n$$

We regard g as a function in $t = (t_1, \ldots, t_d)$ with the parameter vector x. Let $\gamma_1, \ldots, \gamma_d$ be complex numbers. Define the connection ∇ with rational function coefficients and the twisted cohomology group by

$$\nabla = d_t + \sum_{i=1}^{d} \left(\frac{\partial g}{\partial t_i} + \frac{\gamma_i}{t_i} \right), \quad H^d(p^{-1}(x), \nabla) = \frac{\mathbf{C}[t_1^{\pm}, \ldots, t_d^{\pm}] dt_1 \wedge \cdots \wedge dt_d}{\nabla \left(\sum_{i=1}^{d} \mathbf{C}[t_1^{\pm}, \ldots, t_d^{\pm}] \wedge_{j \neq i} dt_j \right)}.$$

Theorem 3 *Let $\gamma = (\gamma_i)$ be generic complex parameter vector and let $\{u_1, \ldots, u_r\}$ be a basis given in Theorem 1. Then the set of rational forms*

$$\varphi_i = \frac{u_i \bullet \exp(g) t^\gamma dt}{\exp(g) t^\gamma}, \quad i = 1, \ldots, r$$

is a basis of the twisted cohomology group $H^d(p^{-1}(x), \nabla)$ when x lies outside of an analytic set.

Proof. Our proof relies on the theory of rapid decay homology groups and confluent A-hypergeometric systems. It follows from Esterov-Takeuchi [9] and the generic condition on the parameter γ that we can form a basis $\mathcal{C}$ from the rapid decay homology cycles $c_x \otimes \exp(g) t^\gamma$ (c_x is the support set) such that $\int_{c_x} \exp(g) t^\gamma dt$, $c_x \otimes \exp(g) t^\gamma \in \mathcal{C}$ span the solution space of the A-hypergeometric system for $\beta_i = -\gamma_i - 1$ on $\mathbf{C}^n \setminus D$ where D is the algebraic set defined by the non-degeneracy condition of g (see, e.g., [9, Def 2.3]). In particular, $\sharp\mathcal{C} = r$ (the normalized volume of A). Hien and Roucairol [18, Th 3.5 and its proof] prove that the integration over the rapid decay cycles and the differentiation with respect to x_i can be exchanged in this case. In other words, we have

$$u_i \bullet \int_{c_x} \exp(g) t^\gamma dt = \int_{c_x} \frac{u_i \bullet \exp(g) t^\gamma}{\exp(g) t^\gamma} \exp(g) t^\gamma dt = \langle \varphi_i, c_x \otimes \exp(g) t^\gamma \rangle.$$

It follows from these two results that the rest of the proof is analogous with that of Theorem 2 under the perfect pairing theorem of the rapid decay homology group and the twisted cohomology group by Hien [17]. Q.E.D.

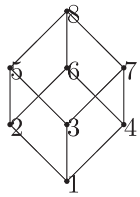

Figure 1: C_{111}

Example 1 Consider the matrix

$$A = \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array}$$
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

(which are the vertices of the order polytope associated to the distributive lattice of Figure 1; see Section 4). The basis given by Theorem 1 is $\{1, \partial_5, \partial_6, \partial_7, \partial_8, \partial_8^2\}$. It is determined by computing a Gröbner basis for the ideal J or $\tilde{J}$ by a computer. Note that this computation is easier than computing the Gröbner basis in R_8 of $H_A(s)$. The corresponding basis of the twisted cohomology group is $\{1, \frac{t_1t_2dt}{Q}, \frac{t_1t_3dt}{Q}, \frac{t_2t_3dt}{Q}, \frac{t_1t_2t_3dt}{Q}, \frac{(t_1t_2t_3)^2dt}{Q^2}\}$, where $Q = x_1 + x_2t_1 + x_3t_2 + x_4t_3 + x_5t_1t_2 + x_6t_1t_3 + x_7t_2t_3 + x_8t_1t_2t_3$ and $dt = dt_1dt_2dt_3$. Note that we assume that parameters α_i, γ_j are generic.

Example 2 Let

$$A' = \begin{array}{c} \begin{array}{ccccccc} x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

The hypergeometric system associated with A' is a confluent system of the previous example. Put $Q = x_2t_1+x_3t_2+x_4t_3+x_5t_1t_2+x_6t_1t_3+x_7t_2t_3+x_8t_1t_2t_3$ and $dt = dt_1dt_2dt_3$. Then the integral $\int_C \exp(Q)t^\gamma dt$, where C is a rapid decay cycle, is a solution. The toric ideal $I_{A'}$ is obtained formally by setting $\partial_1 = 1$ in I_A of Example 1. A basis given by Theorem 1 is $\{1, \partial_5, \partial_6, \partial_7, \partial_8, \partial_8^2\}$. The corresponding basis of the twisted cohomology group is $\{1, t_1t_2dt, t_1t_3dt, t_2t_3dt, t_1t_2t_3dt, (t_1t_2t_3)^2dt\}$.

4 A-hypergeometric Systems for Order Polytopes

For some classes of generic sparse polynomials or A, we can calculate by hand the set of standard monomials for $\tilde{J}$. These are order polytopes associated to posets

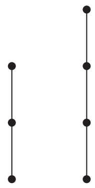

Figure 2: A poset P

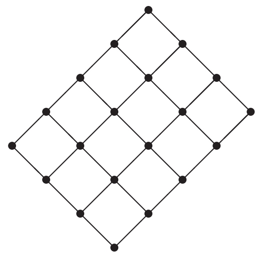

Figure 3: Distributive lattice of $\mathcal{J}(P)$

which decompose into two chains and to posets which are buquets. We note that it yields the celebrated theorem of K. Aomoto on the twisted cohomology group for a hyperplane arrangement in a general position as a special case. We change a gear in this section to combinatorics and symbols P, k, r, s, w, α, β will be redefined and (ideal) J in this section stands for (ideal) $\tilde{J}$ in the previous sections.

First, recall the order polytope of a finite partially ordered set ([15, p. 115]). Let $P = \{v_1, \ldots, v_m\}$ be a finite partially ordered set with $|P| = m$. A *poset ideal* of P is a subset α of P such that if $v \in \alpha$, $w \in P$, and $w \leq v$, then $w \in \alpha$. Thus in particular the empty set and P itself are poset ideals of P. Let $\mathcal{J}(P)$ denote the distributive lattice ([15, p. 118]) consisting of all poset ideals of P, ordered by inclusion. For example, if P is the disjoint union of two chains of length 2 and length 3 shown in Figure 2, then $L = \mathcal{J}(P)$ is the distributive lattice shown in Figure 3.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ denote the standard unit coordinate vectors of $\mathbf{R}^m$. If β is a subset of P, then we write a_β for the $(0,1)$-vector $\sum_{v_i \in \beta} \mathbf{e}_i \in \mathbf{R}^m$. The *order polytope* $\mathcal{O}(P) \subset \mathbf{R}^m$ of P is the convex hull of the finite set $\{a_\alpha : \alpha \in \mathcal{J}(P)\}$. Its dimension is $\dim \mathcal{O}(P) = m$.

Let $K = \mathbf{C}(\{\xi_\alpha\}_{\alpha \in \mathcal{J}(P)})$ denote the rational function field in $|\mathcal{J}(P)|$ variables over $\mathbf{C}$. Let $K[t_1, \ldots, t_m, s]$ be the polynomial ring in $m+1$ variables over K. If β is a subset of P, then we write u_β for the square-free monomial $\prod_{v_i \in \beta} t_i s$. Let $K[\mathcal{O}(P)]$ denote the subalgebra of $K[t_1, \ldots, t_m, s]$ that is generated by those square-free monomials u_β with $\beta \in \mathcal{J}(P)$. The semigroup ring $K[\mathcal{O}(P)]$ was introduced in [14]. We call $K[\mathcal{O}(P)]$ the *toric ring* of $\mathcal{O}(P)$. The Krull dimension of $K[\mathcal{O}(P)]$ is $m+1$.

Let $K[\{y_\alpha\}_{\alpha \in \mathcal{J}(P)}]$ denote the polynomial ring in $|\mathcal{J}(P)|$ variables over K, and define the surjective ring homomorphism $\pi : K[\{y_\alpha\}_{\alpha \in \mathcal{J}(P)}] \to K[\mathcal{O}(P)]$ by setting $\pi(y_\alpha) = u_\alpha$. Its kernel $I_{\mathcal{O}(P)}$ is called the *toric ideal* of $\mathcal{O}(P)$. It is known [14] that $I_{\mathcal{O}(P)}$ is generated by quadratic binomials

$$y_\alpha y_\beta - y_{\alpha \wedge \beta} y_{\alpha \vee \beta}, \tag{11}$$

such that α and β are incomparable in the distributive lattice $\mathcal{J}(P)$. We fix an ordering $<$ of variables of $K[\{y_\alpha\}_{\alpha \in \mathcal{J}(P)}]$ with the property that if $\alpha > \beta$ in $\mathcal{J}(P)$, then $y_\alpha < y_\beta$. Let $<_{\mathrm{rev}}$ be the reverse lexicographic order on

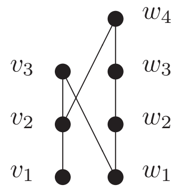

Figure 4: P

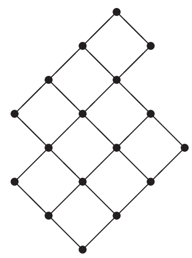

Figure 5: $\mathcal{J}(P)$

$K[\{y_\alpha\}_{\alpha\in\mathcal{J}(P)}]$ induced by the ordering $<$. In [14] it was shown that the set of binomials (11) is the reduced Gröbner basis of $I_{\mathcal{O}(P)}$ with respect to $<_{\rm rev}$. Thus $\mathrm{in}_{<_{\rm rev}}(I_{\mathcal{O}(P)})$ is generated by those square-free quadratic monomials $y_\alpha y_\beta$ such that α and β are incomparable in $\mathcal{J}(P)$.

Let

$$\Theta_i = \sum_{v_i\in\alpha} \xi_\alpha y_\alpha - \eta_i, \qquad 1 \le i \le m,$$

and let

$$\Theta_0 = \sum_{\alpha\in\mathcal{J}(P)} \xi_\alpha y_\alpha - \eta_0,$$

where $\eta_i \in K$. It then follows that the sequence $(\Theta_0, \Theta_1, \ldots, \Theta_m)$ is a system of parameters of both the residue rings $K[\{y_\alpha\}_{\alpha\in\mathcal{J}(P)}]/I_{\mathcal{O}(P)}$ and $K[\{y_\alpha\}_{\alpha\in\mathcal{J}(P)}]/\mathrm{in}_{<_{\rm rev}}(I_{\mathcal{O}(P)})$. The fundamental goal is to find a K-basis of the zero-dimensional residue ring

$$K[\{y_\alpha\}_{\alpha\in\mathcal{J}(P)}]/(\mathrm{in}_{<_{\rm rev}}(I_{\mathcal{O}(P)}), \Theta_0, \Theta_1, \ldots, \Theta_m). \tag{12}$$

In general, however, this is difficult. When P can be decomposed into two chains, a complete answer can be found, as shown below. For example, the poset P of Figure 4 can be decomposed into the chains $v_1 < v_2 < v_3$ and $w_1 < w_2 < w_3 < w_4$.

Now, suppose that a finite poset P can be decomposed into two chains $C_p : v_1 < \cdots < v_p$ of length $p-1$ and $C_q : w_1 < \cdots < w_q$ of length $q-1$, where $p \ge 1$ and $q \ge 1$. Let $\mathcal{L}$ denote the set of those pairs (i,j), where $0 \le i \le p$ and $0 \le j \le q$, for which $\{v_1, \ldots, v_i, w_1, \ldots, w_j\}$ is a poset ideal of P. In particular, $(0,0), (p,q) \in \mathcal{L}$. When $(i,j) \in \mathcal{L}$, we write $\alpha_{i,j}$ for $\{v_1, \ldots, v_i, w_1, \ldots, w_j\}$. For example, $\alpha_{0,0} = \emptyset$ and $\alpha_{p,q} = P$. We then have $L = \{\, \alpha_{i,j} \,:\, (i,j) \in \mathcal{L} \,\}$. When $(i,j) \in \mathcal{L}$, we write $\xi_{i,j}$ for $\xi_{\alpha_{i,j}}$ and $y_{i,j}$ for $y_{\alpha_{i,j}}$. Let

$$\Theta_{i*} = \sum_{i\le k\le p,\, 0\le j\le q,\, (k,j)\in\mathcal{L}} \xi_{k,j} y_{k,j} - \eta_{i*}, \qquad 0 \le i \le p$$

and

$$\Theta_{*j} = \sum_{0\le i\le p,\, j\le \ell\le q,\, (i,\ell)\in\mathcal{L}} \xi_{i,\ell} y_{i,\ell} - \eta_{*j}, \qquad 0 \le j \le q.$$

In particular,

$$\Theta_{0*} = \Theta_{*0} = \sum_{(i,j)\in\mathcal{L}} \xi_{i,j} y_{i,j} - \eta_0,$$

with $\eta_0 = \eta_{0*} = \eta_{*0}$. Let $K[\mathbf{y}] = K[\{y_{i,j}\}_{0\leq i\leq p,\, 0\leq j\leq q,\, (i,j)\in\mathcal{L}}]$ and

$$J = (\,\mathrm{in}_{<_{\mathrm{rev}}}(I_{\mathcal{O}(P)}),\ \{\Theta_{i*}\}_{0\leq i\leq p},\ \{\Theta_{*j}\}_{0\leq j\leq q}\,), \tag{13}$$

where

$$\mathrm{in}_{<_{\mathrm{rev}}}(I_{\mathcal{O}(P)}) = (\{\, y_{i,j}y_{k,\ell} \,:\, i<k,\ \ell<j,\ (i,j)\in\mathcal{L},\ (k,\ell)\in\mathcal{L}\,\}).$$

Then the residue ring (12) is $K[\mathbf{y}]/J$. Let $<_{\mathrm{rev}}$ denote the reverse lexicographic order on $K[\mathbf{y}]$ induced by the ordering of the variables, as follows: $y_{i,j} > y_{k,\ell}$ if either $i+j<k+\ell$ or $i+j=k+\ell$ with $i>k$.

Lemma 1 *In $K[\mathbf{y}]/\mathrm{in}_{<_{\mathrm{rev}}}(J)$,*

$$y_{i,j}y_{i,j'} \;=\; y_{i,j}y_{i',j} \;=\; 0,$$

where $(i,j),(i,j')$ and (i',j) belong to $\mathcal{L}$.

Proof. Let $i<i'$. Then

$$\Theta_{i'*}y_{i,j} - y_{i,j}\Big(\Big(\sum_{i'\leq k,\, j\leq \ell,\, (k,\ell)\in\mathcal{L}} \xi_{k,\ell}y_{k,\ell}\Big) - \eta_{i'*}\Big)$$

belongs to $\mathrm{in}_{<_{\mathrm{rev}}}(I_{\mathcal{O}(P)})$. Hence

$$y_{i,j}\Big(\Big(\sum_{i'\leq k,\, j\leq \ell,\, (k,\ell)\in\mathcal{L}} \xi_{k,\ell}y_{k,\ell}\Big) - \eta_{i'*}\Big)$$

belongs to J. Thus its initial monomial $y_{i,j}y_{i',j}$ belongs to $\mathrm{in}_{<_{\mathrm{rev}}}(J)$.

Let $i=i'$. Let f be the polynomial

$$\Theta_{i*}y_{i,j} - \xi_{i,j}^{-1}\Theta_{*j}\Big(\sum_{0\leq k\leq j-1,\, (i,k)\in\mathcal{L}} \xi_{i,k}y_{i,k}\Big),$$

and write $f=f_1+f_2$, where $f_2\in\mathrm{in}_{<_{\mathrm{rev}}}(I_{\mathcal{O}(P)})$ and where none of the monomials appearing in f_1 belongs to $\mathrm{in}_{<_{\mathrm{rev}}}(I_{\mathcal{O}(P)})$. Then $f_1\in J$ and the initial monomial of f_1 is $y_{i,j}^2$. Hence $y_{i,j}^2\in\mathrm{in}_{<_{\mathrm{rev}}}(J)$. Similarly, $y_{i,j}y_{i,j'}\in\mathrm{in}_{<_{\mathrm{rev}}}(J)$. Q.E.D.

Lemma 2 *For each $0\leq i\leq p$, we write $j_i^\sharp$ for the smallest integer for which $(i,j_i^\sharp)\in\mathcal{L}$. For each $0\leq j\leq q$, we write $i_j^\flat$ for the smallest integer for which $(i_j^\flat,j)\in\mathcal{L}$. Then $y_{i,j_i^\sharp}$ and $y_{i_j^\flat,j}$ belong to $\mathrm{in}_{<_{\mathrm{rev}}}(J)$.*

Proof. Since Θ_{i*} and Θ_{*j} belong to J, their initial monomials $y_{i,j_i^\sharp}$ and $y_{i_j^\flat,j}$ belong to $\mathrm{in}_{<_{\mathrm{rev}}}(J)$. Q.E.D.

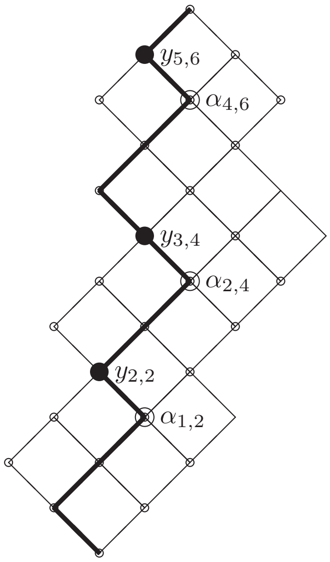

Figure 6:

Let $\mathcal{S}$ denote the set of square-free monomials of $K[\mathbf{y}]$ of the form

$$y_{i_1,j_1} y_{i_2,j_2} \cdots y_{i_r,j_r}, \tag{14}$$

with each $(i_k, j_k) \in \mathcal{L} \setminus (\{ y_{i,j_i^\sharp} : 0 \leq i \leq p \} \cup \{ y_{i_j^\flat,j} : 0 \leq j \leq q \})$ such that

$$0 < i_1 < i_2 < \cdots < i_r \leq p, \;\; 0 < j_1 < j_2 < \cdots < j_r \leq q, \;\; r = 0, 1, 2, \ldots.$$

Theorem 4 *The set of standard monomials of* $\mathrm{in}_{<_{\mathrm{rev}}}(J)$ *is equal to* $\mathcal{S}$.

Proof. In [7, Th. 2.2], it was proven that the number of standard monomials of degree r coincides with the number of maximal chains of $\mathcal{J}(P)$ with r descents. Recall that the descents of a maximal chain

$$\alpha_{0,0} = \alpha_{i_0,j_0} < \alpha_{i_1,j_1} < \cdots < \alpha_{i_{p+q},j_{p+q}} = \alpha_{p,q}$$

of $\mathcal{J}(P)$ are those α_{i_k,j_k} with $1 \leq k < p+q$ such that

$$i_{k-1} = i_k < i_{k+1}, \quad j_{k-1} < j_k = j_{k+1}, \quad j_{k+1} \neq j^\sharp_{i_{k+1}}.$$

Now, given a square-free monomial (14) of degree r, we can associate a unique maximal chain whose descents are

$$\alpha_{i_1-1,j_1}, \alpha_{i_2-1,j_2}, \cdots, \alpha_{i_r-1,j_r},$$

in the obvious way (see Figure 6.)

Hence the number of square-free monomials (14) of degree r is less than or equal to that of standard monomials of degree r. On the other hand, since Lemmata 1 and 2 guarantee that each standard monomial must belong to $\mathcal{S}$, it follows that $\mathcal{S}$ is the set of standard monomials of $\mathrm{in}_{<_{\mathrm{rev}}}(J)$, as desired. Q.E.D.

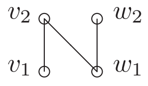

Figure 7: P

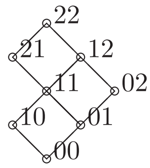

Figure 8: $\mathcal{J}(P)$

Remark 2 When the variables η_{i*} and η_{*j} are 0 in the rational function field, we can work with a system of parameters consisting of homogeneous elements and both Lemma 2 and Lemma 1 are valid without modification. This observation is crucial to our argument of counting the number of standard monomials in the proof of Theorem 4. We also note that ξ_{ij} may be specialized to any nonzero number for $K = \mathbf{C}$ without changing the claims of this section.

Example 3 Let P be the finite poset of Figure 7, and let $L = \mathcal{J}(P)$ be the distributive lattice shown in Figure 8. Then the standard monomials of $\mathrm{in}_{<_{\mathrm{rev}}}(J)$ are 1; $y_{1,1}$; $y_{1,2}$; $y_{2,2}$; and $y_{1,1}y_{2,2}$.

Let us turn to the discussion of A-hypergeometric systems. Let $P_{p,q}$ denote the disjoint union of two chains $C_p : v_1 < \cdots < v_p$ of length $p-1$ and $C_q : w_1 < \cdots < w_q$ of length $q-1$. Let $\alpha_{i,j}$, where $0 \le i \le p$ and $0 \le j \le q$, be the poset ideal $\{v_1, \ldots, v_i, w_1, \ldots, w_j\}$. In particular $\alpha_{0,0} = \emptyset$. This is a special and interesting subclass of poset ideals. We regard the vector a_α, $\alpha \in \mathcal{J}(P_{p,q})$, as a column vector and construct a matrix $A_{p,q}$ with these column vectors and a row vector $(1, 1, \ldots, 1)$. For example, $A_{2,2}$ is

$$\begin{array}{c}
\begin{array}{ccccccccc} 00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \end{array}\\
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1\\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}
\end{array}.$$

By elementary row transformations, we transform the matrix $A_{p,q}$ into the matrix $\bar{A}_{p,q}$ of the form (8) with $k = p+1$, $n_1 = \cdots = n_k = q+1$, and $a_i = 0 \in \mathbf{R}^q$ when $i \equiv 1 \bmod q+1$, $a_{i+1} = e_k \in \mathbf{R}^q$ when $i \equiv k \bmod q+1$. For example, $A_{2,2}$ can be transformed into

$$\bar{A}_{2,2} = \begin{array}{c}
\begin{array}{ccccccccc} 00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \end{array}\\
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}
\end{array}.$$

We note that $A_{p,q}$ and $\bar{A}_{p,q}$ define the same A-hypergeometric system.

The matrix A, which represents a poset P that can be decomposed into two chains (as considered in this section), is obtained by removing some columns from $\bar{A}_{p,q}$. For example, the A that represents Figure 8 is obtained by deleting the seventh column of the matrix $\bar{A}_{2,2}$. Therefore, for A, the sparse polynomials f_j (6) that can be decomposed into two chains are linear in t. In particular, the f_j's for $P_{p,q}$ are in the general linear position. It follows from the integral representation (7) that the $\bar{A}_{p,q}$-hypergeometric system agrees with the Aomoto-Gel'fand system $E(p+1,(p+1)+(q+1))$ [3], because the matrix $\bar{A}_{p,q}$ defines a hyperplane arrangement $V(\prod t_i \prod f_j)$ in a general position. The initial ideal $\mathrm{in}_{<_{\mathrm{rev}}}(I_A)$ is a square-free monomial ideal. In particular, it follows from Corollary 1 that the standard monomials of the ideal J defined in (13) provide a basis of the Pfaffian system for $H_A(s)$ when y_{ij} is replaced by ∂_{ij}.

Let $\mathcal{S}$ be the set of standard monomials given in Theorem 4 for the poset ideal $\mathcal{J}(P)$. Then, the set $\mathcal{S}|_{y_{ij}\to\partial_{ij}}$ gives a basis of the Pfaffian system for the A-hypergeometric system.

From our Theorems 2 and 4 and the correspondence that we have explained above, we have the following theorem (we locally use α and γ to denote complex parameters in this theorem to use the notation in the previous sections).

Theorem 5 *Let A be the matrix in the form (8) representing a poset P that can be decomposed into two chains, and let $\mathcal{S} = \{u_1, \ldots, u_r\}$ be the set of standard monomials given in Theorem 4 with y_{ij} replaced by ∂_{ij}. Set $Q = \prod_{j=1}^{k} f_j(x,t)^{\alpha_j} t^{\gamma}$ and $Q' = Q|_{\alpha_i=\gamma_j=1}$. Then, the set of rational forms*

$$\frac{u_i \bullet Q}{Q} dt_1 \cdots dt_\delta, \quad i = 1, \ldots, r \tag{15}$$

is a basis of the twisted cohomology group (10) (with P replaced by Q) when α_i, γ_j are generic complex numbers and x lies outside of an analytic set.

Remark 3 In the case $P = P_{p,q}$, this theorem is a different presentation of the celebrated work of K. Aomoto, who gave a basis for the twisted cohomology group associated with a hyperplane arrangement in a general position (see, e.g., [3, Theorem 9.6.2]). In a more general result, Orlik and Terao gave bases of twisted cohomology groups associated with hyperplane arrangements in terms of the βnbc basis [25, 6.3]. Our theorem gives bases for twisted cohomology groups in a very different way for a class of hyperplane arrangements obtained by restricting the arrangements in the general position to the $x_{ij} = 0$'s.

Example 4 The A-hypergeometric system associated with Figure 8 is the restriction of $E(3,6)$ to $x_{20} = 0$. Figure 9 illustrates the arrangement that represents it.

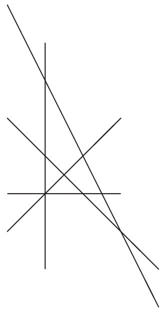

Figure 9: $V(t_1 t_2 \prod f_j)$

5 Rank of a Class of Order Polytopes

We now turn to the discussion of the normalized volume of order polytopes, which stands for the rank of the Pfaffian system with generic parameters. It follows from [27] that the normalized volume of the order polytope $\mathcal{O}(P)$ is equal to $e(P)$, the number of linear extensions of P. Recall that an *antichain* of P is a subset B of P such that if v and w belong to B with $v \neq w$, then v and w are incomparable in P. The *width* of P is the supremum of cardinalities of antichains of P. The *length* of a chain C is $|C| - 1$. The *rank* of P is the supremum of lengths of chains of P.

Lemma 3 *Fix positive integers q and r. Let P be the disjoint union of q chains $C_1, \ldots, C_q$, and assume that the length of each chain C_i with $1 \leq i < q$ is at most $r - 1$. Then there exists a polynomial $f(m)$ in m of degree $r(q-1)$ such that $e(P)$ is at most $f(m)$, where $m = |P|$.*

Proof. Let ℓ_i denote the length of C_i. Then the number of linear extensions of P is

$$e(P) = \binom{m}{\ell_1, \ell_2, \ldots, \ell_q} = \frac{m!}{\ell_1! \ell_2! \ldots \ell_q!}.$$

Since $\ell_q = m - \sum_{i=1}^{q-1} \ell_i \geq m - r(q-1)$, it follows that

$$e(P) \leq \frac{m!}{\ell_q!} \leq \frac{m!}{(m - r(q-1))!}.$$

Let

$$f(m) = m(m-1)(m-2)\cdots(m - r(q-1) + 1),$$

which is a polynomial in m of degree $r(q-1)$. Then $e(P) \leq f(m)$, as required. Q.E.D.

Theorem 6 *Fix positive integers q and r. Let P be a finite partially ordered set, and suppose that there exists a chain C of P such that*

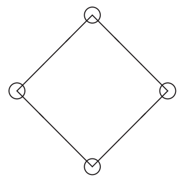

Figure 10: $\mathcal{J}(P_{1,1})$

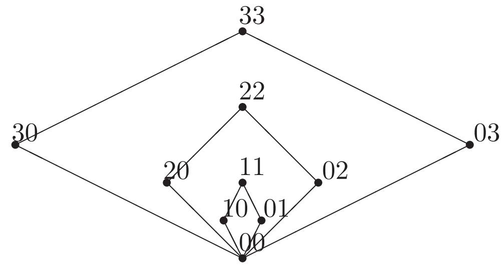

Figure 11: The bouquet of 3 $\mathcal{J}(P_{1,1})$'s

(i) *the width of* $P \setminus C$ *is at most* $q-1$*;*

(ii) *the rank of* $P \setminus C$ *is at most* $r-1$.

Then there exists a polynomial $f(m)$ *in* m *of degree* $r(q-1)$ *such that* $e(P)$ *is at most* $f(m)$*, where* $m = |P|$.

Proof. Since the width of $P \setminus C$ is at most $q-1$, Dilworth's theorem [8] guarantees the existence of $q-1$ chains $C_1, \ldots, C_{q-1}$ of $P \setminus C$, where the length of each C_i is at most $r-1$, such that $P \setminus C = C_1 \cup C_2 \cup \cdots \cup C_{q-1}$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. Hence there exists a partially ordered set Q that is the disjoint union of q chains $C'_1, \ldots, C'_q$, where the length of each C'_i with $1 \leq i < q$ is at most $r-1$, such that there is an order-preserving bijection $\varphi : Q \to P$. Hence $e(P) \leq e(Q)$. Thus the desired result follows from Lemma 3. Q.E.D.

Let us turn to the discussion of A-hypergeometric systems. By Theorem 6, the rank $e(P)$ of the hypergeometric system associated with the order polytope $\mathcal{O}(P)$ ($m = |P|$), has a polynomial growth property with respect to m. This is good news, since the rank determines the complexity of the holonomic gradient method [22].

6 Bouquet

We now wish to introduce a "bouquet" of finite distributive lattices. The associated hypergeometric systems include the Lauricella function F_A as we will see in Example 5. Let $P_1, \ldots, P_q$ be finite posets, where

$$P_i = \{v_1^{(i)}, \ldots, v_{m_i}^{(i)}\}, \qquad 1 \leq i \leq q,$$

and let $L_i = \mathcal{J}(P_i)$ be the distributive lattice consisting of all poset ideals of P_i. The finite meet-semilattice ([28, p. 249, 3.3]) $\bigcup_{i=1}^{q}(\mathcal{J}(P_i))$ is called the *bouquet* of $L_1 = \mathcal{J}(P_1), \ldots, L_q = \mathcal{J}(P_q)$. For example, if $q = 3$ and $P_1 = P_2 = P_3$ are the finite poset $P_{1,1}$ shown in Figure 10, then the Hasse diagram of the bouquet of L_1, L_2, L_3 is shown in Figure 11.

Let $\mathbf{e}_j^{(i)}$, $1 \leq i \leq q$, $1 \leq j \leq m_i$, denote the standard unit coordinate vectors of $\mathbf{R}^m$, where $m = m_1 + \cdots + m_q$. If β is a subset of P_i, then we write a_β for the $(0,1)$-vector $\sum_{v_j^{(i)} \in \beta} \mathbf{e}_j^{(i)} \in \mathbf{R}^m$. In particular, $w_\emptyset$ is the origin of $\mathbf{R}^m$. Let $\mathcal{O}(P_1, \ldots, P_q) \subset \mathbf{R}^m$ denote the convex hull of the finite set

$$\{ a_\alpha \, : \, \alpha \in \bigcup_{i=1}^{q} \mathcal{J}(P_i) \}.$$

Its dimension is $\dim \mathcal{O}(P_1, \ldots, P_q) = m$. In the language of combinatorics, the convex polytope $\mathcal{O}(P_1, \ldots, P_q)$ is called the *free sum* ([6]) of $\mathcal{O}(P_1), \ldots, \mathcal{O}(P_q)$.

Let

$$K = \mathbf{C}(\{ \xi_\alpha \, : \, \alpha \in \bigcup_{i=1}^{q} \mathcal{J}(P_i) \}, \{ \eta_j^{(i)} \, : \, 1 \leq i \leq q, 1 \leq j \leq m_i \}, \eta_0)$$

denote the rational function field in $|\bigcup_{i=1}^{q} \mathcal{J}(P_i) + (m+1)|$ variables over $\mathbf{C}$, and let

$$K[\{ t_j^{(i)} \, : \, 1 \leq i \leq q, 1 \leq j \leq m_i \}, s]$$

be the polynomial ring in $m+1$ variables over K. If β is a subset of P_i, then we write u_β for the square-free monomial $(\prod_{v_j^{(i)} \in \beta} t_j^{(i)})s$. The toric ring $K[\mathcal{O}(P_1, \ldots, P_q)]$ of $\mathcal{O}(P_1, \ldots, P_q)$ is the subalgebra of $K[\{t_j^{(i)}\}, s]$ that is generated by those square-free monomials u_α with $\alpha \in \bigcup_{i=1}^{m} \mathcal{J}(P_i)$. Its Krull dimension is $m+1$.

Let $K[\{ y_\alpha \}_\alpha] = K[\{ y_\alpha \, : \, \alpha \in \bigcup_{i=1}^{m} \mathcal{J}(P_i) \}]$ denote the polynomial ring in $|\bigcup_{i=1}^{m} \mathcal{J}(P_i)|$ variables over K, and define the surjective ring homomorphism

$$\pi : K[\{ y_\alpha \}_\alpha] \to K[\mathcal{O}(P_1, \ldots, P_q)]$$

by setting $\pi(y_\alpha) = u_\alpha$. Its kernel is the toric ideal $I_{\mathcal{O}(P_1,\ldots,P_q)}$ of $\mathcal{O}(P_1, \ldots, P_q)$. It follows that $I_{\mathcal{O}(P_1,\ldots,P_q)}$ is generated by those quadratic binomials

$$y_\alpha y_\beta - y_{\alpha \wedge \beta} y_{\alpha \vee \beta}, \tag{16}$$

where both α and β belong to $\mathcal{J}(P_i)$ for some $1 \leq i \leq q$ and where α and β are incomparable in $\mathcal{J}(P_i)$.

We fix an ordering $<$ of the variables of $K[\{ y_\alpha \}_\alpha]$ with the property that if both α and β belong to $\mathcal{J}(P_i)$ for some $1 \leq i \leq q$ and if $\alpha > \beta$ in $\mathcal{J}(P_i)$, then $y_\alpha < y_\beta$. Let $<_{\text{rev}}$ denote the reverse lexicographic order on $K[\{ y_\alpha \}_\alpha]$ induced by the ordering $<$. It then follows that the set of binomials (16) is the reduced Gröbner basis of $I_{\mathcal{O}(P_1,\ldots,P_q)}$ with respect to $<_{\text{rev}}$. Thus the initial ideal $\text{in}_{<_{\text{rev}}}(I_{\mathcal{O}(P_1,\ldots,P_q)})$ of $I_{\mathcal{O}(P_1,\ldots,P_q)}$ with respect to $<_{\text{rev}}$ is generated by those square-free quadratic monomials $y_\alpha y_\beta$ such that both α and β belong to $\mathcal{J}(P_i)$ for some $1 \leq i \leq q$ and that α and β are incomparable in $\mathcal{J}(P_i)$.

Let

$$\Theta_j^{(i)} = \sum_{v_j^{(i)} \in \alpha} \xi_\alpha y_\alpha - \eta_j^{(i)}, \qquad 1 \leq i \leq q, \; 1 \leq j \leq m_i,$$

and let

$$\Theta_0 = \sum_{\alpha \in \mathcal{J}(P)} \xi_\alpha y_\alpha - \eta_0.$$

It then follows that the sequence

$$(\Theta_0, \Theta_1^{(1)}, \ldots, \Theta_{m_1}^{(1)}, \Theta_1^{(2)}, \ldots, \Theta_{m_2}^{(2)}, \ldots, \Theta_1^{(q)}, \ldots, \Theta_{m_q}^{(q)})$$

is a system of parameters of both the residue rings $K[\{y_\alpha\}_\alpha]/I_{\mathcal{O}(P_1,\ldots,P_q)}$ and $K[\{y_\alpha\}_\alpha]/\mathrm{in}_{<_{\mathrm{rev}}}(I_{\mathcal{O}(P_1,\ldots,P_q)})$.

Now, suppose that each poset P_i can be decomposed into two chains, and write $\mathcal{S}_i$ for the set of standard monomials, which is obtained using Theorem 4, for the residue class ring arising from $\mathcal{O}(P_i)$. By virtue of the fact that Θ_0 plays no rule in the proof of Lemma 1, it follows that the set of standard monomials of

$$K[\{y_\alpha\}_\alpha]/\mathrm{in}_{<_{\mathrm{rev}}}(\mathrm{in}_{<_{\mathrm{rev}}}(I_{\mathcal{O}(P_1,\ldots,P_q)}), \Theta_0, \Theta_1^{(1)}, \ldots, \Theta_{m_q}^{(q)}) \tag{17}$$

with respect to $<_{\mathrm{rev}}$ is a subset of

$$\Big\{ \prod_{i=1}^{q} u_i \,:\, u_i \in \mathcal{S}_i,\, 1 \le i \le q \Big\}. \tag{18}$$

Finally, the computation of the number of standard monomials based on the equality (7) of [6, Theorem 1.4] together with the information on the facets of the order polytopes ([27, p. 10]) guarantee the following theorem.

Theorem 7 *The set of standard monomials of the residue class ring (17) with respect to* $<_{\mathrm{rev}}$ *coincides with the set of square-free monomials (18).*

Example 5 For the bouquet of Figure 11, the set of standard monomials obtained by Theorem 7 (labeling variables as in Figure 11 and replacing y_{ij} by ∂_{ij}) is

$$\{\partial_{11}^{k_1}\partial_{22}^{k_2}\partial_{33}^{k_3} \mid k_i \in \{0,1\}\}.$$

We will show that this bouquet represents the Lauricella hypergeometric function F_A of three variables [5, Chapitre VII]. We note that the twisted cohomology groups associated with the F_A are studied in [20] in a quite different way. We consider A-hypergeometric system associated with the lattice shown in Figure 11. The independent variables of the system will be denoted by $p_{00}, p_{01}, p_{02}, p_{03}, p_{10}, p_{20}, p_{30}, p_{11}, p_{22}, p_{33}$ or simply as $00, 01, 02, 03, 10, 20, 30, 11, 22, 33$ if no confusion arises. The differential operators of the system will be denoted by $\partial_{00}, \partial_{01}, \partial_{02}, \partial_{03}, \partial_{10}, \partial_{20}, \partial_{30}, \partial_{11}, \partial_{22}, \partial_{33}$ or simply as $00, 01, 02, 03, 10, 20, 30, 11, 22, 33$ if no confusion arises. The toric ideal associated with the lattice is generated by

$$\underline{10 \cdot 01} - 00 \cdot 11, \underline{20 \cdot 02} - 00 \cdot 22, \underline{30 \cdot 03} - 00 \cdot 33. \tag{19}$$

The underlined terms are the leading terms for the reverse lexicographic order such that $00 <$ other variables, and the set is a Gröbner basis with this order. Hence, the A-hypergeometric system has a solution of the form

$$p^\gamma f\left(\frac{10\cdot 01}{00\cdot 11}, \frac{20\cdot 02}{00\cdot 22}, \frac{30\cdot 03}{00\cdot 33}\right). \tag{20}$$

Set

$$\begin{aligned} x &= \frac{10\cdot 01}{00\cdot 11}, \\ y &= \frac{20\cdot 02}{00\cdot 22}, \\ z &= \frac{30\cdot 03}{00\cdot 33}. \end{aligned}$$

The differential operator $p_{10}p_{01}p_{00}p_{11}(\underline{\partial_{10}\partial_{01}} - \partial_{00}\partial_{11})$ can be written as $\theta_{10}\theta_{01} - x\theta_{00}\Theta_{11}$, where $\theta_{ij} = p_{ij}\partial_{ij}$ (the Euler operator). We will derive a differential operator that annihilates the function f from this operator. Apply θ_{11} to $p^\gamma f(x,y,z)$. Then, we have $p^\gamma(\gamma_{11} - \theta_x)f$. Apply Θ_{00} to this function. Then, we have

$$\gamma_{00}\gamma_{11}p^\gamma f + p^\gamma\gamma_{11}(-\theta_x - \theta_y - \theta_z)f - \gamma_{00}p^\gamma x f - p^\gamma(-1)x f_x - p^\gamma x(-\theta_x - \theta_y - \theta_z)f_x.$$

This can be factored as

$$p^\gamma(\theta_x + \theta_y + \theta_z - \gamma_{00})(\theta_x - \gamma_{11})f.$$

An analogous calculation leads us to

$$\theta_{10}\theta_{01}p^\gamma f(x,y,z) = p^\gamma(\theta_x + \gamma_{01})(\theta_x + \gamma_{10})f.$$

Therefore, the function $f(x,y,z)$ satisfies

$$((\theta_x + \gamma_{01})(\theta_x + \gamma_{10}) - x(\theta_x + \theta_y + \theta_z - \gamma_{00})(\theta_x - \gamma_{11}))\, f = 0. \tag{21}$$

By analogous calculations, we have

$$((\theta_y + \gamma_{02})(\theta_y + \gamma_{20}) - y(\theta_x + \theta_y + \theta_z - \gamma_{00})(\theta_y - \gamma_{22}))\, f = 0, \tag{22}$$
$$((\theta_z + \gamma_{03})(\theta_x + \gamma_{30}) - z(\theta_x + \theta_y + \theta_z - \gamma_{00})(\theta_z - \gamma_{33}))\, f = 0. \tag{23}$$

By these equations for the function f, we conclude that the function $g = x^{-\gamma_{01}}y^{-\gamma_{02}}z^{-\gamma_{03}}f(x,y,z)$ satisfies the differential equations for the Lauricella function F_A, $n = 3$.

The bouquet of n squares stands for the Lauricella F_A of n variables.

Remark 4 The Lauricella functions belong to the Mellin hypergeometric systems introduced in the 19-th century [3, Appendix 1], [21]. A categorial correspondence between Mellin systems and A-hypergeometric systems is discussed in [30].

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