A Note on Gauge Transformations and Stabile Ordinary Differential Equations

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1 Introduction

We have introduced the notion of stabile ODEs (odrinary differential equations) for obtaining numerical solutions valid when the independent variable x goes to the infinity. In our applications of holonomic gradient method (HGM, see, e.g., [2], [3], [4]), it is a key step to find a stabile ODE [1]. In this short note, we will propose an algorithmic method, which utilizes Gauge transformations, to derive a stabile system. If we can apply this algorithmic method by an algebraic calculation, we can obtain a stabile system with no numerical error. We demonstrate that this approach gives a useful stabile ODE used in [1].

2 Obtaining Stabile Subsystems

We review our definition of a stabile ODE given in [1].

Definition 1. Consider the ODE

$$\left[a_m(x)\frac{d^m}{dx^m} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)\right] \bullet u = 0 \tag{1}$$

where $a_i(x)$ are polynomials in x. We assume that f is a solution of this differential equation. and denote with $f_1(x), \ldots, f_m(x)$ its linearly independent solutions. Then, let $f_i(x)$ be the dominant solution for $x \to \infty$, i.e., $|f_i(x)| \geq |f_j(x)|$, $\forall j$. We refer to the LODE (liear ODE) as *stabile* for $f(x)$ if $\lim_{x\to\infty} \frac{|f_i(x)|}{|f(x)|} < \infty$. Note that a LODE is stabile or not regardless of the selected set of linearly independent solutions. The notion of stabile LODE is defined analogously in the case of a vector-valued function, by replacing $|\cdot|$ with a vector norm $||\cdot||$.

Theorem 1. *Stabile LODEs can be derived algorithmically and numerically as a lower-dimensional subsystem of given LODEs.*

Proof. We give our algorithm. It is enough to give an algorithm in case of a vector valued ODE (a system of ODEs) of the first order.

Let $\mathbf{g} = (g_1, \ldots, g_r)^\mathsf{T}$ be the dominant solution of the LODE $\mathbf{f}' = \mathbf{P}(x)\mathbf{f}$ where $\mathbf{P}$ is an $r \times r$ square matrix valued function. For this LODE, let us consider the following gauge transformation matrix and its inverse:

$$\mathbf{G} = \left(\begin{array}{c|cccc} g_1 & 0 & 0 & \cdots & 0 \\ \hline g_2 & & & & \\ \cdot & & \mathbf{I}_{r-1} & & \\ \cdot & & & & \\ g_r & & & & \end{array}\right), \quad \mathbf{G}^{-1} = \left(\begin{array}{c|cccc} g_1^{-1} & 0 & 0 & \cdots & 0 \\ \hline -g_2 g_1^{-1} & & & & \\ \cdot & & \mathbf{I}_{r-1} & & \\ \cdot & & & & \\ -g_r g_1^{-1} & & & & \end{array}\right). \tag{2}$$

Then, the matrix of the transformed LODE by the Gauge transformation $\mathbf{G}$ can be written as

$$\mathbf{G}^{-1}\mathbf{P}\mathbf{G} - \mathbf{G}^{-1}\mathbf{G}' = \left(\begin{array}{c|cccc} 0 & g_1^{-1}p_{12} & g_1^{-1}p_{13} & \cdots & g_1^{-1}p_{1r} \\ \hline 0 & & & & \\ \cdot & & \mathbf{P}_2 & & \\ \cdot & & & & \\ 0 & & & & \end{array}\right), \tag{3}$$

where $p_{ij} = [\mathbf{P}]_{i,j}$, and $\mathbf{P}_2$ is an $(r-1) \times (r-1)$ matrix with column $(i-1)$ given by

$$(-g_2 g_1^{-1} p_{1i} + p_{2i} \quad -g_3 g_1^{-1} p_{1i} + p_{3i} \quad \ldots \quad -g_r g_1^{-1} p_{1i} + p_{ri})^{\mathsf{T}}. \tag{4}$$

In other words, the original $\mathbf{P}$ is block upper-triangularized by the gauge transformation $\mathbf{G}$.

It follows from the local theory of LODEs that the second dominant solution of the LODE for $\mathbf{F}$ is obtained from the dominant solution of the smaller system $\mathbf{f}_2' = \mathbf{P}_2\mathbf{f}_2$. We can apply this procedure recursively. //

We apply this procedure for the following ODE discussed in [1]

$$\frac{\partial}{\partial x}\mathbf{f}(x,\lambda) = \mathbf{P}(x,\lambda)\,\mathbf{f}(x,\lambda) \quad \text{where } \mathbf{f} \text{ is the 4-dimensional vector} \tag{5}$$

and

$$\mathbf{P}(x,\lambda) = \frac{1}{x\lambda}\begin{pmatrix} a_1 & a_2 & -1 & 0 \\ 0 & a_3 & a_2+1 & -1 \\ a_5 & a_6 & a_7 & n-1 \\ a_8 & a_9 & a_{10} & a_{11} \end{pmatrix}.$$

a_i's are polynomials in x and λ shown in Table 1.

In this case, we can derive a stabile subsystem

$$\frac{\partial}{\partial \varphi}\mathbf{g}(\varphi) = \frac{1}{\varphi}\begin{pmatrix} 0 & 2e^{-\varphi^2+2\varphi\psi}\varphi^{2(k+1)} & 0 \\ 0 & 0 & 1 \\ 0 & -2(2n-1)\varphi\psi & -[4\varphi\psi + 2(n-1)] \end{pmatrix}\mathbf{g}(\varphi). \tag{6}$$

by an algebraic calculation.

Theorem 2. *A gauge transformation yields the* 3 *dimensional subsystem (2), which is stabile, from the* 4 *dimensional system (5), which is not stabile, by block upper-triangularization where* $x = \sqrt{x}$ *and* $\psi = \sqrt{\lambda}$.

Table 1: Elements of matrices $\mathbf{P}(x,\lambda)$ from [1]

	Expression
a_1	$(k+1)\lambda$
a_2	$\lambda - n + 1$
a_3	$(k+1)\lambda + n - 1$
a_5	$(k+1)x\lambda^2$
a_6	$x\lambda^2 - (n-1)\left[x\lambda + (k+1)\lambda + n - 1\right]$
a_7	$-(x+n-1)\lambda + (n-1)(n-2)$
a_8	$-(n-2)(k+1)x\lambda^2$
a_9	$(k-n+3)x\lambda^2 + (n-1)^2\left[x\lambda + (k+1)\lambda + n - 1\right]$
a_{10}	$x\lambda^2 + (n-1)^2(\lambda - n + 2)$
a_{11}	$-x\lambda - (n-1)^2$

Proof. Our proof relies on [1, Appendix I-b] and we use symbols l_i defined in this appendix. Let $\mathcal{I}$ be the left ideal in ring $\mathcal{R}$ generated by operators l_1 and l_2 respectively, and $\mathcal{G}$ be the Gröbner basis of $\mathcal{I}$ for the lexicographic ordering $\partial_\lambda > \partial_x$, which comprises differential operators l_3, l_4, and l_5. Based on the ordering we can identify the leading terms of l_3, l_4, and l_5 as θ_x^3, $\theta_\lambda\theta_x$, and θ_λ^2, respectively, which, in the $(\theta_x,\theta_\lambda)$-plane, correspond to points $(3,0)$, $(1,1)$, and $(0,2)$, respectively. Then, the remaining points in the top-right quadrant under the line connecting these points are: $(0,0)$, $(1,0)$, $(2,0)$, and $(0,1)$. These points yield the so-called *standard monomials* with respect to $\mathcal{G}$ as 1, θ_x, θ_x^2, and $\theta_\lambda\}$, respectively. Please see [5], [3, p. 46] for their definition. Finally, notice again that operator l_3 is free of θ_λ.

Then, using the definition of $\theta_x = x\frac{\partial}{\partial x}$ and expressing θ_x^3 from l_3, we can write the following in the ring of differential operators l_1 and l_2[1]:

$$\frac{\partial}{\partial x}1 = \frac{1}{x}\theta_x, \tag{7}$$

$$\frac{\partial}{\partial x}\theta_x = \frac{1}{x}\theta_x^2, \tag{8}$$

$$\frac{\partial}{\partial x}\theta_x^2 = \frac{1}{x}\theta_x^3 \equiv -\frac{2x-2k-3}{x}\theta_x^2 - \frac{(x-k-1)(x-k+n-2)-x\lambda}{x}\theta_x. \tag{9}$$

Because the right-hand side contains θ_x, θ_x^2, but not θ_λ, we can recast the ODEs in block-diagonalized form as

$$\frac{\partial}{\partial x}\begin{pmatrix}\theta_\lambda\\ 1\\ \theta_x\\ \theta_x^2\end{pmatrix} \equiv \left(\begin{array}{c|ccc} * & * & * & * \\ \hline 0 & & & \\ 0 & & \mathbf{P}_2 & \\ 0 & & & \end{array}\right)\begin{pmatrix}\theta_\lambda\\ 1\\ \theta_x\\ \theta_x^2\end{pmatrix} \quad \text{modulo}\,\mathcal{I}, \tag{10}$$

where symbols $*$ stand for unimportant elements, and $\mathbf{P}_2$ is a 3×3 matrix whose elements can be readily identified from (7), (8), (9).

Now, let $\mathcal{J}$ be the left ideal generated by $\mathcal{I}$ in the ring of differential operators $\mathcal{K}\langle\partial_x,\partial_\lambda\rangle$, where $\mathcal{K}$ is the quotient field of holomorphic functions of (x,λ) on a

[1] $A \equiv B$ or $A \equiv B$ modulo $\mathcal{I}$ means that $A - B \in \mathcal{I}$. Note that if l_1 and l_2 annihilate f then $A \bullet f = B \bullet f$ when $A \equiv B$.

neighborhood of $(1,1)$. Using the following gauge transformation[2]

$$
\begin{aligned}
s_1(x,\lambda) &= \exp(-2\sqrt{x\lambda})x^{-k-1}e^x\theta_x, && (11)\\
s_2(x,\lambda) &= \frac{1}{2}\theta_x(\exp(-2\sqrt{x\lambda})x^{-k-1}e^x\theta_x), && (12)
\end{aligned}
$$

we can transform (10) into

$$
\frac{\partial}{\partial x}\begin{pmatrix}\theta_\lambda\\ 1\\ s_1\\ s_2\end{pmatrix} \equiv \left(\begin{array}{c|ccc} * & * & * & * \\ \hline 0 & & & \\ 0 & & \mathbf{P}_3 & \\ 0 & & & \end{array}\right)\begin{pmatrix}\theta_\lambda\\ 1\\ s_1\\ s_2\end{pmatrix} \quad \text{modulo}\,\mathcal{J}, \tag{13}
$$

where $\mathbf{P}_3$ is a 3×3 matrix.

Further, let $\mathcal{G}'$ be a Gröbner basis of $\mathcal{I}$ with the ordering $\partial_x > \partial_\lambda$. Recall that this basis is characterized by the differential operators l_6 and l_7 shown in [1]. Computing the *normal forms*[5][3, p. 283] of s_1 and s_2 by $\mathcal{G}'$ yields $c_i(x,\lambda)$ and $d_i(x,\lambda)$, $i=0,\ldots,3$, for

$$
\begin{aligned}
s_1 &\equiv \sum_{i=0}^{3} c_i(x,\lambda)\theta_\lambda^i \quad \text{modulo}\,\mathcal{J}, && (14)\\
s_2 &\equiv \sum_{i=0}^{3} d_i(x,\lambda)\theta_\lambda^i \quad \text{modulo}\,\mathcal{J}. && (15)
\end{aligned}
$$

Now, using $c_i(x,\lambda)$ and $d_i(x,\lambda)$, $i=0:3$ to construct the matrix

$$
\mathbf{G} = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ c_0 & c_1 & c_2 & c_3\\ d_0 & d_1 & d_2 & d_3\end{pmatrix}, \tag{16}
$$

and applying the gauge transformation with $\mathbf{G}$ to (5) yields the matrix $\mathbf{P}_3$ in (13). Finally, upon making the variable transformation $\varphi = \sqrt{x}$, differential equation $\frac{\partial}{\partial x}\mathbf{f} = \mathbf{P}_3\mathbf{f}$ yields (6). //

Numerical evidences that the stabile ODE (2) is useful in a performance evaluation of multiple-input multiple-output (MIMO) wireless communications systems under Rician fading channel is given in [1].

References

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[3] T. Hibi (eds), Gröbner Bases — Statistics and Software Systems, Springer, 2013.

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