Nobuki Takayama, 2019.12.05, et-2, Holonomic functions and algorithms

Holonomic function in one variable Let $f(x)$ be a smooth (C^∞) function defined on an open interval U in $\mathbf{R}$. The function f or its analytic continuation is called a holonomic (analytic) function when $\exists L \in \mathbf{C}(x)\langle\partial\rangle$ such that $L \bullet f = 0$ (L is called an annihilator of f).

Example. $\int_0^{+\infty} \exp(-x - \theta x^3) dx$ is a holonomic function, because it is annihilated by an ordinary differential operator with polynomial coefficients.

Theorem

The sum and the product of holonomic functions are holonomic functions. The derivative of any holonomic function is a holonomic function.

Holonomic function of several variables Let $f(x) = f(x_1, \ldots, x_n)$ be a smooth function defined on an open set U in $\mathbf{R}^n$. The function f or its analytic continuation is called a holonomic (analytic) function when there exist n-differential operators L_i, $i = 1, \ldots, n$ of the form

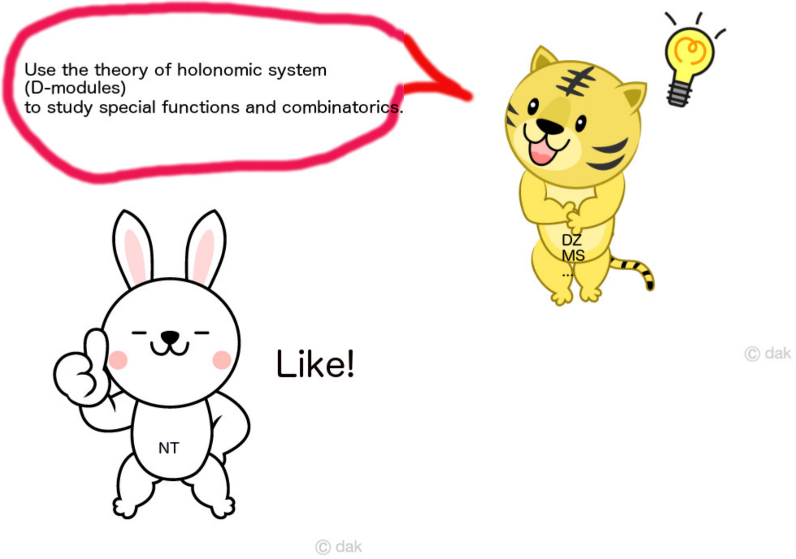
$$L_i = a^i_{m_i}(x)\partial_i^{m_i} + a^i_{m_i-1}(x)\partial_i^{m_i-1} + \cdots + a^i_0(x), \quad \partial_i = \frac{\partial}{\partial x_i} \quad (1)$$

where $a^i_j(x) \in \mathbf{C}(x)$ which annihilate the function f. The following important theorem/project follows from the D-module theory.

Theorem (Zeilberger project, 1990)

If $f(x_1, \ldots, x_n)$ is a holonomic function in x, then the integral $\int_\Omega f(x)dx_n$ is a holonomic function in $(x_1, \ldots, x_{n-1})$ (under some conditions on the set Ω).

Doron Zeilberger: Let's use this fact to prove combinatorial identities and special function identities! We need algorithms for it.


Use the theory of holonomic system
(D-modules)
to study special functions and combinatorics.
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Examples of holonomic functions

Which are holonomic (analytic) functions?

1. $\exp(f(x))$ where f is a rational function,
2. $\frac{1}{\sin x}$ [Hint] Use Th: Any solution of the ordinary differential equation $(a_m(x)\partial^m + \cdots + a_0(x)) \bullet f = 0$, $a_i \in \mathbf{C}[x]$, is holomorphic out of the singular locus $\{x \mid a_m(x) = 0\}$.
3. $\Gamma(x)$, [Hint] $\Gamma(x)$ has poles at $x = -n$, $n \in \mathbf{N}_0$.
4. 2^x,
5. $H(x)$ (Heaviside function),
6. x^a where a is a constant,
7. $|x|$,
8. $\int_{-\infty}^{+\infty} \exp(-xt^6 - t)dt$, $x > 0$.
9. $\exp(\exp(x))$ [Hint] local theory of linear ODE.

Weyl algebra and holonomic ideal   Let D_n be the ring of differential operators of polynomial coefficients. D_n is a subring of $R_n = \mathbf{C}(x)\langle \partial_1, \ldots, \partial_n \rangle$ (the ring of diff op with rational function coefficients). For $L = \sum_{(\alpha,\beta)\in E} a_{\alpha,\beta} x^\alpha \partial^\beta \in D_n$, we define

$$
\begin{aligned}
\mathrm{ord}_{(u,v)}(L) &= \max_{(\alpha,\beta)\in E} (u\alpha + v\beta) &(2)\\
\mathrm{in}_{(u,v)}(L) &= \sum_{\mathrm{ord}(x^\alpha\partial^\beta)=\mathrm{ord}(L),(\alpha,\beta)\in E} a_{\alpha,\beta} x^\alpha \xi^\beta \in \mathbf{C}[x,\xi] &(3)
\end{aligned}
$$

where $u = (1, \ldots, 1)$ and $v = (1, \ldots, 1)$. Example $L = (x_1 - x_2)\partial_1\partial_2 + \partial_1 + \partial_2$. We have $\mathrm{ord}_{(u,v)}(L) = 3$ and $\mathrm{in}_{(u,v)}(L) = (x_1 - x_2)\xi_1\xi_2$ ▮
For a left ideal I of D_n, define $\mathrm{in}_{(u,v)}(I) = \langle \mathrm{in}_{(u,v)}(L) \mid L \in I \rangle$, which is called the (u, v)-initial ideal of I. I is called a holonomic ideal when the (Krull) dimension of $\mathrm{in}_{(u,v)}(I)$ is n. $\dim_{\mathbf{C}(x)} R_n/R_n I$ is the holonomic rank of I.

Some important theorems on holonomic ideal

Exercise Show that when J is a holonomic left ideal, then $I = R_n J$ is a zero-dimensional ideal in R_n. (In other words, the holonomic rank $\dim_{\mathbf{C}(x)} R_n/I$ of I is finite) ∎

When J is a holonomic left ideal such that $J \neq D_n$, we have

Theorem (Bernstein inequality)

$(\mathrm{Krull})\dim\, \mathrm{in}_{(u,v)}(J) \geq n.$

Theorem (Cor. of Kashiwara 1978)

If I is a zero-dimensional left ideal in R_n, then, $I \cap D_n$ is a holonomic ideal.

Note. The ideal $I \cap D_n$ is called the Weyl closure of I. An algorithm to construct generators of the Weyl closure from generators of I was given by H.Tsai (2002). It is implemented in Macaulay 2 (`WeylClosure`).

Supplemental Exercise

1. For $f = \exp(1/(x_1^3 - x_2^2 x_3^2))$, define polynomials p_i and q_i by
$$p_i/q_i = (\partial f/\partial x_i)/f.$$
We have
$q_1 = q_2 = q_3(x_1^3 - x_2^2 x_3^2)^2, p_1 = -3x_1^2, p_2 = 2x_2 x_3^2, p_3 = 2x_2^2 x_3.$
Show that
$$q_i \partial_i - p_i, \quad i = 1, 2, 3$$
generate a zero dimensional ideal I in R_3 but they do not generate a holonomic ideal in D_3.
2. Compute the Weyl closure of I.

An answer

```
loadPackage "Dmodules"
D=QQ[x,y,z,dx,dy,dz, WeylAlgebra=>{x=>dx,y=>dy,z=>dz}];
I = ideal((x^3-y^2*z^2)^2*dx+3*x^2,
          (x^3-y^2*z^2)^2*dy-2*y*z^2,
          (x^3-y^2*z^2)^2*dz-2*y^2*z);
II=inw(I,{1,1,1,1,1,1});
print(dim II);  --- the output 4 implies that it is not holonomic.
J=WeylClosure I;
print(toString(J));
JJ=inw(J,{1,1,1,1,1,1});
print(dim JJ);  --- the output 3 implies that it is holonomic.
```

J contains $-y\partial_x\partial_y + z\partial_x\partial_z, \ldots$.

Holonomic Schwartz distribution A distribution f on $\mathbf{R}^n$ is called a holonomic (Schwartz) distribution when it is annihilated by a holonomic ideal.

Theorem

When $f(x_1, \ldots, x_n)$ is a holonomic distribution, $g(x') = \int_{\mathbf{R}^{n-m}} f(x) dx_{m+1} \cdots dx_n$ is a holonomic distribution of m-variables x' (under some conditions).

Exercise Which are holonomic distributions?

1. $\frac{1}{\sin x}$,
2. $H(x)$ (Heaviside function),
3. $|\sin x|$,
4. $|x|$,
5. $\int_0^1 exp(-x^3 y + x) dx = \int_{\mathbf{R}} exp(-x^3 y + x) H(x) H(1-x) dx$,
6. $\frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\xi x} \frac{\sin \xi}{\xi} d\xi$ [Hint] $2 \sin \xi / \xi$ is the Fourier transform of $H(1-x)H(1+x)$ as a distribution.

Algorithms and implementations To obtain differential equation for the integral $g(x')$, we need an elimination alg. of a left ideal I in D_n and a right ideal:

$$(I + \partial_1 D_n + \cdots + \partial_m D_n)) \cap \mathbf{C}\langle x_{m+1}, \ldots, x_n, \partial_{m+1}, \ldots, \partial_n \rangle$$

1. Creative telescoping (D.Zeilberger, 1980's – 2010's).
2. T (kan/sm1, ..., 1980's — 2000's)
3. T.Oaku, algorithms for b-functions, restrictions, and algebraic local cohomology groups (1997)
4. F.Chyzak's heuristics (2000's), C.Kouchan's heuristics (2010's)*
5. Risa/Asir, Macaulay 2, Singular.
6. T.Oaku (2013), Annihilator of the Heaviside function with the support on a semi-algebraic set.
7. Annihilators and b-functions for f^s (Nabeshima (this conference), ...).

*HolonomicFunctions.m on Mathematica