Funkcialaj Ekvacioj, **62** (2019) 409–424

L^p Boundedness of Higher Order Schrödinger Type Operators

By

Satoko SUGANO

(Kobe City College of Technology, Japan)

Abstract. We consider higher order Schrödinger type operators with nonnegative potentials. We assume that the potential belongs to the reverse Hölder class which includes nonnegative polynomials. We establish estimates of the fundamental solution and show L^p boundedness of some Schrödinger type operators. We use pointwise estimates by the Hardy-Littlewood maximal operator to prove our results.

Key Words and Phrases. Schrödinger operator, Reverse Hölder class, Maximal operator.

2010 *Mathematics Subject Classification Numbers.* 35J10, 42B25.

1. Introduction

Let $V(x)$ be a nonnegative potential and consider the Schrödinger type operators $H_k = (-\Delta)^k + V^k$ on $\boldsymbol{R}^n$, where k is a positive integer and $n \geq 2k+1$. When V is a nonnegative polynomial, Zhong proved estimates of the fundamental solution for H_1 and H_2 and showed some estimates for H_1 and H_2 ([16]). More precisely, he showed the L^p boundedness of the operators VH_1^{-1}, $V^{1/2}\nabla H_1^{-1}$, and $V^{2-j/2}\nabla^j H_2^{-1}$, where $j = 0, 1, 2, 3, 4$. Recently, in [10], the authors showed the L^p boundedness of the operators $V^k H_k^{-1}$, $\nabla^{2k} H_k^{-1}$, and $\nabla^k H_k^{-1/2}$ for nonnegative polynomial potentials V.

For the potential V which belongs to the reverse Hölder class, which includes nonnegative polynomials, Shen generalized Zhong's results on H_1 ([11]). Actually, he established estimates of the fundamental solution for H_1 and showed the L^p estimates for the operators VH_1^{-1}, $V^{1/2}\nabla H_1^{-1}$, $\nabla^2 H_1^{-1}$, and so on. For the operator H_1 with reverse Hölder class potentials, further results have been investigated by many researchers. See [1], [3], [6], [7], and [12], for example. For the operator H_2 with reverse Hölder class potentials, in [13], the author established estimates of the fundamental solution for H_2 and showed the L^p boundedness of the operators $V^{2-j/2}\nabla^j H_2^{-1}$, where $j = 0, 1, 2, 3, 4$. For the operator H_2, further results have been shown by several researchers. See [2] and [9] for example. Recently, in [14], the author established estimates of the fundamental solution for H_{2^m}, where m is a positive integer satisfying $m \geq 2$, and showed the L^p boundedness of the operators $V^{2^m - j/2}\nabla^j H_{2^m}^{-1}$, where j is an integer satisfying $1 \leq j \leq 2^{m+1} - 1$.

As mentioned above, in [10], the authors proved some results on $H_k = (-\Delta)^k + V^k$, where V is a nonnegative polynomial and k is an integer satisfying $k \geq 3$. The purpose of this paper is to show a result on $H_k = (-\Delta)^k + V^k$, where k is an integer satisfying $k \geq 3$, with potentials V which belong to the reverse Hölder class, which includes nonnegative polynomials.

We recall the definitions of the reverse Hölder class (e.g. [11]). We denote by $B(x,r)$ the ball centered at x with radius r.

Definition 1.1 (Reverse Hölder class). Let $V \geq 0$.

(1) For $1 < p < \infty$ one says that $V \in (RH)_p$, if $V \in L^p_{\mathrm{loc}}(\boldsymbol{R}^n)$ and there exists a positive constant C such that

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)^p dy\right)^{1/p} \leq \frac{C}{|B(x,r)|}\int_{B(x,r)} V(y)dy \tag{1.1}$$

holds for every $x \in \boldsymbol{R}^n$ and $0 < r < \infty$.

(2) One says that $V \in (RH)_\infty$, if $V \in L^\infty_{\mathrm{loc}}(\boldsymbol{R}^n)$ and there exists a positive constant C such that

$$\|V\|_{L^\infty(B(x,r))} \leq \frac{C}{|B(x,r)|}\int_{B(x,r)} V(y)dy \tag{1.2}$$

holds for every $x \in \boldsymbol{R}^n$ and $0 < r < \infty$.

Remark 1.2. (1) For $1 < p < \infty$, it is easy to see $(RH)_\infty \subset (RH)_p$.

(2) If $P(x)$ is a polynomial and $\alpha > 0$, then $V(x) = |P(x)|^\alpha$ belongs to $(RH)_\infty$ ([4, page 146]).

(3) If $V \in (RH)_\infty$ then $V^\alpha \in (RH)_\infty$ for every $\alpha > 0$ ([8, Lemma 1]).

Definition 1.3 ([11, Definition 1.3]). Let $V \in (RH)_{n/2}$ and $V \not\equiv 0$. Then it is well-known that there exists a positive ε such that $V \in (RH)_{n/2+\varepsilon}$ ([5, Lemma 2]). Then the function $\rho(x,V)$ is well-defined by

$$\frac{1}{\rho(x,V)} = \sup\left\{r > 0 : \frac{r^2}{|B(x,r)|}\int_{B(x,r)} V(y)dy \leq 1\right\} \tag{1.3}$$

and satisfies $0 < \rho(x,V) < \infty$ for every $x \in \boldsymbol{R}^n$.

Remark 1.4. (1) If $V \in (RH)_\infty$ then there exists a positive constant C such that $V(x) \leq C\rho(x,V)^2$ for a.e. $x \in \boldsymbol{R}^n$ ([11, Remark 2.9]).

(2) If $V \in (RH)_p$, $p \geq n/2$, then there exists a positive constant C such that

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)^p dy\right)^{1/p} \leq C\rho(x,V)^2 \tag{1.4}$$

for every $x \in \boldsymbol{R}^n$ and $0 < r < \infty$ ([13, Remark 2]).

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ denote the multi-index with $\alpha_i \in \boldsymbol{N}$, $i \in \boldsymbol{N}$, $1 \le i \le n$. Define $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $\nabla^j = \partial^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n})$ for $j = |\alpha| = \alpha_1 + \dots + \alpha_n$. For any positive integer j and a function $u \in C^j(\boldsymbol{R}^n)$, denote $\nabla^j u(x) = (\partial^\alpha u(x) : |\alpha| = j)$ and $|\nabla^j u(x)|^2 = \sum_{|\alpha|=j} |\partial^\alpha u(x)|^2$. We denote by $\Gamma_{H_k}(x, y)$ the fundamental solution for H_k, where k is a positive integer. The operator H_k^{-1} is the integral operator with $\Gamma_{H_k}(x, y)$ as its kernel.

Now we state our theorem.

Theorem 1.5. *Let j, k, and n be integers, $k \ge 1$, $0 \le j \le 2k - 1$, and $n \ge 2k + 1$. Suppose that $V \in (RH)_\infty$. Then there exists a constant C such that*

$$\| V^{k-j/2} \nabla^j \{(-\Delta)^k + V^k\}^{-1} f \|_{L^p(\boldsymbol{R}^n)} \le C \|f\|_{L^p(\boldsymbol{R}^n)}, \tag{1.5}$$

where $1 < p \le \infty$ and $f \in C_0^\infty(\boldsymbol{R}^n)$.

Remark 1.6. In Theorem 1.5, the case $k = 2^m$ was shown in [14, Theorem 28].

To prove Theorem 1.5, we need the estimates of the fundamental solution. Let l and m be integers, $m \ge 0$, and $1 \le l \le 2^m$. We consider $H_{2^m+l, 2^{m+1}} = (-\Delta)^{2^m+l} + W^{2^{m+1}}$ on $\boldsymbol{R}^n$, where $W \ge 0$ and $n \ge 2(2^m + l) + 1$. We denote by $\Gamma_{H_{2^m+l, 2^{m+1}}}(x, y)$ the fundamental solution for $H_{2^m+l, 2^{m+1}}$. The operator $H_{2^m+l, 2^{m+1}}^{-1}$ is the integral operator with $\Gamma_{H_{2^m+l, 2^{m+1}}}(x, y)$ as its kernel.

Theorem 1.7. *Let l, m, and n be integers, $m \ge 0$, $1 \le l \le 2^m$, and $n \ge 2(2^m + l) + 1$. Suppose that $W \in (RH)_{2^m n/(2^m+l)}$ and that there exists a positive constant C such that $\rho(x, W) \le C$. Then for any positive integer N there exists a positive constant C_N such that*

$$(0 \le) \Gamma_{H_{2^m+l, 2^{m+1}}}(x, y) \le \frac{C_N}{\{1 + \rho(x, W)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-2(2^m+l)}}. \tag{1.6}$$

Theorem 1.8. *Let j, l, m, and n be integers, $m \ge 0$, $1 \le l \le 2^m$, $1 \le j \le 2(2^m + l) - 1$, and $n \ge 2(2^m + l) + 1$. Suppose that $W \in (RH)_{2^{m+1}n/\{2(2^m+l)-j\}}$ and that there exists a positive constant C such that $\rho(x, W) \le C$. Then for any positive integer N there exists a positive constant C_N such that*

$$|\nabla^j \Gamma_{H_{2^m+l, 2^{m+1}}}(x, y)| \le \frac{C_N}{\{1 + \rho(x, W)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-2(2^m+l)+j}}. \tag{1.7}$$

Remark 1.9. In Theorems 1.7 and 1.8, the case $l = 2^m$ was shown in [14, Theorems 9 and 10] without the assumption $\rho(x, W) \le C$.

The plan of this paper is as follows. In Section 2, we describe some lemmas needed to prove Theorems 1.7 and 1.8. In Section 3, we prove Theorems 1.7 and 1.8. Finally, in Section 4, we prove Theorem 1.5.

Throughout this paper the letter C stands for a constant not necessarily the same at each occurrence.

2. Preliminaries

In this section, we describe some lemmas needed later.

Lemma 2.1 ([11, Lemma 1.4 (b), (c)]).
(1) *Suppose that* $V \in (RH)_{n/2}$. *Then there exist positive constants* C *and* k_0 *such that, for* $x, y \in \boldsymbol{R}^n$,

$$\rho(y, V) \leq C\{1 + \rho(x, V)|x - y|\}^{k_0} \rho(x, V). \tag{2.1}$$

(2) *Suppose that* $V \in (RH)_{n/2}$. *Then there exist positive constants* C *and* k_0 *such that, for* $x, y \in \boldsymbol{R}^n$,

$$\rho(y, V) \geq \frac{C\rho(x, V)}{\{1 + \rho(x, V)|x - y|\}^{k_0/(k_0+1)}}. \tag{2.2}$$

Lemma 2.2 (Caccioppoli type inequality, [14, Lemmas 13 and 15]). *Let* i, j, l, *and* m *be integers,* $m \geq 0$, $1 \leq l \leq 2^m$, *and* $1 \leq i \leq j \leq 2^m + l$. *Assume that* $(-\Delta)^{2^m+l}u + W^{2^{m+1}}u = 0$ *in* $B(x_0, R)$ *for some* $x_0 \in \boldsymbol{R}^n$. *Then there exists a positive constant* C *such that*

$$\begin{aligned}\int_{B(x_0, R/2)} |\nabla^{2^{m+1}+2l-j}u(x)|^2 dx + \int_{B(x_0, R/2)} W(x)^{2^{m+1}} |u(x)|\,|\Delta^{2^m+l-j}u(x)| dx \\ \leq C \sum_{i=1}^{j} \frac{1}{R^{2i}} \int_{B(x_0, R)} |\nabla^{2^{m+1}+2l-j-i}u(x)|^2 dx.\end{aligned} \tag{2.3}$$

Lemma 2.3. *Let* j, l, *and* m *be integers,* $m \geq 0$, $1 \leq l \leq 2^m$, *and* $0 \leq j \leq 2^m + l - 1$. *Assume that* $(-\Delta)^{2^m+l}u + W^{2^{m+1}}u = 0$ *in* $B(x_0, R)$ *for some* $x_0 \in \boldsymbol{R}^n$. *Then there exists a positive constant* C *such that*

$$\int_{B(x_0, R/2)} W(x)^{2^{m+1}} |u(x)|^2 dx \leq C \sum_{j=0}^{2^m+l-1} \frac{1}{R^{2(2^{m+1}-j)}} \int_{B(x_0, R)} |\nabla^j u(x)|^2 dx. \tag{2.4}$$

Proof. Case l is an even number: Let a and b be nonnegative integers satisfying $a + b \leq (2^m + l)/2$. We choose $\eta \in C_0^\infty(B(x_0, R))$ such that $\eta \equiv 1$ on $B(x_0, R/2)$ and $|\nabla^b(\Delta^a \eta)| \leq C/R^{2a+b+2^{m+1}-(2^m+l)}$. Note that, there exists a positive constant $C_{m,l,a,b}$ such that

$$\begin{aligned}\Delta^{(2^m+l)/2}(u(x)\eta(x)^{2^m+l+1}) = \sum_{\substack{a \geq 0, b \geq 0 \\ a+b \leq (2^m+l)/2}} C_{m,l,a,b} \nabla^b(\Delta^{(2^m+l)/2-a-b}u)(x) \\ \cdot \nabla^b(\Delta^a \eta^{2^m+l+1})(x),\end{aligned} \tag{2.5}$$

where

$$(2.6)\quad \nabla^b(\Delta^{(2^m+l)/2-a-b}u)(x)\cdot\nabla^b(\Delta^a\eta^{2^m+l+1})(x)$$

$$= \sum_{i_1,i_2,\dots,i_{(2^m+l)/2}=1}^{n} \frac{\partial^b}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_b}} \cdot\left(\frac{\partial^2}{\partial x_{i_1}^2}\left(\frac{\partial^2}{\partial x_{i_2}^2}\left(\cdots\left(\frac{\partial^2}{\partial x_{i_{(2^m+l)/2-a-b}}^2}u\right)\cdots\right)\right)\right)$$

$$\cdot\frac{\partial^b}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_b}}\left(\frac{\partial^2}{\partial x_{i_1}^2}\left(\frac{\partial^2}{\partial x_{i_2}^2}\left(\cdots\left(\frac{\partial^2}{\partial x_{i_a}^2}\eta^{2^m+l+1}\right)\cdots\right)\right)\right).$$

(See [15, Lemma 2.6].) Multiplying $(-\Delta)^{2^m+l}u + W^{2^{m+1}}u = 0$ by $u\eta^{2^m+l+1}$ and integrating over $\boldsymbol{R}^n$ by integrating by parts, we have

$$(2.7)\quad \int_{B(x_0,R/2)}|\Delta^{(2^m+l)/2}u(x)|^2dx + \int_{B(x_0,R/2)}W(x)^{2^{m+1}}|u(x)|^2dx$$

$$\le C\int_{B(x_0,R)}\sum_{\substack{a\ge 0,b\ge 0\\ 1\le a+b\le(2^m+l)/2}}|\Delta^{(2^m+l)/2}u(x)| \cdot|\nabla^b(\Delta^{(2^m+l)/2-a-b}u)(x)\cdot\nabla^b(\Delta^a\eta^{2^m+l+1})(x)|dx.$$

Let ε be a positive real number which will be determined later. Then the right hand side of (2.7) is bounded by

$$(2.8)\quad C\int_{B(x_0,R)}\sum_{\substack{a\ge 0,b\ge 0\\ 1\le a+b\le(2^m+l)/2}}\sqrt{\varepsilon}|\Delta^{(2^m+l)/2}u(x)|\cdot\frac{1}{\sqrt{\varepsilon}} \cdot\frac{1}{R^{2a+b+2^{m+1}-(2^m+l)}}|\nabla^b(\Delta^{(2^m+l)/2-a-b}u)(x)|dx$$

$$\le C\int_{B(x_0,R)}\sum_{\substack{a\ge 0,b\ge 0\\ 1\le a+b\le(2^m+l)/2}}\left(\varepsilon|\Delta^{(2^m+l)/2}u(x)|^2 +\frac{1}{\varepsilon}\cdot\frac{1}{R^{2\{2a+b+2^{m+1}-(2^m+l)\}}}|\nabla^b(\Delta^{(2^m+l)/2-a-b}u)(x)|^2\right)dx.$$

Then choosing ε such that $C\varepsilon\{(2^m+l)/2\}\{(2^m+l)/2+1\}/2=1$ we arrive at the desired inequality.

Case l is an odd number: Let a and b be nonnegative integers satisfying $a+b\le(2^m+l-1)/2$. We choose $\eta\in C_0^\infty(B(x_0,R))$ such that $\eta\equiv 1$ on $B(x_0,R/2)$ and $|\nabla^{b+1}(\Delta^a\eta)|\le C/R^{2a+b+2^{m+1}-(2^m+l-1)}$. Multiplying $(-\Delta)^{2^m+l}u+$

$W^{2^{m+1}}u = 0$ by $u\eta^{2^m+l+1}$ and integrating over $\mathbf{R}^n$ by integrating by parts, we have

$$(2.9)\qquad \int_{B(x_0,R/2)} |\nabla(\Delta^{(2^m+l-1)/2}u)(x)|^2 dx + \int_{B(x_0,R/2)} W(x)^{2^{m+1}}|u(x)|^2 dx$$
$$\leq C\int_{B(x_0,R)} \sum_{\substack{a\geq 0, b\geq 0\\ 1\leq a+b\leq (2^m+l-1)/2}} |\nabla(\Delta^{(2^m+l-1)/2}u)(x)|$$
$$\cdot |\nabla^{b+1}(\Delta^{(2^m+l-1)/2-a-b}u)(x)\cdot \nabla^b(\Delta^a\eta^{2^m+l+1})(x)|dx$$
$$+ C\int_{B(x_0,R)} \sum_{\substack{a\geq 0, b\geq 0\\ a+b\leq (2^m+l-1)/2}} |\nabla(\Delta^{(2^m+l-1)/2}u)(x)|$$
$$\cdot |\nabla^{b}(\Delta^{(2^m+l-1)/2-a-b}u)(x)\cdot \nabla^{b+1}(\Delta^a\eta^{2^m+l+1})(x)|dx.$$

Then by the same argument as the case k is an even number, we arrive at the desired inequality. □

Lemma 2.4 (cf. [14, Lemma 17]). *Let j, l, m, and n be integers, $m\geq 0$, $1\leq l\leq 2^m$, $n\geq 2(2^m+l)+1$, and $0\leq j\leq 2^m+l-1$. Assume that $(-\Delta)^{2^m+l}u + W^{2^{m+1}}u = 0$, $u\geq 0$, in $B(x_0,R)$ for some $x_0\in\mathbf{R}^n$. Then there exists a positive constant C such that*

$$(2.10)\qquad \sup_{y\in B(x_0,R/2)}|u(y)| \leq C\sum_{j=0}^{2^m+l-1} R^j\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)}|\nabla^j u(x)|^2 dx\right)^{1/2}.$$

Proof. We can prove Lemma 2.4 by the same way as in the proof of [14, Lemma 17]. We omit the details. □

Lemma 2.5 ([14, Lemma 19]). *Let m and n be integers, $m\geq 0$, and $n\geq 2^{m+1}+1$. Suppose that $W\in(RH)_{n/2}$. Then there exists a positive constant C such that*

$$(2.11)\qquad \|\rho(\cdot,W)^{2^{m+1}}H^{-1}_{2^m,2^m}f\|_{L^p(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)},$$

where $1\leq p\leq\infty$.

Lemma 2.6 ([14, Lemma 20]). *Let j, m, and n be integers, $m\geq 0$, $n\geq 2^{m+1}+1$, and $1\leq j\leq 2^{m+1}-1$. Suppose that $W\in(RH)_{q_0}$ for some q_0 satisfying $n/2\leq q_0 < 2^m n/(2^{m+1}-j)$. Then for $1\leq p\leq p_0$ there exists a positive constant C such that*

$$(2.12)\qquad \|\rho(\cdot,W)^{2^{m+1}-j}\nabla^j H^{-1}_{2^m,2^m}f\|_{L^p(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)},$$

where $1/p_0 = 2^m/q_0 - (2^{m+1}-j)/n$.

Let l and m be integers satisfying $m \geq 0$ and $1 \leq l \leq 2^m$. For the case $q_0 = n/2$ in Lemma 2.6, if $1 \leq j \leq 2^m + l - 1$ and $n \geq 2(2^m + l) + 1$, letting $p = 2$ we have

Corollary 2.7. *Let j, l, m, and n be integers, $m \geq 0$, $1 \leq l \leq 2^m$, $n \geq 2(2^m + l) + 1$, and $1 \leq j \leq 2^m + l - 1$. Suppose that $W \in (RH)_{n/2}$. Then there exists a positive constant C such that*

$$\|\rho(\cdot, W)^{2^{m+1}-j} \nabla^j H_{2^m,2^m}^{-1} f\|_{L^2(\boldsymbol{R}^n)} \leq C \|f\|_{L^2(\boldsymbol{R}^n)}. \tag{2.13}$$

3. Proofs

In this section, we prove Theorems 1.7 and 1.8. To prove Theorem 1.7, we need the following lemmas.

Lemma 3.1. *Let j, l, m, and n be integers, $m \geq 0$, $1 \leq l \leq 2^m$, $n \geq 2(2^m + l) + 1$, and $1 \leq j \leq 2(2^m + l) - 1$. Suppose that $W \in (RH)_{q_0}$ for some q_0 satisfying $n/2 \leq q_0 < 2^{m+1} n / \{2(2^m + l) - j\}$. Assume also that $(-\Delta)^{2^m + l} u + W^{2^{m+1}} u = 0$ in $B(x_0, R)$ for some $x_0 \in \boldsymbol{R}^n$. Then there exists a positive constant C such that*

$$\left(\int_{B(x_0, R/2)} |\nabla^j u(x)|^t dx \right)^{1/t} \tag{3.1}$$
$$\leq C R^{2^{m+1} n / q_0 - 2(2^m + l)} \{1 + R^{2(2^m + l)} \rho(x_0, W)^{2^{m+2}}\} \sup_{y \in B(x_0, R)} |u(y)|,$$

where $1/t = 2^{m+1}/q_0 - \{2(2^m + l) - j\}/n$.

Proof. We show Lemma 3.1 by a method similar to the one used in the proof of [11, Lemma 4.6]. Let a and b be nonnegative integers satisfying $1 \leq a + b \leq 2^m + l$. We choose $\eta \in C_0^\infty(B(x_0, R))$ such that $\eta \equiv 1$ on $B(x_0, 3R/4)$ and $|\nabla^b (\Delta^a \eta)| \leq C / R^{2a+b}$. We denote by $\Gamma_0(x, y)$ the fundamental solution for $(-\Delta)^{2^m + l}$. It is known that there exists a positive constant C such that $0 \leq \Gamma_0(x, y) \leq C |x - y|^{2(2^m + l) - n}$. (See [10].) Note that, there exists a positive constant $C_{m,l,a,b}$ such that

$$u(x)\eta(x) = \int_{\boldsymbol{R}^n} \Gamma_0(x, y) \left(-W(y)^{2^{m+1}} u(y)\eta(y) + (-1)^{2^m + l} \right. \tag{3.2}$$
$$\left. \cdot \sum_{\substack{a \geq 0, b \geq 0 \\ 1 \leq a + b \leq 2^m + l}} C_{m,l,a,b} \nabla^b (\Delta^{2^m + l - a - b} u)(y) \cdot \nabla^b (\Delta^a \eta)(y) \right) dy.$$

Then integrating by parts, for $x \in B(x_0, R/2)$, we have

$$(3.3) \qquad |\nabla^j u(x)| \le C \int_{B(x_0,R)} \frac{W(y)^{2^{m+1}} |u(y)|\,|\eta(y)|}{|x-y|^{n-2(2^m+l)+j}}\,dy + \frac{C}{R^{n-2(2^m+l)+j+b+2(2^m+l-a-b)+2a+b}} \int_{B(x_0,R)} |u(y)|dy$$

$$\le C \sup_{y \in B(x_0,R)} |u(y)| \cdot \int_{B(x_0,R)} \frac{W(y)^{2^{m+1}}}{|x-y|^{n-2(2^m+l)+j}}\,dy + \frac{C}{R^{n+j}} \int_{B(x_0,R)} |u(y)|dy.$$

It then follows from the well-known theorem on fractional integrals that

$$(3.4) \quad \left(\int_{B(x_0,R/2)} |\nabla^j u(x)|^t dx \right)^{1/t}$$

$$\le C \sup_{y \in B(x_0,R)} |u(y)| \left(\int_{B(x_0,R)} W(x)^{q_0} dx \right)^{2^{m+1}/q_0} + CR^{2^{m+1}n/q_0 - 2(2^m+l)} \sup_{y \in B(x_0,R)} |u(y)|$$

$$\le CR^{2^{m+1}n/q_0 - 2(2^m+l)} \{1 + R^{2(2^m+l)} \rho(x_0, W)^{2^{m+2}}\} \sup_{y \in B(x_0,R)} |u(y)|,$$

where $1/t = 2^{m+1}/q_0 - \{2(2^m+l) - j\}/n$ and we have used Remark 1.4 (2). Then the proof is complete. □

If $n \ge 2(2^m+l)+1$ and $1 \le j \le 2^m + l - 1$, then letting $q_0 = 2^m n/(2^m+l)$ in Lemma 3.1 we have

Corollary 3.2. *Let j, l, m, and n be integers, $m \ge 0$, $1 \le l \le 2^m$, $n \ge 2(2^m+l)+1$, and $1 \le j \le 2^m+l-1$. Suppose that $W \in (RH)_{2^m n/(2^m+l)}$ and that $(-\Delta)^{2^m+l} u + W^{2^{m+1}} u = 0$ in $B(x_0,R)$ for some $x_0 \in \boldsymbol{R}^n$. Then there exists a positive constant C such that*

$$(3.5) \qquad \left(\frac{1}{|B(x_0,R/2)|} \int_{B(x_0,R/2)} |\nabla^j u(x)|^2 dx \right)^{1/2} \le \frac{C\{1 + R^{2(2^m+l)} \rho(x_0,W)^{2^{m+2}}\}}{R^j} \sup_{y \in B(x_0,R)} |u(y)|.$$

In Corollary 3.2, if we assume that there exists a positive constant C such that $\rho(x_0, W) \le C$ then we have

Corollary 3.3. *Let j, l, m, and n be integers, $m \ge 0$, $1 \le l \le 2^m$, $n \ge 2(2^m + l) + 1$, and $1 \le j \le 2^m + l - 1$. Suppose that $W \in (RH)_{2^m n/(2^m+l)}$ and that $(-\Delta)^{2^m+l}u + W^{2^{m+1}}u = 0$ in $B(x_0, R)$ for some $x_0 \in \boldsymbol{R}^n$. Assume also that there exists a positive constant C such that $\rho(x_0, W) \le C$. Then there exists a positive constant C' such that*

$$\left(\frac{1}{|B(x_0, R/2)|}\int_{B(x_0, R/2)} |\nabla^j u(x)|^2 dx\right)^{1/2} \le \frac{C'\{1 + R\rho(x_0, W)\}^{2(2^m+l)}}{R^j} \sup_{y \in B(x_0, R)} |u(y)|. \tag{3.6}$$

Lemma 3.4. *Let j, l, m, and n be integers, $m \ge 0$, $1 \le l \le 2^m$, $n \ge 2(2^m + l) + 1$, and $0 \le j \le 2^m + l - 1$. Suppose that $W \in (RH)_{n/2}$. Then there exists a positive constant C such that*

$$\sum_{j=0}^{2^m+l-1} \int_{\boldsymbol{R}^n} \rho(x, W)^{2(2^{m+1}-j)} |\nabla^j u(x)|^2 dx \le C\left(\int_{\boldsymbol{R}^n} |\Delta^{2^m} u(x)|^2 dx + \int_{\boldsymbol{R}^n} W(x)^{2^{m+1}} |u(x)|^2 dx\right), \tag{3.7}$$

where $u \in C_0^\infty(\boldsymbol{R}^n)$.

Proof. By the case $p = 2$ in Lemma 2.5 and Corollary 2.7, we have

$$\begin{aligned} \sum_{j=0}^{2^m+l-1} \int_{\boldsymbol{R}^n} \rho(x, W)^{2(2^{m+1}-j)} |\nabla^j u(x)|^2 dx &\le C \int_{\boldsymbol{R}^n} |\{(-\Delta)^{2^m} + W^{2^m}\} u(x)|^2 dx \\ &\le C\left(\int_{\boldsymbol{R}^n} |\Delta^{2^m} u(x)|^2 dx + \int_{\boldsymbol{R}^n} W(x)^{2^{m+1}} |u(x)|^2 dx\right). \quad \square \end{aligned} \tag{3.8}$$

Lemma 3.5. *Let l, m, and n be integers, $m \ge 0$, $1 \le l \le 2^m$, and $n \ge 2(2^m + l) + 1$. Suppose that $W \in (RH)_{2^{m+1} n/(2^m+l)}$ and that $(-\Delta)^{2^m+l}u + W^{2^{m+1}}u = 0$ in $B(x_0, R)$ for some $x_0 \in \boldsymbol{R}^n$. Then for any positive integer N there exists a positive constant C_N such that*

$$(3.9)\qquad \sup_{y\in B(x_0,R/2)}|u(y)| \le \frac{C_N}{\{1+R\rho(x_0,W)\}^N} \cdot \sum_{j=0}^{2^m+l-1} R^j\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)}|\nabla^j u(x)|^2dx\right)^{1/2}.$$

Proof. Let $\eta \in C_0^\infty(B(x_0,R/2^{2^m}))$ such that $\eta \equiv 1$ on $B(x_0,R/2^{2^m+1})$, $|\nabla^j\eta| \le C/R^j$, where j is an integer satisfying $1 \le j \le 2^{m+1}-1$. Applying Lemma 3.4 to $u\eta$ we have

$$(3.10)\qquad \sum_{j=0}^{2^m+l-1}\int_{B(x_0,R/2^{2^m+1})}\rho(x,W)^{2(2^{m+1}-j)}|\nabla^j u(x)|^2dx$$
$$\le C\left(\int_{B(x_0,R/2^{2^m})}|\Delta^{2^m}u(x)|^2dx + \int_{B(x_0,R/2^{2^m})}W(x)^{2^{m+1}}|u(x)|^2dx\right)$$
$$= C(I_1+I_2).$$

We estimate I_1. We show that, for all integer l satisfying $1 \le l \le 2^m$,

$$(3.11)\qquad I_1 \le C\sum_{k=0}^{2^m+l-1}\frac{1}{R^{2(2^{m+1}-k)}}\int_{B(x_0,R/2^{l-1})}|\nabla^k u(x)|^2dx.$$

First we show the case $l = 2^m$. From the case $l = 2^m$ and $j = 2^{m+1}$ in Lemma 2.2, we deduce that

$$(3.12)\qquad I_1 \le C\sum_{i=1}^{2^{m+1}}\frac{1}{R^{2i}}\int_{B(x_0,R/2^{2^m-1})}|\nabla^{2^{m+1}-i}u(x)|^2dx$$
$$= C\sum_{k=0}^{2^{m+1}-1}\frac{1}{R^{2(2^{m+1}-k)}}\int_{B(x_0,R/2^{2^m-1})}|\nabla^k u(x)|^2dx.$$

This means that (3.11) is true for $l = 2^m$. Let a be an integer satisfying $0 \le a \le 2^m-2$. We assume that (3.11) is true for $l = 2^m-a$ and show the case $l = 2^m-a-1$. From the inductive assumption we deduce that

$$(3.13)\qquad I_1 \le C\sum_{k=0}^{2^{m+1}-a-1}\frac{1}{R^{2(2^{m+1}-k)}}\int_{B(x_0,R/2^{2^m-a-1})}|\nabla^k u(x)|^2dx.$$

Note that, (3.13) is equivalent to

$$I_1 \le C\left(\frac{1}{R^{2(a+1)}}\int_{B(x_0,\,R/2^{2^m-a-1})}|\nabla^{2^{m+1}-a-1}u(x)|^2dx\right.\tag{3.14}$$

$$+\frac{1}{R^{2(a+2)}}\int_{B(x_0,\,R/2^{2^m-a-1})}|\nabla^{2^{m+1}-a-2}u(x)|^2dx$$

$$\left.+\cdots+\frac{1}{R^{2\cdot 2^{m+1}}}\int_{B(x_0,\,R/2^{2^m-a-1})}|u(x)|^2dx\right)$$

$$= C(J_{a+1}+J_{a+2}+\cdots+J_{2^{m+1}}).$$

From the case $l = 2^m - a - 1$ and $j = 2^{m+1} - a - 1$ in Lemma 2.2, we deduce that

$$J_{a+1} \le \frac{C}{R^{2(a+1)}}\sum_{i=1}^{2^{m+1}-a-1}\frac{1}{R^{2i}}\int_{B(x_0,\,R/2^{2^m-a-2})}|\nabla^{2^{m+1}-a-1-i}u(x)|^2dx\tag{3.15}$$

$$=\frac{C}{R^{2(a+1)}}\sum_{k=0}^{2^{m+1}-a-2}\frac{1}{R^{2(2^{m+1}-a-1-k)}}\int_{B(x_0,\,R/2^{2^m-a-2})}|\nabla^k u(x)|^2dx$$

$$= C\sum_{k=0}^{2^{m+1}-a-2}\frac{1}{R^{2(2^{m+1}-k)}}\int_{B(x_0,\,R/2^{2^m-a-2})}|\nabla^k u(x)|^2dx.$$

Combining (3.14) with (3.15) we have

$$I_1 \le C\sum_{k=0}^{2^{m+1}-a-2}\frac{1}{R^{2(2^{m+1}-k)}}\int_{B(x_0,\,R/2^{2^m-a-2})}|\nabla^k u(x)|^2dx.\tag{3.16}$$

This means (3.11) is true for $l = 2^m - a - 1$. Hence (3.11) is true for all integer l satisfying $1 \le l \le 2^m$. From (3.10), (3.11), and Lemma 2.3 it follows for each integer l satisfying $1 \le l \le 2^m$ that

$$\sum_{j=0}^{2^m+l-1}\int_{B(x_0,\,R/2^{2^{m+1}})}\rho(x,W)^{2(2^{m+1}-j)}|\nabla^j u(x)|^2dx\tag{3.17}$$

$$\le C\sum_{j=0}^{2^m+l-1}\frac{1}{R^{2(2^{m+1}-j)}}\int_{B(x_0,\,R)}|\nabla^j u(x)|^2dx.$$

From Lemma 2.1 (2) it follows for each integer j satisfying $1 \le j \le 2^m + l - 1$ that

$$(3.18)\qquad \int_{B(x_0,R/2^{2^{m+1}})} |\nabla^j u(x)|^2 dx \le \frac{C\{1+R\rho(x_0,W)\}^{2(2^{m+1}-j)k_0/(k_0+1)}}{\{R\rho(x_0,W)\}^{2(2^{m+1}-j)}} \cdot \frac{R^{2^{m+2}}}{R^{2j}} \cdot \sum_{i=0}^{2^m+l-1} \frac{R^{2i}}{R^{2^{m+2}}} \int_{B(x_0,R)} |\nabla^i u(x)|^2 dx.$$

Then we have

$$(3.19)\qquad \sum_{j=0}^{2^m+l-1} R^j \left(\frac{1}{|B(x_0,R/2^{2^{m+1}})|} \int_{B(x_0,R/2^{2^{m+1}})} |\nabla^j u(x)|^2 dx \right)^{1/2} \le \frac{C}{\{1+R\rho(x_0,W)\}^{2(2^{m+1}-j)/(k_0+1)}} \cdot \sum_{i=0}^{2^m+l-1} R^i \left(\frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} |\nabla^i u(x)|^2 dx \right)^{1/2}.$$

Repeating above argument, for any positive integer K we have

$$(3.20)\qquad \sum_{j=0}^{2^m+l-1} R^j \left(\frac{1}{|B(x_0,R/2^{(2^{m+1})K})|} \int_{B(x_0,R/2^{(2^{m+1})K})} |\nabla^j u(x)|^2 dx \right)^{1/2} \le \frac{C_K}{\{1+R\rho(x_0,W)\}^{2(2^{m+1}-j)K/(k_0+1)}} \cdot \sum_{j=0}^{2^m+l-1} R^j \left(\frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} |\nabla^j u(x)|^2 dx \right)^{1/2}.$$

Then using Lemma 2.4 and (3.20), we arrive at the desired inequality. □

Combining Corollary 3.3 with Lemma 3.5, we have

Lemma 3.6. *Let l, m, and n be integers, $m \ge 0$, $1 \le l \le 2^m$, and $n \ge 2(2^m+l)+1$. Suppose that $W \in (RH)_{2^m n/(2^m+l)}$ and that $(-\Delta)^{2^m+l} u + W^{2^{m+1}} u = 0$ in $B(x_0,R)$ for some $x_0 \in \boldsymbol{R}^n$. Assume also that there exists a positive constant C such that $\rho(x_0,W) \le C$. Then for any positive integer N there exists a positive constant C_N such that*

$$(3.21)\qquad \sup_{y \in B(x_0,R/2)} |u(y)| \le \frac{C_N}{\{1+R\rho(x_0,W)\}^N} \sup_{y \in B(x_0,R)} |u(y)|.$$

Now we are ready to give

Proof of Theorem 1.7. Fix $x_0, y_0 \in \boldsymbol{R}^n$ and put $R = |x_0 - y_0|$. Then $u(x) = \Gamma_{H_{2^m+l,\,2^{m+1}}}(x, y_0)$ is a solution of $(-\Delta)^{2^m+l}u + W^{2^{m+1}}u = 0$ on $B(x_0, R/2)$. We note that there exists a positive constant C such that

$$0 \le \Gamma_{H_{2^m+l,\,2^{m+1}}}(x, y) \le \frac{C}{|x-y|^{n-2(2^m+l)}}. \tag{3.22}$$

(See [10].) Using (3.21) and (3.22), we arrive at the desired inequality. □

Next we prove Theorem 1.8. We arrive at Theorem 1.8 combining the following Lemma 3.7 with Lemma 3.6.

Lemma 3.7. *Let j, l, m, and n be integers, $m \ge 0$, $1 \le l \le 2^m$, $n \ge 2(2^m + l) + 1$, and $1 \le j \le 2(2^m + l) - 1$. Suppose that $W \in (RH)_{2^{m+1}n/\{2(2^m+l)-j\}}$ and that $(-\Delta)^{2^m+l}u + W^{2^{m+1}}u = 0$ in $B(x_0, R)$ for some $x_0 \in \boldsymbol{R}^n$. Assume also that there exists a positive constant C such that $\rho(x_0, W) \le C$. Then there exist positive constants C_j and C_j' such that*

$$\sup_{y \in B(x_0, R/2)} |\nabla^j u(y)| \le \frac{C_j\{1 + R\rho(x_0, W)\}^{C_j'}}{R^j} \sup_{y \in B(x_0, R)} |u(y)|. \tag{3.23}$$

Proof. Let a and b be nonnegative integers satisfying $1 \le a + b \le 2^m + l$. We choose $\eta \in C_0^\infty(B(x_0, R))$ such that $\eta \equiv 1$ on $B(x_0, 3R/4)$ and $|\nabla^b(\Delta^a\eta)| \le C/R^{2a+b}$. We use (3.2); by integration by parts, we have

$$|\nabla^j u(x_0)| \le C\int_{B(x_0,R)} \frac{W(y)^{2^{m+1}}|u(y)|}{|x_0 - y|^{n-2(2^m+l)+j}}\,dy + \frac{C}{R^{n+j}}\int_{B(x_0,R)} |u(y)|dy. \tag{3.24}$$

Since $W \in (RH)_{2^{m+1}n/\{2(2^m+l)-j\}}$, it follows that $W \in (RH)_q$ for some $q > 2^{m+1}n/\{2(2^m + l) - j\}$. We choose r such that $2^{m+1}/q + 1/r = 1$ and $r > 1$. By Hölder's inequality we have

$$\begin{aligned}
|\nabla^j u(x_0)| &\le CR^n\left(\frac{1}{R^n}\int_{B(x_0,R)} W(y)^q dy\right)^{2^{m+1}/q} \\
&\quad\cdot\left(\frac{1}{R^n}\int_{B(x_0,R)} \frac{dy}{|x_0-y|^{\{n-2(2^m+l)+j\}r}}\right)^{1/r} \sup_{y\in B(x_0,R)}|u(y)| \\
&\quad+\frac{C}{R^{n+j}}\int_{B(x_0,R)}|u(y)|dy \\
&\le \frac{C\{1 + R^{2(2^m+l)}\rho(x_0, W)^{2^{m+2}}\}}{R^j}\sup_{y\in B(x_0,R)}|u(y)|,
\end{aligned} \tag{3.25}$$

where we have used Remark 1.4 (2). From (3.25) we have for all $y \in B(x_0, R/2)$,

$$|\nabla^j u(y)| \le \frac{C\{1 + R^{2(2^m+l)}\rho(y, W)^{2^{m+2}}\}}{R^j} \sup_{x \in B(y, R/4)} |u(x)|. \tag{3.26}$$

Using Lemma 2.1 (1) and the fact that $\rho(x_0, W) \le C$ we have

$$\sup_{y \in B(x_0, R/2)} |\nabla^j u(y)| \le \frac{C\{1 + R\rho(x_0, W)\}^{2^{m+2}k_0+2(2^m+l)}}{R^j} \sup_{y \in B(x_0, R)} |u(y)|. \tag{3.27}$$

Then the proof is complete. □

At the end of Section 3, we state a remark on the estimate of the fundamental solution. In Corollary 3.2, if we add the assumption $R \ge 1$ then we have

Corollary 3.8. *Let j, l, m, and n be integers, $m \ge 0$, $1 \le l \le 2^m$, $n \ge 2(2^m + l) + 1$, and $1 \le j \le 2^m + l - 1$. Suppose that $W \in (RH)_{2^m n/(2^m+l)}$, $R \ge 1$, and that $(-\Delta)^{2^m+l}u + W^{2^{m+1}}u = 0$ in $B(x_0, R)$ for some $x_0 \in \boldsymbol{R}^n$. Then there exists a positive constant C such that*

$$\left(\frac{1}{|B(x_0, R/2)|}\int_{B(x_0, R/2)} |\nabla^j u(x)|^2 dx\right)^{1/2} \le \frac{C\{1 + R\rho(x_0, W)\}^{2^{m+2}}}{R^j} \sup_{y \in B(x_0, R)} |u(y)|. \tag{3.28}$$

Using Corollary 3.8 we have an estimate of the fundamental solution under the assumption $|x - y| \ge 1$ instead of $\rho(x, W) \le C$.

Theorem 3.9. *Let l, m, and n be integers, $m \ge 0$, $1 \le l \le 2^m$, and $n \ge 2(2^m + l) + 1$. Suppose that $W \in (RH)_{2^m n/(2^m+l)}$. Then for any positive integer N there exists a positive constant C_N such that*

$$(0 \le)\Gamma_{H_{2^m+l,\,2^{m+1}}}(x, y) \le \frac{C_N}{\{1 + \rho(x, W)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-2(2^m+l)}}, \tag{3.29}$$

where $|x - y| \ge 1$.

4. Proof of Theorem 1.5

Theorem 1.5 follows easily from the following lemma.

Lemma 4.1. *Let j, l, m, and n be integers, $m \ge 0$, $1 \le l \le 2^m$, $n \ge 2(2^m + l) + 1$, and $0 \le j \le 2(2^m + l) - 1$. Suppose that $W \in (RH)_{2^{m+1}n/\{2(2^m+l)-j\}}$.*

Then there exists a positive constant C such that

$$(4.1)\quad |\rho(x,W)^{2^{m+2}-2^{m+1}j/(2^m+l)}\nabla^j\{(-\Delta)^{2^m+l}+W^{2^{m+1}}\}^{-1}f(x)| \le CMf(x),$$

where $f \in C_0^\infty(\boldsymbol{R}^n)$ and M is the Hardy-Littlewood maximal operator.

Proof. Let $r = 1/\rho(x,W)$. It follows from Theorem 1.7 for $j=0$ and from Theorem 1.8 for $j \ge 1$ that

$$(4.2)\quad |\rho(x,W)^{2^{m+2}-2^{m+1}j/(2^m+l)}\nabla^j H_{2^m+l,2^{m+1}}^{-1}f(x)|$$

$$\le C_N\rho(x,W)^{2(2^m-l)+j-2^{m+1}j/(2^m+l)}$$

$$\cdot\int_{\boldsymbol{R}^n}\frac{\rho(x,W)^{2(2^m+l)-j}|f(y)|}{\{1+\rho(x,W)|x-y|\}^N|x-y|^{n-2(2^m+l)+j}}\,dy$$

$$= CC_N\sum_{i=-\infty}^{\infty}\int_{2^{i-1}r<|x-y|\le 2^i r}\frac{|f(y)|dy}{r^{2(2^m+l)-j}(1+r^{-1}|x-y|)^N|x-y|^{n-2(2^m+l)+j}}$$

$$\le CC_N\sum_{i=-\infty}^{\infty}\frac{(2^{i-1})^{2(2^m+l)-j}}{(1+2^{i-1})^N}\cdot\frac{1}{(2^{i-1}r)^n}\int_{|x-y|\le 2^i r}|f(y)|dy$$

$$\le CC_N\sum_{i=-\infty}^{\infty}\frac{(2^i)^{2(2^m+l)-j}}{(1+2^i)^N}Mf(x).$$

Then choosing $N \ge 2(2^m+l)-j+1$ we have

$$(4.3)\quad |\rho(x,W)^{2^{m+2}-2^{m+1}j/(2^m+l)}\nabla^j H_{2^m+l,2^{m+1}}^{-1}f(x)| \le CMf(x).$$

Then the proof is complete. □

Proof of Theorem 1.5. Let j, l, m, and n be integers, $m \ge 0$, $1 \le l \le 2^m$, $n \ge 2(2^m+l)+1$, and $0 \le j \le 2(2^m+l)-1$. Let $V = W^{2^{m+1}/(2^m+l)}$. From Remark 1.2 (3) if $V \in (RH)_\infty$ then $W \in (RH)_\infty$. We note that $W \in (RH)_\infty$ implies "$W \in (RH)_{2^{m+1}n/\{2(2^m+l)-j\}}$ and there exists a constant C such that $W(x) \le C\rho(x,W)^2$". Using Lemma 4.1 and the fact that the Hardy-Littlewood maximal operator is bounded on $L^p(\boldsymbol{R}^n)$, $1 < p \le \infty$, we have

$$(4.4)\quad \|W^{2^{m+1}-2^m j/(2^m+l)}\nabla^j\{(-\Delta)^{2^m+l}+W^{2^{m+1}}\}^{-1}f\|_{L^p(\boldsymbol{R}^n)} \le C\|f\|_{L^p(\boldsymbol{R}^n)},$$

where $1 < p \le \infty$. Using $W = V^{(2^m+l)/2^{m+1}}$ and letting $k = 2^m+l$, we arrive at the desired inequality. □

Acknowledgements. The author is grateful to the referee for useful comments. This work was supported by Grant-in-Aid for Scientific Research (C) (No. 25400152), Japan Society for the Promotion of Science.

References

[1] Cao, J., Chang, D.-C., Yang, D. and Yang, S., Boundedness of second order Riesz transforms associated to Schrödinger operators on Musielak-Orlicz-Hardy spaces, Commun. Pure Appl. Anal., **13** (2014), 1435–1463.

[2] Cao, J., Liu, Y. and Yang, D., Hardy spaces $H^1_{\mathscr{L}}(\boldsymbol{R}^n)$ associated to Schrödinger type operators $(-\Delta)^2 + V^2$, Houston J. Math., **36** (2010), 1067–1095.

[3] Dziubański, J. and Zienkiewicz, J., Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality, Rev. Mat. Iberoam., **15** (1999), 279–296.

[4] Fefferman, C. L., The uncertainty principle, Bull. Amer. Math. Soc., **9** (1983), 129–206.

[5] Gehring, F. W., The L^p-integrability of the partial derivatives of a quasi-conformal mapping, Acta Math., **130** (1973), 265–277.

[6] Kurata, K. and Sugano, S., A remark on estimates for uniformly elliptic operators on weighted L^p spaces and Morrey spaces, Math. Nachr., **209** (2000), 137–150.

[7] Kurata, K. and Sugano, S., Estimates of the fundamental solution for magnetic Schrödinger operators and their applications, Tohoku Math. J., **52** (2000), 367–382.

[8] Kurata, K. and Sugano, S., Fundamental solution, eigenvalue asymptotics and eigenfunctions of degenerate elliptic operators with positive potentials, Studia Math., **138** (2000), 101–119.

[9] Liu, Y. and Dong, J., Some estimates of higer order Riesz transform related to Schrödinger type operators, Potential Anal., **32** (2010), 41–55.

[10] Liu, Y. and Huang, J., L^p estimates for the Schrödinger type operators, Appl. Math. J. Chinese Univ., **26** (2011), 412–424.

[11] Shen, Z., L^p estimates for Schrödinger operators with certain potentials, Ann. Inst. Fourier (Grenoble), **45** (1995), 513–546.

[12] Sugano, S., Estimates for the operators $V^{\alpha}(-\Delta + V)^{-\beta}$ and $V^{\alpha}\nabla(-\Delta + V)^{-\beta}$ with certain non-negative potentials V, Tokyo J. Math., **21** (1998), 441–452.

[13] Sugano, S., L^p estimates for some Schrödinger type operators and a Calderón-Zygmund operator of Schrödinger type, Tokyo J. Math., **30** (2007), 179–197.

[14] Sugano, S., Estimates of the fundamental solution for higher order Schrödinger type operators and their applications, J. Funct. Spaces Appl., 2013 (2013), Article ID 435480, 11 pages.

[15] Sugano, S., A Calderón-Zygmund operator of higher order Schrödinger type, Math. Nachr., **289** (2016), 1946–1960.

[16] Zhong, J., Harmonic analysis for some Schrödinger type operators, Ph. D. Thesis, Princeton Univ., 1993.

nuna adreso:
Kobe City College of Technology
8-3 Gakuen-higashimachi, Nishi-ku
Kobe, 651-2194
Japan
E-mail: sugano@kobe-kosen.ac.jp

(Recevita la 24-an de aŭgusto, 2017)
(Revizita la 24-an de novembro, 2017)

L^p Boundedness of Higher Order Schrödinger Type Operators

Satoko Sugano

Funkcialaj Ekvacioj, 62 (2019) 409--424.

Funkcialaj Ekvacioj , December 24, 2019