1 Newton 法

Let f_i be the i-th element of a column vector f. f_i is a function of n variables $(x_1, \ldots, x_n)$. x and α be column vectors. We have

$$f_i(x) = f_i(\alpha) + \sum_j \frac{\partial f_i}{\partial x_j}(\alpha)(x_j - \alpha_j) + \frac{1}{2}\sum_{j,k} \frac{\partial^2 f_i}{\partial x_j \partial x_k}(\alpha)(x_j - \alpha_j)(x_k - \alpha_k) + O((x-\alpha)^3) \tag{1}$$

where $O((x-\alpha)^3) = \sum_{n,|n|=3} O(\prod_i (x_i - \alpha_i)^{n_i})$. Since $|x|^2 < \varepsilon^2$ implies $|x_i|^2 < \varepsilon^2$, $O((x-\alpha)^3) \leq O(|x-\alpha|^3)$.

次が Newton 法の式.

$$\mathrm{Jac}(f)(x(n))(x(n+1) - \alpha) = \mathrm{Jac}(f)(x(n))(x(n) - \alpha) - f(x(n)) \tag{2}$$

ベクトルの場合の Newton 法を証明するために, ベクトルの i 番目の成分を考える. It follows from (1) that

$$f_i(x(n)) = f_i(\alpha) + \sum_j \frac{\partial f_i}{\partial x_j}(\alpha)(x(n)_j - \alpha_j) + \frac{1}{2}\sum_{j,k} \frac{\partial^2 f_i}{\partial x_j \partial x_k}(\alpha)(x(n)_j - \alpha_j)(x(n)_k - \alpha_k) + O((x(n)-\alpha)^3) \tag{3}$$

$$\mathrm{Jac}(f)(x)_{ij} = \frac{\partial f_i}{\partial x_j}(\alpha) + \sum_k \frac{\partial^2 f_i}{\partial x_j \partial x_k}(\alpha)(x_k - \alpha_k) + O(|x-\alpha|^2). \tag{4}$$

$\mathrm{Jac}(f)(x(n))(x(n) - \alpha)$ の i 成分は

$$\sum_j \mathrm{Jac}(f)(x(n))_{ij}(x(n)_j - \alpha_j)$$

なので, 上の展開を代入すると

$$\sum_j \frac{\partial f_i}{\partial x_j}(\alpha)(x(n)_j - \alpha_j) + \sum_{kj} \frac{\partial^2 f_i}{\partial x_j \partial x_k}(\alpha)(x(n)_k - \alpha_k)(x(n)_j - \alpha_j) + O(|x-\alpha|^3). \tag{5}$$

2 次のところに 1/2 がないことに注意.

(2) の右辺は (5) 引く (3). その i 成分は, 2 次の係数が 1 と 1/2 なので

$$O(|x(n) - \alpha|^2).$$

Jacobian の冪級数展開を考えて (todo)

$$\mathrm{Jac}(f)(x(n))^{-1} = \mathrm{Jac}(f)(\alpha)^{-1} + O(|x(n) - \alpha|)$$

よって

$$x(n+1) - \alpha = \left(\mathrm{Jac}(f)(\alpha)^{-1} + O(|x(n) - \alpha|)\right) O(|x(n) - \alpha|^2) \tag{6}$$

つまり

$$|x(n+1) - \alpha| = O(|x(n) - \alpha|^2)$$

2 Example

$$f = (x^2 + y^2 - 4, xy - 1)^T$$

$$\mathrm{Jac}(f) = \begin{pmatrix} 2x & 2y \\ y & x \end{pmatrix}$$

Refer to `p-0511.rr`.