Restriction algorithms for holonomic systems and their applications

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- CGM4T [3] V.Chestnov, F.Gasparotto, M.K.Mandal, P.Mastrolia, S.J.Matsubara-Heo, H.J.Munch, N.Takayama, Macaulay Matrix for Feynman Integrals: Linear Relations and Intersection Numbers, `https://doi.org/10.1007/JHEP09(2022)187`, Journal of high energy physics, 2022, 187(2022).
- CMMT2023 [4] Restrictions of Pfaffian Systems for Feynman Integrals, `https://arxiv.org/abs/2305.01585`.

Numerical solver for ODE's

Numerical evaluation of integrals at a few points

Holonomic systems (ODE's) for them (cohomology; GKZ; $\mathcal{D}$ restriction . . .; Computer algebra)

Definite integrals defined by Feynman diagrams. Normalizing constants in statistics

Physics, statistics: values of integrals, relations among integrals. OpenXM-hgm [6]

What is the restriction ideal (restriction to a linear space)

$\mathcal{D}_m = \mathbb{C}\langle z_1, \ldots, z_m, \partial_1, \ldots, \partial_m \rangle$ and I is a left ideal of $\mathcal{D}_m$. The restriction ideal of I to $z_1 = c_1, \ldots, z_{m'} = c_{m'}$ is

$$\left(I + \sum_{j=1}^{m'} (z_j - c_j)\mathcal{D}_m \right) \cap \mathbb{C}\langle z_{m'+1}, \ldots, z_m, \partial_{m'+1}, \ldots, \partial_m \rangle \quad (1)$$

If a function $F(z)$ is annihilated by I and holomorphic around $z = c$, then

> $F(c_1, \ldots, c_{m'}, z_{m'+1}, \ldots, z_m)$ is annihilated by the restriction ideal.

It is the elimination of $z_1, \ldots, z_{m'}$, $\partial_1, \ldots, \partial_{m'}$. Algorithms for elimination have been studied in the Gröbner basis theory, but ...

Example

$$\partial_x - \partial_y, \tag{2}$$
$$\partial_y - (x\partial_x + y\partial_y + a) \tag{3}$$

Let $f(x, y)$ be a solution of it. Can we find an ODE for $f(x, 0)$?
Answer:

$$\partial_x - (x\partial_x + a)$$

Taking $y \to 0$ (2), (3) does not give the answer.
The left ideal I in $D = \mathrm{C}\langle x, y, \partial_x, \partial_y\rangle$ generated by (2), (3) contains

$$\partial_x - (x\partial_x + y\partial_y + a)$$

Taking $y \to 0$ gives the answer; $I + xD$ contains the answer.
$f(x, y) = (1 - x - y)^{-a}$.

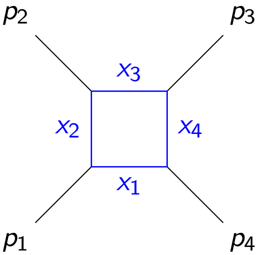
Example 1

massless box model.

$$\mathcal{G} = \mathcal{U}_G + \mathcal{F}_G = x_1 + x_2 + x_3 + x_4 + x_1 x_3 + z x_2 x_4$$

$$I(d_0, \nu; z) = \mathrm{const} \cdot (-s)^{d_0/2 - \varepsilon - |\nu|} \cdot \int_{[0,+\infty]^4} \mathcal{G}^{\varepsilon - d_0/2} \prod_{i=1}^{4} x_i^{\nu_i} \frac{dx_i}{x_i}$$

$z = \frac{p_2 \cdot p_3}{p_1 \cdot p_2}$, $s = 2 p_1 \cdot p_2$.



GKZ system ⇒ restriction ⇒ ODE(Pfaff eq) for FI

$$G = z_1 x_1 + z_2 x_2 + z_3 x_3 + z_4 x_4 + z_5 x_1 x_3 + z_6 x_2 x_4$$

The integral $F(z) = \int_C G^{\beta_5} x^{-\beta} \frac{dx}{x}$ satisfies the GKZ system

$$\sum_{j=1}^{6} a_{ij} z_j \partial_j - \beta_i, \partial^u - \partial^v, \quad (Au = Av)$$

where

$$A = (a_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

If we can compute the restriction of the GKZ system to $z_1 = \cdots = z_5 = 1$ then we can obtain a system of PDE's of the Lee-Pomenransky integral of

$$\mathcal{G} = x_1 + x_2 + x_3 + x_4 + x_1 x_3 + z_6 x_2 x_4$$

Algorithms and implementations

1. Restriction to a linear space Oaku1997 [2]. `nk_restriction.rr`(Nakayama, Nishiyama on Risa/Asir), Singular (V.Levandovsky), Macaulay2 (A.Leykin, H.Tsai). Several algorithms of computing b-function $\mathbf{C}[z_1\partial_1 + \cdots + z_{m'}\partial_{m'}] \cap I$. $(-1, 1)$-Gröbner basis. Hard, e.g., GKZ of $9 - 5$-variables (1L1m box).
2. We can get a Pfaffian system of restriction for larger systems CMMT2023 [4]. (1) Moser reduction, ... (2) Restriction to a point (probabilistic algorithm) and Macaulay matrix. It also gives a Pfaffian system of restriction to hypersurfaces. [CMMT2023] V.Chestnov, S.J.Matsubara-Heo, H.J.Munch, N.T., Restrictions of Pfaffian Systems for Feynman Integrals, `https://arxiv.org/abs/2305.01585`. `mt_mm.rr` .

Current computer algebra systems compute restrictions for (small) inputs skip

$F(1,1,1,1,1,z_6)$ with $\beta = (d,d,d,d,e)$ satisifes the following ODE.

```
import("mt_gkz.rr")$ import("nk_restriction.rr")$
Xm_noX=1$
A = [
        [1,0,0,0,1,0],
        [0,1,0,0,0,1],
        [0,0,1,0,1,0],
        [0,0,0,1,0,1],
        [1,1,1,1,1,1]
];
Beta = [d,d,d,d,e];
F0=sm1.gkz([A,Beta])[0];
F=base_replace(F0,[[x1,x1+1],[x2,x2+1],[x3,x3+1],[x4,x4+1],[x5,x5+1]]);
dp_gr_print(1);  // to be verbose
G=nk_restriction.restriction_ideal(F,[x1,x2,x3,x4,x5,x6],
 [dx1,dx2,dx3,dx4,dx5,dx6],[1,1,1,1,1,0]);

Output F0:
[x5*dx5+x1*dx1-d,x6*dx6+x2*dx2-d,x5*dx5+x3*dx3-d,x6*dx6+x4*dx4-d,
 x6*dx6+x5*dx5+x4*dx4+x3*dx3+x2*dx2+x1*dx1-e,dx1*dx3*dx6-dx2*dx4*dx5]
Output G:
[(x6^3+x6^2)*dx6^3+((-6*d+e+3)*x6^2+(-6*d+2*e+3)*x6)*dx6^2+
 ((9*d^2+(-2*e-6)*d+e+1)*x6+9*d^2+(-6*e-6)*d+e^2+2*e+1)*dx6-4*d^3+e*d^2]
```

From restriction ideal to restriction (module)

$$\frac{\mathcal{D}_m}{I + \sum_{j=1}^{m'}(z_j - c_j)\mathcal{D}_m} = \frac{\mathcal{D}_m}{\sum_{j=1}^{m'}(z_j - c_j)\mathcal{D}_m} \otimes_{\mathcal{D}_m} \mathcal{D}_m/I. \qquad (4)$$

(Suppose $c_j = 0$ for simplicity.)
$Y = \mathbb{C}^m$. $\mathcal{D}_Y = \mathbb{C}\langle z_1, \ldots, z_m, \partial_1, \ldots, \partial_m\rangle$. $J = \langle z_1, \ldots, z_{m'}\rangle$, $m' \leq m$ or $J = \langle L\rangle$, L is an irreducible polynomial.
$Y' = V(J)$. $\mathcal{O}_{Y'} = \mathbb{C}[z]/J$, $\mathcal{O}_Y = \mathbb{C}[z]$. The restriction $\mathcal{N}$ of $\mathcal{M} = \mathcal{D}_Y/\mathcal{I}$ to Y' is

$$\mathcal{N} = \mathcal{O}_{Y'} \otimes_{\mathcal{O}_Y} \mathcal{M}$$

What is a Pfaffian system of the restriction $\mathcal{N} = \mathcal{O}_{Y'} \otimes_{\mathcal{O}_Y} \mathcal{M}$?

$\mathcal{R}_{Y'} = \mathrm{frac}(\mathcal{O}(Y'))\langle \partial_{m'+1}, \ldots, \partial_m \rangle$. Rational restriction

$$\mathcal{R}_{Y'} \otimes_{\mathcal{D}_{Y'}} \mathcal{N}$$

When $\mathcal{M}$ is holonomic, the rational restriction is a finite dimensional vector space over the field $\mathrm{frac}(\mathcal{O}(Y'))$.
$\{s_1, \ldots, s_r\}$ be a basis (standard monomials, `RStd`).

$$\partial_i \begin{pmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_r \end{pmatrix} - P_i \begin{pmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_r \end{pmatrix} \quad \text{in } \mathcal{R}_{Y'} \otimes_{\mathcal{D}_{Y'}} \mathcal{N}.$$

P_i's are $r \times r$ matrix of $\mathrm{frac}(\mathcal{O}(Y')$ entries.

Algorithm for rational restriction (probalistic)

1. Choose a generic point in Y' (probalistic). Compute the restriction to the point to obtain a set of the standard monomials S (`RStd`) for the rational restriction (approximate, probalistic).
2. Find Q_i, P_i, q_i of Theorems 5, 6 by a Macaulay matrix method with increasing the order of ∂ in Q_i.

skip Restriction to a hypersurface: $J = \langle L \rangle$. $I = \langle f_1, \ldots, f_\mu \rangle$.

Theorem 2 (Th6)

S: RStd (standard monomials of the restriction). There exist $r \times \mu$ matrix Q_i of entries in $\mathcal{D}_Y$, $r \times r$ matrix P_i of entries in $\mathbb{C}[z]$, polynomial $q_i \in \mathbb{C}[z]$ such that

$$q_i \partial_i S = P_i S + : Q_i \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_\mu \end{pmatrix} : \mod L \tag{5}$$

holds in $\mathcal{D}_Y$. Here, $\mathcal{D}_Y \ni \sum c_\alpha(z) \partial^\alpha = 0 \mod L$ means that L divides $c_\alpha(z)$.

$q_i \partial_i - P_i$ is the Pfaffian system of the rational restriction $\mathcal{N}$ to $V(L)$ of $\mathcal{M}$ modulo L.

Implementation in mt_mm.rr

$F_2(a,b_1,b_2,c_1,c_2;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(1)_m(1)_n(c_1)_m(c_2)_n}x^m y^n$. We restrict the rank 4 system for Appell function F_2 to $x=0$.
$(a)_k := a(a+1)\cdots(a+k-1)$.

```
import("mt_mm.rr")$
Ideal = [(-x^2+x)*dx^2+(-y*x)*dx*dy+((-a-b1-1)*x+c1)*dx-b1*y*dy-b1*a,
         (-y^2+y)*dy^2+(-x*y)*dy*dx+((-a-b2-1)*y+c2)*dy-b2*x*dx-b2*a]$
Xvars = [x,y]$
//Rule for a probabilistic determination
//     of RStd (Std for the restriction)
Rule=[[y,y+1/3],[a,1/2],[b1,1/3],[b2,1/5],[c1,1/7],[c2,1/11]]$
Ideal_p = base_replace(Ideal,Rule);
RStd=mt_mm.restriction_to_pt_by_linsolv(Ideal_p,Gamma=2,KK=4,[x,y]);
RStd=reverse(map(dp_ptod,RStd[0],[dx,dy]));
Id = map(dp_ptod,Ideal,poly_dvar(Xvars))$
MData = mt_mm.find_macaulay(Id,RStd,Xvars | restriction_var=[x]);
P2 = mt_mm.find_pfaffian(MData,Xvars,2 | use_orig=1);
```

Pfaffian system $\partial_y S - P_2 S$, $S = (1, \partial_y)^T$,

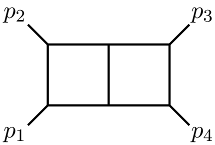
$$P_2 = \begin{pmatrix} 0 & 1 \\ \frac{-b_2 a}{y(y-1)} & \frac{-(a+b_2+1)y+c_2}{y(y-1)} \end{pmatrix}$$

Larger example(2 loop 0 mass doublebox)

A of the GKZ system:

$$\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0\\
1&0&0&0&1&1&1&0&0&0&0&0&0&0&0&1&1&0&0&0&0&1&1&1&1&0\\
0&1&0&0&1&0&0&1&1&1&1&0&0&0&0&0&0&1&1&0&0&1&1&0&0&1\\
0&0&0&0&0&1&0&1&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&1&0&1&0&0&0&0&1&1&1&0&1&0&1&1&1&0&0&1&0\\
0&0&1&0&0&0&0&0&0&1&0&1&0&1&0&0&1&0&1&1&0&0&1&1&1&0\\
0&0&0&1&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&1&0&0&0&0&1
\end{pmatrix} \tag{6}$$

The holonomic rank (dim of solutions) of the GKZ system is 238.
We need to restrict it to $z_i = 1$, $1 \leq i \leq 25$ $z_{26} = y$.



The rank of the rational restriction is 12 (< 238)

$y = 1/7$, $\gamma = 3$, $k = 5$, Specialize parameters to random numbers and the echelon form is computed with mod $p = 100000007$. (`generic_gauss_elim_mod`).

$$[\partial_y \partial_{z_{15}}, \partial_{z_{23}} \partial_y, \partial_{z_{24}} \partial_y, \partial_y^2, \partial_{z_{13}}, \partial_{z_{15}}, \partial_{z_{21}}, \partial_{z_{22}}, \partial_{z_{23}}, \partial_{z_{24}}, \partial_y, 1]$$

Output S=`RStd` consisting of 12 elements. Computing rank 12 ODE (Pfaffian system) by the Macaulay matrix method (Th 5).
Timing on t-PC(AMD EPYC 7552 48-Core Processor `*` 4 `@` 1.5GHz, 1T memory)
23,815s (Guess `RStd`, $132,145 \times 33,649$ matrix)
502s (Macaulay matrix, 2926×10775 matrix)
89,021s (rational reconstruction, FiniteFlow. About 20min by distributed computation)
`http://www.math.kobe-u.ac.jp/OpenXM/Math/amp-Restriction/ref.html`

Example of the restriction to $V(L)$

$$F_4(a,b,c_1,c_2;x,y) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b)_{i+j}}{(1)_i(1)_j(c_1)_i(c_2)_j} x^i y^j. \qquad (7)$$

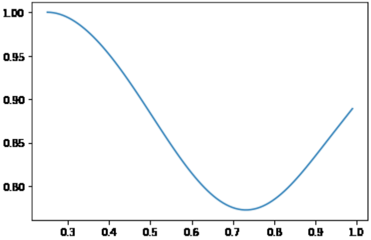
$$f_1 = \theta_x(\theta_x + c_1 - 1) - x(\theta_x + \theta_y + a)(\theta_x + \theta_y + b), \qquad (8)$$

$$f_2 = \theta_y(\theta_y + c_2 - 1) - y(\theta_x + \theta_y + a)(\theta_x + \theta_y + b). \qquad (9)$$

generates the rank 4 holonomic ideal. The singular locus is

$$xy((x-y)^2 - 2(x+y) + 1) = 0 \qquad (10)$$

Let parameters be generic. Compute the restriction to
$L = (x-y)^2 - 2(x+y) + 1 = 0$.
Ans: `RStd` is $(1, \partial_x, \partial_y)$ (rank 3, ODE).
Put $(a, b, c_1, c_2) = (-2/3, 1/3, 1/3, 1/3)$. Give initial value $(1, 0, 0)^T$ at $(x, y) = (1/4, 1/4)$. Numerically solve the ODE as `https://colab.research.google.com/drive/1UQIOo4B2qz_6BNUbbzP1XDkxZPZhOfJQ?usp=sharing` Todo, include Prog:amp22/Data2/2023-03-29-rest-hs.rr in mt_mm.rr


1.00
0.95
0.90
0.85
0.80
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Summary

We give an algorithm to obtain the Pfaffian system (ODE's) of the rational restriction of a holonomic $\mathcal{D}$-module.

1. The algorithm can find rational restrictions, e.g., the GKZ system of $26-8$ variables (2L0m model).
2. The algorithm can also find rational restrictions to hypersurfaces, e.g., Appell system F_4 to $(x-y)^2-2(x+y)+1=0$.

Technical details

What is the restriction? Toward more general setting.

$\mathcal{D}_m = \mathbb{C}\langle z_1, \ldots, z_m, \partial_1, \ldots, \partial_m \rangle$ and I is a left ideal of $\mathcal{D}_m$. The *restriction*[1] of the $\mathcal{D}_m$-module $\mathcal{D}_m/I$ to a linear subspace $z_1 = c_1, \ldots, z_{m'} = c_{m'}$, for some constant $c_i \in \mathbb{C}$, is defined by

$$\frac{\mathcal{D}_m}{\sum_{j=1}^{m'}(z_j - c_j)\mathcal{D}_m} \otimes_{\mathcal{D}_m} \mathcal{D}_m/I = \frac{\mathcal{D}_m}{I + \sum_{j=1}^{m'}(z_j - c_j)\mathcal{D}_m}. \qquad (11)$$

A left $\mathcal{D}_m$-module is called holonomic if the Hilbert polynomial w.r.t. a good filtration is of degree m. When $\mathcal{D}_m/I$ is holonomic, then I is called holonomic ideal. The solution space of holonomic I (the space of functions annihilated by I) is finite dimensional space.

Theorem 3

(J.Bernstein 1972, see, e.g., dojo [1][Th 6.10.8, p.303].) *If $\mathcal{D}_m/I$ is a holonomic $\mathcal{D}_m$-module, then the restriction defined above is holonomic $\mathcal{D}_{m-m'}$-module.*

[1] It is an algebraic counterpart of the restriction of solutions of differential equations to the linear subspace.

What is the restriction? Tensor expression.

$Y = \mathbb{C}^m$. $\mathcal{D}_Y = \mathbb{C}\langle z_1, \ldots, z_m, \partial_1, \ldots, \partial_m \rangle$. $J = \langle z_1, \ldots, z_{m'} \rangle$, $m' \leq m$ or $J = \langle L \rangle$, L is an irreducible polynomial. $Y' = V(J)$. $\mathcal{O}_{Y'} = \mathbb{C}[z]/J$, $\mathcal{O}_Y = \mathbb{C}[z]$. The restriction $\mathcal{N}$ of $\mathcal{M} = \mathcal{D}_Y/\mathcal{I}$ to Y' is

$$\mathcal{N} = \mathcal{O}_{Y'} \otimes_{\mathcal{O}_Y} \mathcal{M}$$

Lemma 4

(well-known) *$\mathcal{M}$ is holonomic. An maximal open strata $W \subset Y'$ of $\mathcal{N}$ contains the origin. $S = \{s_i\}$ is a finite set of monomials of ∂ and is a basis of $\mathbb{C}$-vector space $\mathcal{D}_Y/(\mathcal{I} + \sum_{i=1}^m z_i \mathcal{D}_Y)$. Then, $1 \otimes s_i \in \mathcal{N}|_W$, $s_i \in S$, is a basis of $\mathcal{N}|_W$ as $\mathcal{O}_{Y'}$-module.*

The (holonomic) rank (the dimension of holomorphic solutions at generic points) of $\mathcal{N} \leq$ rank of $\mathcal{M}$.

What is a Pfaffian system of the restriction $\mathcal{N} = \mathcal{O}_{Y'} \otimes_{\mathcal{O}_Y} \mathcal{M}$? $\mathcal{M} = \mathcal{D}_Y / I$

$\mathcal{R}_{Y'} = \mathrm{frac}(\mathcal{O}(Y'))\langle \partial_{m'+1}, \ldots, \partial_m \rangle$. Rational restriction

$$\mathcal{R}_{Y'} \otimes_{\mathcal{D}_{Y'}} \mathcal{N}$$

When $\mathcal{M}$ is holonomic, the rational restriction is a finite dimensional vector space over the field $\mathrm{frac}(\mathcal{O}(Y'))$.
$\{s_1, \ldots, s_r\}$ be a basis.

$$\partial_i \begin{pmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_r \end{pmatrix} - P_i \begin{pmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_r \end{pmatrix} \quad \text{in } \mathcal{R}_{Y'} \otimes_{\mathcal{D}_{Y'}} \mathcal{N}.$$

P_i's are $r \times r$ matrix of $\mathrm{frac}(\mathcal{O}(Y')$ entries.

Note:

$$i : Y' \to Y$$

Since Y and Y' are ringed space, there exists a map

$$i^\sharp : \mathcal{O}_Y \to \mathcal{O}_{Y'}$$

The product of $g \in \mathcal{O}_Y$ and $h \in \mathcal{O}_{Y'}$ is defined by $g(i^\sharp h)$.
When $\mathcal{O}_{Y'} = \mathbb{C}[z]/J$, the natural definition of $i^\sharp$ gives $g(i^\sharp h)$ is gh in $\mathcal{O}_{Y'}$ when $h \in \mathbb{C}[z]$.

What is the stratification associated to $\mathcal{N}$?

What is a Pfaffian system?

Example: Appell function F_2.

$$F_2(a, b_1, b_2, c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(1)_m(1)_n(c_1)_m(c_2)_n} x^m y^n$$

$$\partial_x \begin{pmatrix} F_2 \\ \partial_x F_2 \\ \partial_y F_2 \\ \partial_y^2 F_2 \end{pmatrix} = P_1 \begin{pmatrix} F_2 \\ \partial_x F_2 \\ \partial_y F_2 \\ \partial_y^2 F_2 \end{pmatrix}$$

$$\partial_y \begin{pmatrix} F_2 \\ \partial_x F_2 \\ \partial_y F_2 \\ \partial_y^2 F_2 \end{pmatrix} = P_2 \begin{pmatrix} F_2 \\ \partial_x F_2 \\ \partial_y F_2 \\ \partial_y^2 F_2 \end{pmatrix}$$

P_i's are 4×4 matrix of rational function entries. $\{1, \partial_x, \partial_y, \partial_y^2\}$ is the set of the standard monomials of $\mathcal{R}_Y I$ where I is the annihilating left ideal of F_2 in $\mathcal{D}_Y$ and $\mathcal{R}_Y = \mathbb{C}(x, y)\langle \partial_x, \partial_y \rangle$.

Let r be the holonomic rank of the restriction $\mathcal{N}$.
$\mathcal{R}_{Y'} = \mathbb{C}(z_{m'+1}, \ldots, z_m)\langle \partial_{m'+1}, \ldots, \partial_m \rangle$.
Meaning of

$$: \ell : \big|_{z_1 = \ldots = z_{m'} = 0}$$

. Move ∂_i's of ℓ to right and z_j's to the left (by the relation $[\partial_i, z_j] = \delta_{ij}$). And put $z_1 = \ldots = z_{m'} = 0$ in the normally ordered expression. For example,

$$: \partial_1 z_1 : \big|_{z_1=0} = \ (z_1 \partial_1 + 1)\big|_{z_1=0} = 1.$$

$\mathcal{I} = \langle f_1, \ldots, f_\mu \rangle$. $\mathcal{M} = \mathcal{D}_Y / \mathcal{I}$.

Theorem 5

We regard S in Lemma 4 as a column vector. There exist $r \times \mu$ matrix Q_i of entries in $\mathcal{R}_{Y'}[\partial_1, \ldots, \partial_{m'}]$ and $r \times r$ matrix P_i of entries in $\mathbb{C}(z') := \mathbb{C}(z_{m'+1}, \ldots, z_m)$ such that

$$\partial_i S = P_i S + : Q_i (f_1, \ldots, f_\mu)^T : \big|_{z_1 = \ldots = z_{m'} = 0}$$

holds in $\mathcal{R}_{Y'}[\partial_1, \ldots, \partial_{m'}]$.

Restriction to a hypersurface: $J = \langle L \rangle$

Theorem 6

There exist $r \times \mu$ matrix Q_i of entries in $\mathcal{D}_Y$, $r \times r$ matrix P_i of entries in $\mathbb{C}[z]$, polynomial $q_i \in \mathbb{C}[z]$ such that

$$q_i \partial_i S = P_i S + : Q_i (f_1, \ldots, f_\mu)^T : \quad \mathrm{mod}\, L \tag{12}$$

holds in $\mathcal{D}_Y$. Here, $\mathcal{D}_Y \ni \sum c_\alpha(z) \partial^\alpha = 0 \bmod L$ means that L divides $c_\alpha(z)$.

$q_i \partial_i - P_i$ is the Pfaffian system of the restriction $\mathcal{N}$ of $\mathcal{M}$.

Algorithm for rational restriction (probalistic)

1. Choose a point in $W \subset Y'$ (probalistic). Compute the restriction to the point to obtain a set of the standard monomials S (`RStd`) for the rational restriction (approximate, probalistic).
2. Find Q_i, P_i, q_i of Theorems 5, 6 by the method of undermined coefficients of ∂ with increasing the order of ∂ in Q_i.

Step 1: Find standard monomials S for rational restriction

(approximate, probalistic $c(k) = \sharp\{\partial^\alpha \mid |\alpha| \le k\}$.
$v_k(\ell)$: the vector of coefficients of ℓ as a polynomial in ∂.

Algorithm 1

(Rational restriction to $z_1 = \cdots = z_m = 0$)
Input: generators $\{f_1, \ldots, f_\mu\}$ of the holonomic $\mathcal{I} \subset \mathcal{D}_Y$. The dimension of the space of holomorphic solutions r at the origin $z = (z_1, \ldots, z_m) = 0$. An intger γ such that $\gamma \ge \max(s_0, s_1)$.
Output: $\mathbb{C}$-basis (S of Lemma 4) at the origin.

$w = (1, \ldots, 1)$
$k = \gamma$
repeat
  $J = \mathbb{C} \cdot \left\{ v_k \left(: \partial^\alpha f_i : \big|_{z=0} \right) \,\middle|\, \mathrm{ord}_{(-w,w)} \prod_{j=1}^m \partial_j^{\alpha_j} f_i \le k,\ \alpha \in \mathbb{N}_0^m \right\} \subseteq \mathbb{C}^{c(k)}$
  $k := k + 1$
until $\dim \mathbb{C}^{c(\gamma)} / J \cap \mathbb{C}^{c(\gamma)} = r$
return A vector space basis of $\mathbb{C}^{c(\gamma)} / J \cap \mathbb{C}^{c(\gamma)}$.

$$\mathrm{ord}_{(-w,w)}(z^\alpha \partial^\beta) = -w \cdot \alpha + w \cdot \beta$$

Example

$x = z_1, y = z_2$. $\mathcal{I} = \{2y\partial_x + 3x^2\partial_y,\ 2x\partial_x + 3y\partial_y - 3\}$ has a solution $(y^2 - x^3)^{1/2}$. We will compute the restriction of $\mathcal{D}/\mathcal{I}$ to $(x, y) = (1, 2)$ approximately. Change of coordinate $x := x + 1$, $y := y + 2$. $w = (1, 1)$, $m = 2$, Apply the algorithm with $\gamma = 1$, $c(\gamma) = 2$, $r = 1$.

$$\begin{aligned} f_1 &= 2(y+2)\partial_x + 3(x+1)^2\partial_y \\ f_2 &= 2(x+1)\partial_x + 3(y+2)\partial_y - 3. \end{aligned}$$

The generators of J at $k = 1$ are

$$v_k(f_1|_{x=y=0}) = (4, 3, 0), v_k(f_2|_{x=y=0}) = (2, 6, -3).$$

Each entiries of v_k is indexed by $(\partial_x, \partial_y, 1)$. $\begin{pmatrix} 4 & 3 & 0 \\ 2 & 6 & -3 \end{pmatrix}$ is transformed to $\begin{pmatrix} -2 & 0 & -1 \\ 0 & -3 & 2 \end{pmatrix}$. The rank of $\mathbb{C}^3/J \cap \mathbb{C}^3$ is 1, then $\{1\}$ is S.

What is the Macaulay matrix

$A = (1, 2)$.

$$z_1\partial_1 + 2z_2\partial_2 - \beta_1 =: E$$
$$\underline{\partial_1^2} - \partial_2$$

∂_1	∂_2^2	$\partial_1\partial_2$	∂_2	1	
z_1	0	0	$-2z_1$	β_1	E
0	$2z_2$	z_1	$\beta_1 - 2$	0	$\partial_2 E$
$1 - \beta_1$	0	$2z_2$	$-z_1$	0	$\partial_1 E$

$$\partial_2 E = z_1\partial_1\partial_2 + 2z_2\partial_2^2 + 2\partial_2 - \beta_1\partial_2$$

Implementation in mt_mm.rr

$F_2(a, b_1, b_2, c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(1)_m(1)_n(c_1)_m(c_2)_n} x^m y^n$. We restrict the system for F_2 to $x = 0$. $(a)_k := a(a+1)\cdots(a+k-1)$.

```
import("mt_mm.rr")$
Ideal = [(-x^2+x)*dx^2+(-y*x)*dx*dy+((-a-b1-1)*x+c1)*dx-b1*y*dy-b1*a,
         (-y^2+y)*dy^2+(-x*y)*dy*dx+((-a-b2-1)*y+c2)*dy-b2*x*dx-b2*a]$
Xvars = [x,y]$
//Rule for a probabilistic determination
//     of RStd (Std for the restriction)
Rule=[[y,y+1/3],[a,1/2],[b1,1/3],[b2,1/5],[c1,1/7],[c2,1/11]]$
Ideal_p = base_replace(Ideal,Rule);
RStd=mt_mm.restriction_to_pt_by_linsolv(Ideal_p,Gamma=2,KK=4,[x,y]);
RStd=reverse(map(dp_ptod,RStd[0],[dx,dy]));
Id = map(dp_ptod,Ideal,poly_dvar(Xvars))$
MData = mt_mm.find_macaulay(Id,RStd,Xvars | restriction_var=[x]);
P2 = mt_mm.find_pfaffian(MData,Xvars,2 | use_orig=1);
```

Pfaffian system $\partial_y S - P_2 S$, $S = (1, \partial_y)^T$,

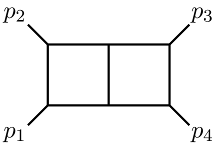
$$P_2 = \begin{pmatrix} 0 & 1 \\ \frac{-b_2 a}{y(y-1)} & \frac{-(a+b_2+1)y+c_2}{y(y-1)} \end{pmatrix}$$

Larger example(2 loop 0 mass doublebox)

A of the GKZ system:

$$
\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0\\
1&0&0&0&1&1&1&0&0&0&0&0&0&0&0&1&1&0&0&0&0&1&1&1&1&0\\
0&1&0&0&1&0&0&1&1&1&1&0&0&0&0&0&0&1&1&0&0&1&1&0&0&1\\
0&0&0&0&0&1&0&1&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&1&0&1&0&0&0&0&1&1&1&0&1&0&1&1&1&0&0&1&0\\
0&0&1&0&0&0&0&0&0&1&0&1&0&1&0&0&1&0&1&1&0&0&1&1&1&0\\
0&0&0&1&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&1&0&0&0&0&1
\end{pmatrix} \tag{13}
$$

The holonomic rank of the GKZ system is 238. We need to restrict it to $z_i = 1$, $1 \leq i \leq 25$ $z_{26} = y$.



The rank of the rational restriction is 12 (< 238)

$y = 1/7$, $\gamma = 3$, $k = 5$, Specialize parameters to random numbers and the echelon form is computed with mod $p = 100000007$. (`generic_gauss_elim_mod`).

$$[\partial_y \partial_{z_{15}}, \partial_{z_{23}} \partial_y, \partial_{z_{24}} \partial_y, \partial_y^2, \partial_{z_{13}}, \partial_{z_{15}}, \partial_{z_{21}}, \partial_{z_{22}}, \partial_{z_{23}}, \partial_{z_{24}}, \partial_y, 1]$$

Output S=`RStd` consisting of 12 elements. Computing rank 12 ODE (Pfaffian system) by the Macaulay matrix method (Th 5).
Timing on t-PC(AMD EPYC 7552 48-Core Processor * 4 @ 1.5GHz, 1T memory)
23,815s (Guess `RStd`, $132,145 \times 33,649$ matrix)
502s (Macaulay matrix, 2926×10775 matrix)
89,021s (rational reconstruction, FiniteFlow. About 20min by distributed computation)
`http://www.math.kobe-u.ac.jp/OpenXM/Math/amp-Restriction/ref.html`

Example of the restriction to $V(L)$

$$F_4(a,b,c_1,c_2;x,y) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b)_{i+j}}{(1)_i(1)_j(c_1)_i(c_2)_j} x^i y^j. \qquad (14)$$

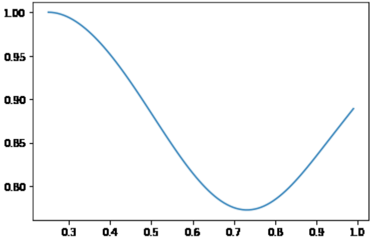
$$f_1 = \theta_x(\theta_x + c_1 - 1) - x(\theta_x + \theta_y + a)(\theta_x + \theta_y + b), \qquad (15)$$

$$f_2 = \theta_y(\theta_y + c_2 - 1) - y(\theta_x + \theta_y + a)(\theta_x + \theta_y + b). \qquad (16)$$

generates the rank 4 holonomic ideal. The singular locus is

$$xy((x-y)^2 - 2(x+y) + 1) = 0 \qquad (17)$$

Let parameters be generic. Compute the restriction to
$L = (x-y)^2 - 2(x+y) + 1 = 0$.
Ans: RStd is $(1, \partial_x, \partial_y)$ (rank 3, ODE).
Put $(a, b, c_1, c_2) = (-2/3, 1/3, 1/3, 1/3)$. Give initial value $(1, 0, 0)^T$ at $(x, y) = (1/4, 1/4)$. Numerically solve the ODE as https://colab.research.google.com/drive/1UQIOo4B2qz_6BNUbbzP1XDkxZPZhOfJQ?usp=sharing Todo, include Prog:amp22/Data2/2023-03-29-rest-hs.rr in mt_mm.rr


1.00
0.95
0.90
0.85
0.80
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

[1] T.Hibi et al, Gröbner bases — statistics and software systems, 2013, Springer

[2] T.Oaku, Algorithms for the *b*-functions, restrictions, and algebraic local cohomology groups of *D*-modules, Advances in Applied Mathematics 19 (1997), 61–105.
See also papers which cite this paper. You can find generalizations and improvements.

[3] https://doi.org/10.1007/JHEP09(2022)187.

[4] https://arxiv.org/abs/2305.01585.

[5] FiniteFlow, multivariate functional reconstruction using finite fields and dataflow graphs.
https://github.com/peraro/finiteflow

[6] References for HGM, http://www.math.kobe-u.ac.jp/OpenXM/Math/hgm/ref-hgm.html