

問 5.1 次の重積分を計算せよ (累次積分).

$$(1) \int_D \sin(x+y) dx dy \quad (D: x, y \geq 0, x+y \leq \pi/2) \quad (2) \int \int_D x dx dy \quad (D: \sqrt{x} + \sqrt{y} \leq 1)$$

$$(3) \int_D x^2 y^2 dx dy \quad (D: |x| + |y| \leq 1) \quad (4) \int_D \sqrt{xy-x^2} dx dy \quad (D: 0 \leq x \leq 1, x \leq y \leq 2x)$$

解説 積分値を  $I$  とおく. いずれも累次積分で求める.

(1)

$$I = \int_0^{\pi/2} dx \int_0^{\pi/2-x} \sin(x+y) dy = \int_0^{\pi/2} dx [-\cos(x+y)]_{y=0}^{y=\pi/2-x} = \int_0^{\pi/2} \cos x dx = 1.$$

(2)

$$\int \int_D x dx dy = \int_0^1 dx \int_0^{(1-\sqrt{x})^2} x dy = \int_0^1 x(1-\sqrt{x})^2 dx = \int_0^1 (x^2 - 2x^{3/2} + x) dx = 1/30.$$

(3)

$$I = 4 \int_0^1 dx \int_0^{1-x} x^2 y^2 dy = 4 \int_0^1 dx [x^2 y^3 / 3]_0^{1-x} = 4 \int_0^1 x^2 (1-x)^3 / 3 dx = 1/45.$$

(4)

$$\int \int_D \sqrt{xy-x^2} dx dy = \int_0^1 dx \int_x^{2x} \sqrt{x} \sqrt{y-x} dy = \int_0^1 dx [\sqrt{x}(2/3)(y-x)^{3/2}]_{y=x}^{y=2x} = \int_0^1 (2/3)x^2 dx = 2/9.$$

問 5.2 次の重積分を計算せよ (変数変換, 広義積分).

$$(1) \int_{x^2+y^2 \leq 1} e^{x^2+y^2} dx dy \quad (2) \int_{x^2/a^2+y^2/b^2 \leq 1} (x^2+y^2) dx dy$$

$$(3) \int_{x^2+y^2+z^2 \leq 1} \frac{1}{\sqrt{1-x^2-y^2-z^2}} dx dy dz \quad (4) \int_{x,y \geq 0} \frac{1}{(ax+by+c)^\alpha} dx dy \quad (\alpha > 2, a, b, c > 0)$$

解説 積分値を  $I$  とおく.

(1)  $x = r \cos \theta, y = r \sin \theta$  とおくと,  $dx dy = r dr d\theta$  であるので

$$I = \int_0^{2\pi} d\theta \int_0^1 e^{r^2} r dr = \int_0^{2\pi} [e^{r^2}/2]_{r=0}^{r=1} = \pi(e-1).$$

(2)  $x = ar \cos \theta, y = br \sin \theta$  とおくと,  $dx dy = abr dr d\theta$  であるので

$$I = ab \int_0^{2\pi} d\theta \int_0^1 dr (a^2 \cos^2 \theta + b^2 \sin^2 \theta) r^3 = (ab/4) \int_0^{2\pi} (a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta = \pi ab(a^2 + b^2)/4.$$

(3)  $x = r \sin \theta \cos \varphi, y = r \sin \theta \sin \varphi, z = r \cos \theta$  とおくと,  $dx dy dz = r^2 \sin \theta dr d\theta d\varphi$  であるので

$$I = \int_0^1 dr \int_0^{2\pi} d\varphi \int_0^\pi d\theta \frac{r^2 \sin \theta}{\sqrt{1-r^2}} = 4\pi \int_0^1 dr \frac{r^2}{\sqrt{1-r^2}}$$

$$= 4\pi \left( [-r\sqrt{1-r^2}]_0^1 + \int_0^1 \sqrt{1-r^2} dr \right) = 4\pi [2^{-1}(r\sqrt{1-r^2} + \sin^{-1} r)]_0^1 = \pi^2.$$

ここで、第 3 の等号では部分積分を、第 4 の等号では

$$\int \sqrt{a^2-x^2} dx = \frac{1}{2} \left( x\sqrt{a^2-x^2} + a^2 \sin^{-1}(x/a) \right)$$

を用いた。

(4)  $K_n : 0 \leq x \leq n, 0 \leq y \leq n$  とすると  $\{K_n\}_{n=1}^{\infty}$  は  $D : x \geq 0, y \geq 0$  の近似列である .

$I_n = \int_{K_n} (ax + by + c)^{-\alpha} dx dy$  とすると ,

$$\begin{aligned} I_n &= \int_0^n dy \int_0^n (ax + by + c)^{-\alpha} dx = \int_0^n dy \frac{1}{(1-\alpha)a} [(ax + by + c)^{1-\alpha}]_{x=0}^{x=n} \\ &= \frac{1}{(1-\alpha)a} \int_0^n ((bx + c + an)^{1-\alpha} - (by + c)^{1-\alpha}) dy \\ &= \frac{1}{(1-\alpha)(2-\alpha)ab} [(by + c + an)^{2-\alpha}]_{y=0}^{y=n} - \frac{1}{(1-\alpha)(2-\alpha)ab} [(by + c)^{2-\alpha}]_{y=0}^{y=n} \\ &= \frac{1}{(1-\alpha)(2-\alpha)ab} ((an + bn + c)^{2-\alpha} - (an + c)^{2-\alpha} - (bn + c)^{2-\alpha} + c^{2-\alpha}). \end{aligned}$$

ここで  $\beta < 0$  のとき  $\lim_{z \rightarrow \infty} z^\beta = 0$  であることを用いると

$$\int_D \frac{1}{(ax + by + c)^\alpha} dx dy = \lim_{n \rightarrow \infty} I_n = \frac{c^{2-\alpha}}{(1-\alpha)(2-\alpha)ab}.$$

馴れてくると次のように計算することがある .

$$\begin{aligned} I &= \int_0^\infty dy \int_0^\infty (ax + by + c)^{-\alpha} dx = \int_0^\infty \left[ \frac{1}{(1-\alpha)a} (ax + by + c)^{1-\alpha} \right]_{x=0}^{x=\infty} dy \\ &= \int_0^\infty \frac{1}{(\alpha-1)a} (by + c)^{1-\alpha} dy = \frac{c^{2-\alpha}}{(\alpha-1)(\alpha-2)ab}. \end{aligned}$$