

IMPROVED ERROR ESTIMATES FOR THE DAVENPORT–HEILBRONN THEOREMS

MANJUL BHARGAVA, TAKASHI TANIGUCHI, AND FRANK THORNE

ABSTRACT. We improve the error terms in the Davenport–Heilbronn theorems on counting cubic fields to $O(X^{2/3+\epsilon})$. This improves on separate and independent results of the authors and Shankar and Tsimerman [BST13, TT13b]. The present paper uses the analytic theory of Shintani zeta functions, and streamlines and simplifies the proof relative to [TT13b]. We also give a second proof that uses a “discriminant-reducing identity” from [BST13] and translates it into the language of zeta functions. We also provide a version of our theorem that counts cubic fields satisfying an arbitrary finite set of local conditions, or even suitable infinite sets of local conditions, where the dependence of the error term on these conditions is described explicitly and significantly improves [BST13, TT13b]. As we explain, these results lead to quantitative improvements in various arithmetic applications.

1. INTRODUCTION

The purpose of this paper is to improve the error terms in the Davenport–Heilbronn theorems to $O(X^{2/3+\epsilon})$:

Theorem 1.1. *Let $N_3^\pm(X)$ denote the number of isomorphism classes of cubic fields F satisfying $0 < \pm\text{Disc}(F) < X$. Then*

$$N_3^\pm(X) = C^\pm \frac{1}{12\zeta(3)} X + K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} X^{5/6} + O(X^{2/3}(\log X)^{2.09}), \quad (1)$$

where $C^+ = 1$, $C^- = 3$, $K^+ = 1$, and $K^- = \sqrt{3}$.

Theorem 1.2. *We have, for any $\epsilon > 0$, that*

$$\sum_{0 < \pm D < X} \#\text{Cl}(\mathbb{Q}(\sqrt{D}))[3] = \frac{3 + C^\pm}{\pi^2} X + K^\pm \frac{8\zeta(1/3)}{5\Gamma(2/3)^3} \prod_p \left(1 - \frac{p^{1/3} + 1}{p(p+1)}\right) X^{5/6} + O(X^{2/3+\epsilon}), \quad (2)$$

where the sum ranges over fundamental discriminants D , the expression $\text{Cl}(\mathbb{Q}(\sqrt{D}))[3]$ denotes the 3-torsion subgroup of the class group of $\mathbb{Q}(\sqrt{D})$, the product is over all primes p , and the constants C^\pm and K^\pm are as in Theorem 1.1.

Theorem 1.2 may be viewed as a counting theorem for cubic fields whose discriminant is fundamental. Indeed, as subgroups of $\text{Cl}(\mathbb{Q}(\sqrt{D}))$ of index 3 are in bijection with cubic fields of discriminant D (see, e.g. [BST13, Section 8.1]), Theorem 1.2 is equivalent to

$$N_{3,\text{fund}}^\pm(X) = \frac{C^\pm}{2\pi^2} X + K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)^3} \prod_p \left(1 - \frac{p^{1/3} + 1}{p(p+1)}\right) X^{5/6} + O(X^{2/3+\epsilon}), \quad (3)$$

where $N_{3,\text{fund}}^\pm(X)$ denotes the number of isomorphism classes of cubic fields F such that $\text{Disc}(F)$ is fundamental and $0 < \pm\text{Disc}(F) < X$.

In each of Theorems 1.1 and 1.2, the first main term is due to Davenport and Heilbronn [DH71], while the second main term was conjectured by Datskovsky and Wright

[DW88, p. 125] and Roberts [Rob01] and proven in [BST13] and [TT13b]. The latter works in turn built on the successively improved error terms obtained in [DH71], [Bel97], and [BBP10]. All of these works, including this one, approached these problems by relating them to counting certain $\mathrm{GL}_2(\mathbb{Z})$ -orbits of integer-coefficient binary cubic forms (though see also [Hou19, Zha13, ST14] for alternative but related approaches). We refer to [BST13] for a complete, self-contained account of this connection, and to Section 2 for a briefer summary of the results we will need.

We also obtain a variation of Theorem 1.1 that counts cubic fields satisfying certain specified sets of *local conditions*, significantly improving the error terms in the corresponding results of [BST13, TT13b] and their dependences on these local conditions. As we will discuss, our earlier results saw applications to the arithmetic of cubic fields and their L -functions, and our improvements yield corresponding improvements to a number of these applications.

For a prime p , let Σ_p denote a set of (isomorphism classes of) étale cubic algebras over \mathbb{Q}_p . We call Σ_p a *local specification* at p , and say that a cubic field F satisfies Σ_p if $F \otimes_{\mathbb{Q}} \mathbb{Q}_p \in \Sigma_p$. We say that Σ_p is *ordinary* if it is equal to either A_p , the set of all étale cubic algebras over \mathbb{Q}_p , or A'_p , the set of all étale cubic algebras over \mathbb{Q}_p that are not totally ramified.

Note that a cubic *splitting type* at a prime p is also a local specification at p for a cubic field. There are five ways in which a prime can split in a cubic field. For each of the three unramified splitting types at p , namely, (111), (12), and (3), there is a unique étale cubic algebra over \mathbb{Q}_p having that splitting type. For each of the two ramified splitting types, $(1^2 1)$ and (1^3) , the number of étale cubic algebras over \mathbb{Q}_p having that splitting type depends on p .

Let $\Sigma = (\Sigma_p)_p$ be a collection of local specifications such that Σ is *ordinary at p* , i.e., Σ_p is *ordinary*, for all but finitely many primes p . We wish to asymptotically count the total number of cubic fields F of absolute discriminant less than X that agree with this collection of local specifications, i.e., $F \otimes \mathbb{Q}_p \in \Sigma_p$ for all p . We have the following theorem:

Theorem 1.3. *Let $\Sigma = (\Sigma_p)_p$ denote a collection of local cubic specifications ordinary at all but finitely many primes p , and let $N_3^{\pm}(X, \Sigma)$ denote the number of isomorphism classes of cubic fields F satisfying $0 < \pm \mathrm{Disc}(K) < X$ and $F \otimes \mathbb{Q}_p \in \Sigma_p$ for all p . Then*

$$N_3^{\pm}(X, \Sigma) = C^{\pm}(\Sigma) \frac{1}{12\zeta(3)} X + K^{\pm}(\Sigma) \frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} X^{5/6} + E(\Sigma) O_{\epsilon}(X^{2/3+\epsilon}), \quad (4)$$

where the constants $C^{\pm}(\Sigma)$, $K^{\pm}(\Sigma)$, and $E(\Sigma)$ are defined by

$$C^{\pm}(\Sigma) := C^{\pm} \prod_p C_p(\Sigma_p), \quad K^{\pm}(\Sigma) := K^{\pm} \prod_p K_p(\Sigma_p), \quad E(\Sigma) := \prod_p E_p(\Sigma_p), \quad (5)$$

$$C_p(\Sigma_p) := \frac{\sum_{F \in \Sigma_p} \frac{1}{\mathrm{Disc}_p(F)} \frac{1}{|\mathrm{Aut}(F)|}}{\sum_{F \in A_p} \frac{1}{\mathrm{Disc}_p(F)} \frac{1}{|\mathrm{Aut}(F)|}}, \quad K_p(\Sigma_p) := \frac{\sum_{F \in \Sigma_p} \frac{1}{\mathrm{Disc}_p(F)} \frac{1}{|\mathrm{Aut}(F)|} \int_{\mathcal{O}_F \setminus p\mathcal{O}_F} [\mathcal{O}_F : \mathbb{Z}_p[x]]^{2/3} dx}{\sum_{F \in A_p} \frac{1}{\mathrm{Disc}_p(F)} \frac{1}{|\mathrm{Aut}(F)|} \int_{\mathcal{O}_F \setminus p\mathcal{O}_F} [\mathcal{O}_F : \mathbb{Z}_p[x]]^{2/3} dx}, \quad (6)$$

and

$$E_p(\Sigma_p) := \begin{cases} 1 & \text{if } \Sigma_p = A_p, A'_p, \text{ or } A_p - A'_p; \\ p^{2/3} & \text{if } \Sigma_p \text{ is a union of splitting types and } \Sigma_p \neq A_p, A'_p, A_p - A'_p; \\ p^{8/3} & \text{otherwise.} \end{cases} \quad (7)$$

The expressions for $C^\pm(\Sigma)$ and $K^\pm(\Sigma)$, in terms of products of local densities of étale cubic algebras in Σ_p , were formulated in [DW88, p. 118] and [BST13, Theorem 7], respectively, and can be evaluated explicitly for any given Σ_p . If Σ_p corresponds to one of the five cubic splitting types \mathcal{S}_p , then the quantities $C_p(\mathcal{S}_p)$, $K_p(\mathcal{S}_p)$, and $E_p(\mathcal{S}_p)$ are given by the following table:

Splitting type \mathcal{S}_p	Notation	$C_p(\mathcal{S}_p)$	$K_p(\mathcal{S}_p)$	$E_p(\mathcal{S}_p)$
Totally split	(111)	$\frac{p^2/6}{p^2+p+1}$	$\frac{(1+p^{-1/3})^3}{6} \cdot \frac{1-p^{-1/3}}{(1-p^{-5/3})(1+p^{-1})}$	$p^{2/3}$
Partially split	(21)	$\frac{p^2/2}{p^2+p+1}$	$\frac{(1+p^{-1/3})(1+p^{-2/3})}{2} \cdot \frac{1-p^{-1/3}}{(1-p^{-5/3})(1+p^{-1})}$	$p^{2/3}$
Inert	(3)	$\frac{p^2/3}{p^2+p+1}$	$\frac{1+p^{-1}}{3} \cdot \frac{1-p^{-1/3}}{(1-p^{-5/3})(1+p^{-1})}$	$p^{2/3}$
Partially ramified	(1 ² 1)	$\frac{p}{p^2+p+1}$	$\frac{(1+p^{-1/3})^2}{p} \cdot \frac{1-p^{-1/3}}{(1-p^{-5/3})(1+p^{-1})}$	$p^{2/3*}$
Totally ramified	(1 ³)	$\frac{1}{p^2+p+1}$	$\frac{1+p^{-1/3}}{p^2} \cdot \frac{1-p^{-1/3}}{(1-p^{-5/3})(1+p^{-1})}$	1^*

TABLE 1. Splitting types, main term constants, and error term constants.

If a local specification Σ_p is a union of such splitting types, then the constants $C_p(\Sigma_p)$ and $K_p(\Sigma_p)$ are the sum of those for the respective splitting types, while the error constant $E_p(\Sigma_p)$ is the maximum (except in the ordinary cases $\Sigma_p = A_p$ or A'_p , where we may take $E_p(\Sigma_p) = 1$). By construction, the constants $C_p(\mathcal{S}_p)$ and $K_p(\mathcal{S}_p)$ for the five splitting conditions \mathcal{S}_p each sum to 1.

The asterisks indicate error terms that can be improved by averaging, as we will describe shortly.

Local completions not determined by splitting types. To complete the description of Theorem 1.3, we describe the constants $C_p(\Sigma_p)$, $K_p(\Sigma_p)$, and $E_p(\Sigma_p)$ in the cases that Σ_p consists of a single étale cubic extension F of \mathbb{Q}_p that is not completely specified by its splitting type \mathcal{S}_p .

- If $p \neq 2$, then the “partially ramified” condition \mathcal{S}_p corresponds to two choices for $F \otimes \mathbb{Q}_p$. If Σ_p consists of only one of these, then $C_p(\Sigma_p) = \frac{1}{2}C_p(\mathcal{S}_p)$ and $K_p(\Sigma_p) = \frac{1}{2}K_p(\mathcal{S}_p)$.
- If $p \equiv 1 \pmod{3}$, then the “totally ramified” condition \mathcal{S}_p corresponds to three choices for $F \otimes \mathbb{Q}_p$. If Σ_p consists of only one of these, then $C_p(\Sigma_p) = \frac{1}{3}C_p(\mathcal{S}_p)$ and $K_p(\Sigma_p) = \frac{1}{3}K_p(\mathcal{S}_p)$.
- If $p = 2$, then the “partially ramified” condition \mathcal{S}_2 corresponds to six choices for $F \otimes_{\mathbb{Q}} \mathbb{Q}_p$, each of which is of the form $\mathbb{Q}_2 \times E$ where E is a ramified quadratic extension of \mathbb{Q}_2 . Generating polynomials for these six possibilities are given in Section 8, together with constants c_2 giving the proportion of quadratic number fields F_2 such that $F_2 \otimes_{\mathbb{Q}} \mathbb{Q}_2 \simeq E$. If Σ_2 consists of only one of these choices, then $C_2(\Sigma_2) = 3c_2C_2(\mathcal{S}_2)$ and $K_p(\Sigma_p) = 3c_2K_2(\mathcal{S}_2)$.
- Finally, if $p = 3$, then the “totally ramified” condition \mathcal{S}_3 corresponds to nine choices for $F \otimes_{\mathbb{Q}} \mathbb{Q}_p$; we refer to [TT13b, Section 6.2] for a list of generating polynomials for these nine extensions and the constants that $C_3(\mathcal{S}_3)$ and $K_3(\mathcal{S}_3)$ should be multiplied by in the case that Σ_3 consists only of one of these nine extensions.

We may always take $E_p(\Sigma_p) = 8/3$, although with additional work this could be improved.

Theorem 1.3 can also be extended to count *nonmaximal* cubic rings satisfying suitable local specifications (see Remark 6.3).

Averaged error terms. As indicated by the asterisks in the table, we may obtain stronger bounds on $E(\Sigma)$ on average when such local conditions are imposed over ranges of primes. Let U be any positive integer, and for each $p \mid U$ let Σ_p be an arbitrary local specification. For each pair r, t of positive squarefree integers such that $(rt, U) = 1$, we complete this to a collection $\Sigma^{r,t} = (\Sigma_p)_p$ of local specifications over all p satisfying the following conditions:

- If $p \mid r$, then Σ_p consists of all partially ramified cubic extensions of \mathbb{Q}_p .
- If $p \mid t$, then Σ_p consists of all totally ramified cubic extensions of \mathbb{Q}_p .
- If $p \nmid Urt$, then Σ_p is ordinary, i.e., $\Sigma_p = A_p$ or A'_p .

Theorem 1.4. *For each collection of cubic local specifications $\Sigma = (\Sigma_p)_p$, let $E(X, \Sigma)$ denote the error term in estimating $N_3^\pm(X, \Sigma)$ using (4). Then, for any $\epsilon > 0$, we have*

$$\sum_{r \leq R} \sum_{t \leq T} |E(X_{r,t}, \Sigma^{r,t})| \ll X^{2/3+\epsilon} R^{2/3} \prod_{p \mid U} E_p(\Sigma_p) \quad (8)$$

for each $R, T, X > 0$ and any $X_{r,t}$ with $X_{r,t} \leq X$; here the sum is over squarefree integers r and t with $(U, rt) = 1$.

In other words, we may sum over ranges of r and t “for free”. Indeed, Theorem 1.3 follows immediately from Theorem 1.4 by applying the bound on the right-hand side of (8) to each individual summand on the left.

Levels of distribution. Our results may be interpreted as “level of distribution” estimates for cubic fields with respect to local conditions. We expect the following two corollaries to perhaps be the most useful in applications; variations can be deduced in the same way.

Corollary 1.5. *For each collection of cubic local specifications $\Sigma = (\Sigma_p)_p$, let $E(X, \Sigma)$ denote again the error in estimating $N_3^\pm(X, \Sigma)$ using (4). Then for each $\epsilon, A > 0$, we have*

$$\sum_{q < X^{1/5-\epsilon}} \mu^2(q) \sum_{\Sigma \pmod{q}} |E(X, \Sigma)| \ll_{\epsilon, A} \frac{X}{(\log X)^A}, \quad (9)$$

where the inner sum is over all collections $\Sigma = (\Sigma_p)_p$ such that if $p \mid q$, then Σ_p corresponds to one of the 32 subsets of the five splitting types, and if $p \nmid q$, then $\Sigma_p = A_p$.

This is immediate from Theorem 1.3: there are $\ll X^\epsilon$ choices of Σ for any given q , and for each such choice we have $|E(X, \Sigma)| \ll X^{2/3+\epsilon} q^{2/3}$; this error may be summed over all q up to $X^{1/5-\epsilon}$ and remain within the upper bound of (9). Moreover, if we instead sum over all collections Σ where $\Sigma_p = A_p$ or $\Sigma_p = A'_p$ for each $p \nmid q$, independently, then Corollary 1.5 still holds.

The following corollary follows similarly from Theorem 1.4.

Corollary 1.6. *Let $N_3^\pm(X, q)$ denote the number of cubic fields F such that $q \mid \text{Disc}(F)$ and $0 < \pm \text{Disc}(F) < X$. Let $E(X, q)$ denote the error term in estimating $N_3^\pm(X, q)$ using (4). Then, for each $\epsilon, A > 0$, we have*

$$\sum_{q < X^{1/2-\epsilon}} \mu^2(q) |E(X, q)| \ll_{\epsilon, A} \frac{X}{(\log X)^A}. \quad (10)$$

The same holds if one counts only cubic fields having squarefree discriminant.

Applications. Our previous results in [BST13, TT13b] were applied by various authors to obtain further results concerning the arithmetic of cubic fields and their L -functions. In some cases, our improvements lead to quantitative results there. Here are several examples:

1. Let $N_6(X; S_3)$ be the number of S_3 -sextic fields L with $|\text{Disc}(L)| < X$. Asymptotics for $N_6(X; S_3)$ were obtained independently by Belabas and Fouvry [BF10] and the first author and Wood [BW08]. The second and third authors obtained a power saving error term in [TT14], proving that

$$N_6(X; S_3) = cX^{1/3} + O(X^{1/3 - \frac{5}{447} + \epsilon})$$

for an explicit constant c .

We can now improve this error term to $O(X^{2/7 + \epsilon})$, only slightly larger than a conjectured secondary term of order $X^{5/18}$. In [TT14, Theorem 2.2], we now have $(\alpha, \beta) = (-1, 2/3 + \epsilon)$ on average, which means that the choice $Q = X^{1/7}$ is admissible in and just below [TT14, (2.7)].

2. Let A be an abelian group with minimal prime divisor of $|A|$ greater than 5. In [Wan], Wang obtains an asymptotic formula counting degree $3|A|$ extensions with Galois group $S_3 \times A$, of the form

$$N(S_3 \times A) = C_1 X^{1/|A|} + C_2 X^{5/6|A|} + O(X^{5/6|A| - \delta}),$$

for explicit constants C_1 , C_2 , and δ .

Her formula for the discriminant of an $S_3 \times A$ -extension is determined by that of the S_3 - and A -subextensions, and the most subtle part is an arithmetic factor divisible by those primes ramified in the S_3 - and A -subextensions. This makes it necessary to count the number of S_3 -cubic fields with prescribed ramification behavior.

As of this writing, Wang's work relies on a version of Theorem 1.3 in an earlier draft of this paper, in which we had obtained results stronger than those of [TT13b], but weaker than Theorem 1.4. She informs us that our improvements may allow her to relax the condition on the minimal prime divisor of $|A|$.

3. In [MT], McGown and Tucker studied the statistics of *genus numbers* of cubic fields. Following [MT], the genus field of a number field F is defined to be the maximal extension F'/F that is (a) unramified at all finite primes and (b) a compositum of the form Fk^* with k^* absolutely abelian. The *genus number* g_F is then defined to be $[F' : F]$.

Let $N_{\text{genus}}^{\pm}(X)$ denote the count of cubic fields F with $0 < \pm \text{Disc}(F) < X$ such that $g_F = 1$. (Equivalently, this counts cubic fields for which there is no nontrivial extension satisfying (a) and (b) above.) Then McGown and Tucker prove that

$$N_{\text{genus}}^{\pm}(X) = \frac{29C^{\pm}}{324\zeta(2)} \prod_{p \equiv 2 \pmod{3}} \left(1 + \frac{1}{p(p+1)} \right) X + O(X^{16/17 + \epsilon}).$$

By using Theorem 1.4 instead of the results of [TT13b] in their proof, the error term above can be improved to $O(X^{2/3 + \epsilon})$, with (as expected) a secondary term of order $X^{5/6}$.

4. Cho and Kim [CK15], Shankar, Södergren, and Templier [SST19], and Yang [Yan09] all obtained one-level density results for the Artin L -functions associated to cubic fields. Such results take the following form. For each cubic number field F , let $L(s, \rho_F) = \zeta_F(s)/\zeta(s)$ be

its associated Artin L -function. In [SST19, Section 2], it is proved that

$$\lim_{X \rightarrow \infty} \frac{1}{N_3(X)} \sum_{|\text{Disc}(F)| < X} \sum_{L(\frac{1}{2} + i\gamma_F, \rho_F) = 0} f\left(\frac{\gamma_F \mathcal{L}}{2\pi}\right) = \widehat{f(0)} - \frac{f(0)}{2}, \quad (11)$$

where: the inner sum is over nontrivial zeroes $\frac{1}{2} + i\gamma_F$ of $L(s, \rho_F)$, not necessarily assumed to lie on the $\frac{1}{2}$ -line; $\mathcal{L} \approx \log(X)$ is a normalizing factor; and f is a Paley-Wiener function whose Fourier transform is smooth and supported in $(-\frac{4}{41}, \frac{4}{41})$.

Their proof relies on the “explicit formula” relating the left side of (11) to the prime power coefficients of the L -functions $L(s, \rho_F)$. These coefficients are determined by the splitting behavior of primes in these cubic fields F ; therefore, our results imply asymptotic density results (with power-saving error terms) for the values of these coefficients on average over F .

These authors relied on the results of [BBP10, TT13b] which are improved here. Applying Theorem 1.4 in their proof, we immediately improve their range of support on the Fourier transform to $(-\frac{2}{7}, \frac{2}{7})$. (See also [CK15] and [Yan09] for very similar results, with [CK15] also allowing for twisting by an automorphic form.)

5. Using the secondary term in Theorem 1.3 and in [BST13, TT13b], Cho, Fiorilli, Lee, and Södergren [CFLS21] refined the latter one-level density estimates to exhibit a secondary term in that counting function as well. Moreover, they proved that even a tiny improvement to our error term in Theorem 1.1 would lead to a larger than expected omega result in the Ratios Conjecture of Conrey, Farmer, and Zirnbauer [CFZ08]. They also proved that for a single prime p , improving the error in Theorem 1.3 to $O(p^\omega X^\theta)$ with $\omega + \theta < \frac{1}{2}$ would contradict the Generalized Riemann Hypothesis. This is thus a conditional omega result establishing a limitation on the strength of the error estimates that can actually hold.

Method of proof. Our proofs apply the theory of *Shintani zeta functions*, which were introduced by Sato and Shintani in their landmark works [SS74, Shi72]. The theory was extended to the adelic setting by Datskovsky and Wright [Wri85, DW86], and developed further by the second and third authors in [TT13a]. In [TT13b], the theory was utilized to obtain (1) and (2) with error terms of $O(X^{7/9+\epsilon})$ and $O(X^{18/23+\epsilon})$, respectively.

In this paper, we will in fact give two variants of the improved estimates in Theorem 1.1:

- Our simplest proof applies Landau’s method for estimating partial sums of Dirichlet series having analytic continuation and a functional equation. While this was also utilized in [TT13b], we will deploy Landau’s method in a more effective manner that keeps track of the dependence on the densities of the local conditions being considered (equivalently, the dependence on the residues of the associated Shintani zeta functions), and also takes advantage of the average behavior of the Fourier transforms of the local conditions. This not only substantially lowers the error terms, but also greatly simplifies the proof. The method enables us to prove Theorems 1.1 and 1.2 simultaneously, with error terms of $O(X^{2/3+\epsilon})$. This is carried out in Section 5.
- We also give a proof of Theorem 1.1 using a “discriminant-reducing identity”. Such an identity was used in the first author’s work with Shankar and Tsimerman [BST13], obtaining (1) with an error term of $O(X^{13/16+\epsilon})$. We translate the identity into the language of Shintani zeta functions, and deploy it within our use of Landau’s method. We also give a more precise treatment of the error term in this proof, replacing the

$O(X^\epsilon)$ with $O((\log X)^\alpha)$ for any $\alpha > -\frac{1}{2} + \frac{5}{3^{3/5}} = 2.0864\dots$. This is carried out in Section 7.

The second proof variant is in some ways more complicated, since in the end we must resort to Landau’s method anyhow. However, it avoids several messy computations (most of which were carried out in [TT13a]), and it showcases how such a “discriminant-reducing identity” may be used. Indeed, similar arguments could potentially be applied to prove Theorem 1.2 using the recent discriminant-reducing identities due to O’Dorney [O’D17] (which in turn build on the identities of Ohno [Ohn97] and Nakagawa [Nak98]), though we do not pursue that here. We suspect that these methods may have further applications as well.

In the interest of brevity, we will describe the first proof in detail, and then will give a full account only of those elements of the second proof that are new. In particular, the second proof leads to a different (but equivalent) computation of the main terms in our main results, the details of which we will omit.

Our proof of Theorems 1.3 and 1.4, which count cubic fields with local conditions, is based on our first proof method, and we explain the details in Section 6. We go to some efforts to optimize the dependence of the error terms on these local conditions. For example, with $U = 1$, the error term of $O(X^{2/3+\epsilon}R^{2/3})$ in Theorem 1.4 significantly improves upon the error of $O(X^{7/9+\epsilon}(RT)^{25/9})$ in [TT13b, Theorem 1.3], while also holding in greater generality. This quantitative improvement is reflected in the substantial corresponding quantitative improvements in the various applications discussed above. The proof applies exponential sum formulas of Mori [Mor10] and the second and third authors [TT13a] together with a further averaging technique to treat the error terms as effectively as possible.

There are a variety of results similar to the Davenport–Heilbronn theorems in the literature, together with a number of applications, and we hope that the framework formulated in this article will provide useful tools in those situations as well.

Organization of the paper. We begin in Section 2 by recalling necessary background material on binary cubic forms and the associated Shintani zeta functions. In Section 3, we apply the method of Landau, as formulated in the work of Lowry-Duda and the second and third authors [LDTT], to estimate partial sums of coefficients of congruence Shintani zeta functions. (This takes the place of “Davenport’s Lemma” in geometry-of-numbers approaches.) We provide some additional preliminaries regarding reducible rings and uniformity estimates in Section 4.

In Section 5, we then give our first proof of Theorems 1.1 and 1.2, with error terms of $O(X^{2/3+\epsilon})$. In Section 6, we prove Theorem 1.4, and thus Theorem 1.3, along similar lines, though with some additional setup and notation required. In Section 7, we prove our discriminant-reducing identity, and use it to give a second proof of Theorem 1.1, this time with the stated error term of $O(X^{2/3}(\log X)^{2.09})$.

Finally, in Section 8, we prove a result counting *quadratic* fields with local conditions, which is also needed in the proofs of our main theorems. A similar such result was proved by Ellenberg, Pierce, and Wood in [EPW17]. We generalize their result slightly by allowing local specifications that are not simply given by local splitting types, and we improve the error term in their work by arranging the relevant sum in a manner that enables an application of the Pólya–Vinogradov inequality.

2. BACKGROUND: BINARY CUBIC FORMS AND SHINTANI ZETA FUNCTIONS

Following [DH71, BST13, TT13b], we count cubic fields by means of the *Levi–Delone–Faddeev correspondence*, which relates binary cubic forms to cubic rings, and the *Davenport–Heilbronn correspondence*, which describes maximality conditions for these rings in terms of congruence conditions on the cubic forms. After discussing the relationship between cubic fields and cubic rings, we recall these two correspondences (and the basic definitions). We then introduce the Shintani zeta functions which we use to prove our counting theorems.

2.1. Cubic fields, orders, and rings. A *cubic ring* (over \mathbb{Z}) is a commutative ring that is free of rank 3 as a \mathbb{Z} -module. Its *discriminant* is the determinant of the trace form $\langle x, y \rangle = \text{Tr}(xy)$. Cubic fields are in bijection with their maximal orders, and more generally so are cubic *étale algebras* (products of number fields of total degree 3). The discriminant of a cubic étale algebra F is, by definition, equal to the discriminant of its maximal order \mathcal{O}_F .

Our methods most directly count cubic rings, and a cubic ring R is the maximal order in a cubic field if and only if it is: (i) an integral domain; and (ii) a maximal cubic ring (i.e., there is no other cubic ring R' properly containing R). Note that if R is an integral domain, then it is automatically nondegenerate (i.e., $\text{Disc}(R) \neq 0$). Maximality may be checked locally: a cubic ring R is maximal if and only if $R \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is maximal as a cubic ring over \mathbb{Z}_p for all p .

2.2. Binary cubic forms. The lattice of *integral binary cubic forms* is defined by

$$V(\mathbb{Z}) := \{au^3 + bu^2v + cuv^2 + dv^3 : a, b, c, d \in \mathbb{Z}\}, \quad (12)$$

and the *discriminant* of $f(u, v) = au^3 + bu^2v + cuv^2 + dv^3 \in V(\mathbb{Z})$ is given by the equation

$$\text{Disc}(f) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd. \quad (13)$$

The group $\text{GL}_2(\mathbb{Z})$ acts on $V(\mathbb{Z})$ by

$$(\gamma \cdot f)(u, v) = \frac{1}{\det \gamma} f((u, v) \cdot \gamma). \quad (14)$$

A cubic form f is *irreducible* if $f(u, v)$ is irreducible as a polynomial over \mathbb{Q} , and *nondegenerate* if $\text{Disc}(f) \neq 0$.

The correspondence of Levi [Lev14] and Delone–Faddeev [DF64], as further extended by Gan, Gross, and Savin [GGS02] to include the degenerate case, is as follows:

Theorem 2.1 ([Lev14, DF64, GGS02]). *There is a canonical, discriminant-preserving bijection between the set of $\text{GL}_2(\mathbb{Z})$ -orbits on $V(\mathbb{Z})$ and the set of isomorphism classes of cubic rings. Under this correspondence, irreducible cubic forms correspond to orders in cubic fields, and if a cubic form f corresponds to a cubic ring R , then $\text{Stab}_{\text{GL}_2(\mathbb{Z})}(f)$ is isomorphic to $\text{Aut}(R)$.*

The Davenport–Heilbronn maximality condition is the following:

Proposition 2.2 ([DH71]). *Under the Levi–Delone–Faddeev correspondence, a cubic ring R is maximal if and only if any corresponding cubic form f belongs to the set $U_p \subset V(\mathbb{Z})$ for all p , defined by the following conditions:*

- the cubic form f is not a multiple of p ; and
- there is no $\text{GL}_2(\mathbb{Z})$ -transformation of $f(u, v) = au^3 + bu^2v + cuv^2 + dv^3$ such that a is a multiple of p^2 and b is a multiple of p .

We say that a cubic form f is *maximal at p* if $f \in U_p$.

Davenport and Heilbronn’s proof of the main term in (1) can be summarized as follows. One obtains an asymptotic formula for the number of cubic rings of bounded discriminant by counting lattice points in a fundamental domain for the action of $\mathrm{GL}_2(\mathbb{Z})$ on binary cubic forms f over \mathbb{R} , subject to the constraint $|\mathrm{Disc}(f)| < X$. The fundamental domain may be chosen so that almost all rings that are not integral domains correspond to forms with $a = 0$, and so these may be excluded from the count. One then multiplies this asymptotic by the product of all the local densities of the sets U_p . This yields a heuristic argument for the main term in (1), and some careful analysis allows one to convert this heuristic into a proof.

We also recall (e.g., from [BST13, Section 8.2]) the notion of the *content* of a cubic ring and a binary cubic form. The content $\mathrm{ct}(R)$ of a cubic ring R is the largest integer n such that $R = \mathbb{Z} + nR'$ for some cubic ring R' ; the content of a binary cubic form is the gcd of its coefficients. As explained in [BST13], a cubic form f and its corresponding cubic ring R have the same content.

We also require the following result on overrings and subrings of a cubic ring having a given squarefree index:

Lemma 2.3. *Let q be a squarefree integer.*

- (i) *Let R be a cubic ring that is nonmaximal at each prime divisor of q and whose content is coprime to q . Then R is contained in an overring R' with index q .*
- (ii) *For any ring R' , the number of R contained in R' with index q is bounded above by*

$$\prod_{\substack{p|q \\ p \nmid \mathrm{ct}(R')}} 3 \prod_{\substack{p|q \\ p \mid \mathrm{ct}(R')}} (p+1).$$

Proof. This follows from [BBP10, Lemma 2.4] and [TT13b, Lemma 3.5]. □

2.3. Local conditions for rings and forms. Let R be a maximal cubic ring corresponding to a binary cubic form f (well defined up to $\mathrm{GL}_2(\mathbb{Z})$ -equivalence). We have just seen in Proposition 2.2 that, for each prime p , the form f is required to satisfy certain congruence conditions modulo p^2 .

Let f be a binary cubic form in $V(\mathbb{Z})$ with content prime to p . The *splitting type* of f modulo p is the combinatorial data describing the number of roots of f in $\mathbb{P}^1(\overline{\mathbb{F}}_p)$, together with multiplicities and degrees of their fields of definition. For example, we say that f has splitting type (1^21) if it has a double root and a single root in $\mathbb{P}^1(\mathbb{F}_p)$, or (3) if it is irreducible over \mathbb{F}_p . The possible splitting types are those listed in Table 1. The basic result is that, for the five splitting types enumerated in Table 1, a cubic field F has the specified splitting type if and only if the binary cubic form f corresponding to the maximal order of F has the analogous splitting type, indicated with the same notation.

With this in mind, to each splitting type (mod p) we may associate the characteristic function $\Phi : V(\mathbb{Z}) \rightarrow \{0, 1\}$ of those $f \in V(\mathbb{Z})$ that have that splitting type and are in the set U_p of Proposition 2.2. This function factors through $V(\mathbb{Z}/p^2\mathbb{Z})$ to define a function $\Phi_{p^2} : V(\mathbb{Z}/p^2\mathbb{Z}) \rightarrow \{0, 1\}$. Moreover, for the splitting types $(111), (21), (3)$, this function factors through $V(\mathbb{Z}/p\mathbb{Z})$ to define a function $\Phi_p : V(\mathbb{Z}/p\mathbb{Z}) \rightarrow \{0, 1\}$. (Indeed, any f with an unramified splitting type (mod p) satisfies $p \nmid \mathrm{Disc}(f)$, and hence is automatically in U_p .)

More generally, let Σ_p denote a cubic local specification at p . The algebras in Σ_p are also then all defined by congruence conditions modulo p^a for some $a > 0$. (We may always take

$a \leq 2$ when $p > 3$.) In this case, too, we may associate to Σ_p the characteristic function $\Phi_{p^a} : V(\mathbb{Z}/p^a\mathbb{Z}) \rightarrow \{0, 1\}$ of those $f \in U_p$ that correspond to the maximal orders of cubic algebras in Σ_p .

Let $\Sigma = (\Sigma_p)_p$ denote any collection of cubic local specifications. We define the *conductor* M of Σ by $M := \prod_p p^a$, where the product is over all primes p for which Σ_p is not ordinary and a is the minimal integer for which the associated characteristic function factors through $V(\mathbb{Z}/p^a\mathbb{Z})$. We define $\Phi_M := \otimes_{p|M} \Phi_{p^a}$.

With this definition, a binary cubic form f corresponds to a maximal cubic ring R for which $R \otimes \mathbb{Q}$ satisfies the local specifications of Σ if and only if (a) $\Phi_M(f) = 1$, (b) $f \in U_p$ for each prime $p \nmid M$, and (c) f does not have splitting type (1^3) at any prime $p \nmid M$ for which $\Sigma_p = A'_p$. For further details and explanation, see [BST13, Section 4], [TT13a, Section 5], and [TT13b, Section 6.2].

2.4. Shintani zeta functions. Our proof will apply the theory of *Shintani zeta functions* [Shi72] associated to the space of integral binary cubic forms, defined by the Dirichlet series

$$\xi^\pm(s) := \sum_{\substack{x \in \mathrm{GL}_2(\mathbb{Z}) \backslash V(\mathbb{Z}) \\ \pm \mathrm{Disc}(x) > 0}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}, \quad (15)$$

which (as we recall below) enjoys an analytic continuation and a functional equation.

By the work of Datskovsky and Wright [Wri85, DW86] (see also [TT13a] and F. Sato [Sat89]), we may also consider the following generalization. Let m be any positive integer, let $\Phi_m : V(\mathbb{Z}/m\mathbb{Z}) \rightarrow \mathbb{C}$ be any function such that $\Phi_m(\gamma x) = \Phi_m(x)$ for all $\gamma \in \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$, and define

$$\xi^\pm(s, \Phi_m) = \sum_n a^\pm(\Phi_m, n) n^{-s} := \sum_{\substack{x \in \mathrm{GL}_2(\mathbb{Z}) \backslash V(\mathbb{Z}) \\ \pm \mathrm{Disc}(x) > 0}} \frac{1}{|\mathrm{Stab}(x)|} \Phi_m(x) |\mathrm{Disc}(x)|^{-s}. \quad (16)$$

The *dual* zeta functions $\xi^{*,\pm}(s, \Psi_m) = \sum_n a^{*,\pm}(\Psi_m, n) n^{-s}$ are defined, for each $\mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$ -invariant function $\Psi_m : V^*(\mathbb{Z}/m\mathbb{Z}) \rightarrow \mathbb{C}$, by a variant of (16): the sum is now over all $\mathrm{GL}_2(\mathbb{Z})$ -orbits in the dual lattice $V^*(\mathbb{Z})$. To define the discriminant of an element of $V^*(\mathbb{Z})$, note that there is a $\mathrm{GL}_2(\mathbb{Z})$ -equivariant embedding $\iota : V^*(\mathbb{Z}) \hookrightarrow V(\mathbb{Z})$, whose image consists of those binary cubic forms whose middle two coefficients are divisible by 3. Following Shintani, we identify $V^*(\mathbb{Z})$ with its image in $V(\mathbb{Z})$, and define the discriminant of an element of $V^*(\mathbb{Z})$ via this embedding.

We define $\widehat{\Phi}_m(x) : V^*(\mathbb{Z}/m\mathbb{Z}) \rightarrow \mathbb{C}$ by the usual Fourier transform formula

$$\widehat{\Phi}_m(x) := \frac{1}{m^4} \sum_{y \in V(\mathbb{Z}/m\mathbb{Z})} \Phi_m(y) \exp\left(2\pi i \cdot \frac{[x, y]}{m}\right), \quad (17)$$

and lift $\widehat{\Phi}_m(x)$ to a function on $V^*(\mathbb{Z})$. We use ι to regard $\widehat{\Phi}_m(x)$ as a function on $V(\mathbb{Z})$, and write $\widehat{\Phi}_m(x) = 0$ for all $x \in V(\mathbb{Z})$ not in the image of ι . With this convention, we have $\xi^{*,\pm}(s, \widehat{\Phi}_m) = \xi^\pm(s, \widehat{\Phi}_m)$.

The functional equation has a particularly nice form when diagonalized, as observed in [DW86]. To state the result, we write $T = \begin{pmatrix} \sqrt{3} & 1 \\ \sqrt{3} & -1 \end{pmatrix}$ and $\xi(s, \Phi_m) = \begin{pmatrix} \xi^+(s, \Phi_m) \\ \xi^-(s, \Phi_m) \end{pmatrix}$, and

introduce a gamma factor $\Delta(s) := \begin{pmatrix} \Delta^+(s) & 0 \\ 0 & \Delta^-(s) \end{pmatrix}$, where

$$\begin{aligned} \Delta^+(s) &:= \left(\frac{2^4 3^6}{\pi^4}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2} + \frac{1}{2}\right) \Gamma\left(\frac{s}{2} - \frac{1}{12}\right) \Gamma\left(\frac{s}{2} + \frac{1}{12}\right), \\ \Delta^-(s) &:= \left(\frac{2^4 3^6}{\pi^4}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2} + \frac{1}{2}\right) \Gamma\left(\frac{s}{2} + \frac{5}{12}\right) \Gamma\left(\frac{s}{2} + \frac{7}{12}\right). \end{aligned}$$

Theorem 2.4. *The Shintani zeta functions $\xi^\pm(s, \Phi_m)$ converge absolutely for $\Re(s) > 1$, have analytic continuation to all of \mathbb{C} , are holomorphic except for simple poles at $s = 1$ and $s = 5/6$, and satisfy the functional equation*

$$\Delta(1-s) \cdot T \cdot \xi(1-s, \Phi_m) = m^{4s} \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix} \cdot \Delta(s) \cdot T \cdot \xi(s, \widehat{\Phi}_m). \quad (18)$$

The residues are given by

$$\text{Res}_{s=1} \xi^\pm(s, \Phi_m) = \alpha^\pm \mathcal{A}(\Phi_m) + \beta \mathcal{B}(\Phi_m) \quad \text{and} \quad \text{Res}_{s=5/6} \xi^\pm(s, \Phi_m) = \gamma^\pm \mathcal{C}(\Phi_m), \quad (19)$$

where

$$\alpha^+ = \frac{\pi^2}{72}, \quad \alpha^- = \frac{\pi^2}{24}, \quad \beta = \frac{\pi^2}{24}, \quad \gamma^+ = \frac{\pi^2 \zeta(1/3)}{9\Gamma(2/3)^3}, \quad \gamma^- = \sqrt{3}\gamma^+, \quad (20)$$

and for functions of the form $\Phi_m = \otimes_{p^a \parallel m} \Phi_{p^a}$, we have

$$\mathcal{A}(\Phi_m) = \prod_{p^a \parallel m} \mathcal{A}(\Phi_{p^a}), \quad \mathcal{B}(\Phi_m) = \prod_{p^a \parallel m} \mathcal{B}(\Phi_{p^a}), \quad \text{and} \quad \mathcal{C}(\Phi_m) = \prod_{p^a \parallel m} \mathcal{C}(\Phi_{p^a}), \quad (21)$$

with $\mathcal{A}(\Phi_{p^a})$, $\mathcal{B}(\Phi_{p^a})$, and $\mathcal{C}(\Phi_{p^a})$ given explicitly as follows in the cases of interest below:

(i) For the characteristic function $\Phi_{p^2} : V(\mathbb{Z}/p^2\mathbb{Z}) \rightarrow \{0, 1\}$ of those x that correspond to non-maximal cubic rings over \mathbb{Z}_p :

$$\mathcal{A}(\Phi_{p^2}) = p^{-2} + p^{-3} - p^{-5}, \quad \mathcal{B}(\Phi_{p^2}) = 2p^{-2} - p^{-4}, \quad \mathcal{C}(\Phi_{p^2}) = p^{-5/3} + p^{-2} - p^{-11/3}. \quad (22)$$

(ii) For the characteristic function $\Phi_{p^2} : V(\mathbb{Z}/p^2\mathbb{Z}) \rightarrow \{0, 1\}$ of those x with $p^2 \mid \text{Disc}(x)$:

$$\mathcal{A}(\Phi_{p^2}) = \mathcal{B}(\Phi_{p^2}) = 2p^{-2} - p^{-4}, \quad \mathcal{C}(\Phi_{p^2}) = p^{-5/3} + 2p^{-2} - p^{-8/3} - p^{-3}. \quad (23)$$

(iii) For the characteristic function $\Phi_p : V(\mathbb{Z}/p\mathbb{Z}) \rightarrow \{0, 1\}$ of those x with $p \mid \text{Disc}(x)$:

$$\mathcal{A}(\Phi_p) = \mathcal{B}(\Phi_p) = p^{-1} + p^{-2} - p^{-3}, \quad \mathcal{C}(\Phi_p) = p^{-1} + p^{-4/3} - p^{-7/3}. \quad (24)$$

(iv) For the characteristic function $\Phi_{p^a} : V(\mathbb{Z}/p^a\mathbb{Z}) \rightarrow \{0, 1\}$ corresponding to any of the five local splitting types \mathcal{S}_p :

$$\mathcal{A}(\Phi_{p^a}) = C(\mathcal{S}_p) \left(1 - \frac{1}{p^2}\right) \left(1 - \frac{1}{p^3}\right), \quad \mathcal{C}(\Phi_{p^a}) = K(\mathcal{S}_p) \left(1 - \frac{1}{p^2}\right) \left(1 - \frac{1}{p^{5/3}}\right), \quad (25)$$

where $C(\mathcal{S}_p)$ and $K(\mathcal{S}_p)$ are as in Table 1. We further have that $\mathcal{B}(\Phi_{p^a}) = \delta \cdot \mathcal{A}(\Phi_{p^2})$, where $\delta = 3, 1, 0, 1, 0$ for each of these five splitting types, respectively.

(v) For the function $\Phi_p : V(\mathbb{Z}/p\mathbb{Z}) \rightarrow \{0, 1, 2, 3, p+1\}$, where $\Phi_p(x)$ equals the number of roots of x in $\mathbb{P}^1(\mathbb{F}_p)$:

$$\mathcal{A}(\Phi_p) = 1 + p^{-1}, \quad \mathcal{B}(\Phi_p) = 2, \quad \mathcal{C}(\Phi_p) = 1 + p^{-1/3}. \quad (26)$$

Parts (i)–(v) above are stated in this form in Propositions 8.15, 8.15, 8.14, 8.6 & 8.13, and 8.12 of [TT13a], respectively. (See also Proposition 5.3 and Theorem 6.2 of [DW86].)

We offer some remarks on normalization and how to deduce Theorem 2.4 from the literature. We have defined the zeta functions in terms of $\mathrm{GL}_2(\mathbb{Z})$ -orbits, rather than $\mathrm{SL}_2(\mathbb{Z})$ -orbits on $V(\mathbb{Z})$ as used in [Shi72] and [TT13b]; hence the Shintani zeta function defined here is equal to half of that defined in [Shi72, TT13b]. In [TT13b], the m^{4s} appeared in each term of the dual zeta function, instead of in the functional equation. The functional equation of the zeta functions may be found in [Sat89, Theorem Q] or [TT13a, Theorem 4.3], in the form $\xi(1-s, \Phi_m) = m^{4s} M(s) \xi(s, \widehat{\Phi}_m)$, where $M(s)$ is defined by

$$M(s) := \frac{3^{6s-2}}{2\pi^{4s}} \Gamma(s)^2 \Gamma\left(s - \frac{1}{6}\right) \Gamma\left(s + \frac{1}{6}\right) \begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3 \sin \pi s & \sin 2\pi s \end{pmatrix}.$$

This definition differs from [TT13a] by a factor of 3^{3s} , but coincides with that of Shintani, because of our choice of normalization of the discriminant function on $V^*(\mathbb{Z})$. The diagonalized functional equation (18) is obtained by plugging in the identity

$$\Delta(1-s) \cdot T \cdot M(s) = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix} \cdot \Delta(s) \cdot T, \quad (27)$$

which (after a change of variables) is due to Datskovsky and Wright [DW86, Proposition 4.1].

3. ERROR TERMS FOR SHINTANI ZETA FUNCTIONS WITH LOCAL CONDITIONS VIA LANDAU'S METHOD

Let $\xi(s) := \sum_n a(n)n^{-s}$ be a “zeta function” with “good analytic properties”; suppose, for example, that it converges absolutely in the half-plane $\Re(s) > 1$, has a simple pole at $s = 1$ and no other poles on the line $\Re(s) = 1$, enjoys a meromorphic continuation to \mathbb{C} , and satisfies a functional equation of the “usual shape”. Then a classical method of Landau [Lan12, Lan15] establishes a power-saving estimate of the form

$$\sum_{n < X} a(n) = X \cdot \mathrm{Res}_{s=1}(\xi(s)) + o_\xi(X),$$

where the o -function depends on ξ . In our case of interest, we have the following version:

Theorem 3.1. *Let $\xi^\pm(s, \Phi_m) := \sum_n a^\pm(\Phi_m, n)n^{-s}$ be the Shintani zeta function associated to a nonnegative $\mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$ -invariant function $\Phi_m : V(\mathbb{Z}/m\mathbb{Z}) \rightarrow \mathbb{C}$, and let $\xi^\pm(s, \widehat{\Phi}_m) := \sum_n a^\pm(\widehat{\Phi}_m, n)n^{-s}$ be the dual zeta function. Let*

$$\delta_1 = \delta_1(\Phi_m) := \mathrm{Res}_{s=1} \xi^\pm(s, \Phi_m) \quad (28)$$

and

$$\widehat{\delta}_1 = \widehat{\delta}_1(\Phi_m) := m^4 \cdot \sup_N \frac{1}{N} \sum_{\alpha \in \{\pm\}} \sum_{n < N} a^\alpha(|\widehat{\Phi}_m|, n). \quad (29)$$

For a parameter $X > 0$, assume the following two technical conditions:

$$|\mathrm{Res}_{s=5/6} \xi^\pm(\Phi_m, s)| \ll X^{1/6} |\mathrm{Res}_{s=1} \xi^\pm(\Phi_m, s)|, \quad (30)$$

$$\widehat{\delta}_1 \ll \delta_1 X. \quad (31)$$

Then

$$N^\pm(X, \Phi_m) := \sum_{n < X} a^\pm(\Phi_m, n) = \sum_{\sigma \in \{1, \frac{5}{6}\}} \frac{X^\sigma}{\sigma} \cdot \mathrm{Res}_{s=\sigma} \xi^\pm(s, \Phi_m) + O\left(X^{3/5} \delta_1^{3/5} (\widehat{\delta}_1)^{2/5}\right), \quad (32)$$

where the implied constant does not depend on Φ_m .

In [LDTT], Lowry-Duda and the second and third authors gave a “uniform version” of Landau’s method, closely following the exposition of Chandrasekharan and Narasimhan [CN62], where the implied constants in (32) depend only on the “shape of the functional equation”, and not on Φ_m . (This was also done in a more *ad hoc* manner in [TT13b].) The proof is a bit easier if we incorporate the Datskovsky-Wright diagonalization (although this is not required; see [SS74, Theorem 3]). Here we summarize the proof in [LDTT] while describing how to accommodate the diagonalization.

Proof. We apply Landau’s method to the diagonalized zeta functions $\sqrt{3}\xi^+(s, \Phi_m) \pm \xi^-(s, \Phi_m)$; we write $\xi(s, \Phi_m) = \sum_n a(\Phi_m, n)n^{-s}$ for either of them. Theorem 2.4 gives (separate) functional equations relating them to $\xi(s, \widehat{\Phi}_m) := \sum a(\widehat{\Phi}_m, n) = \sqrt{3}\xi^+(s, \widehat{\Phi}_m) \pm \xi^-(s, \widehat{\Phi}_m)$,

For each positive integer k , a variant of Perron’s formula states that

$$\frac{1}{\Gamma(k+1)} \sum_{n < X} a(\Phi_m, n)(X-n)^k = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \xi(s, \Phi_m) \frac{X^{s+k}}{s(s+1)\cdots(s+k)} ds. \quad (33)$$

Shifting (33) to a line to the left of the critical strip, the two main terms in (32) come from the poles of $\xi(s)$. As $\xi(s)$ grows polynomially on this line as $|\Im(s)| \rightarrow \infty$, for k sufficiently large the integral in (33) will converge absolutely.

We then use the functional equation, expand the dual zeta function $\xi(s, \widehat{\Phi}_m)$ as an absolutely convergent Dirichlet series, and switch the order of summation and integration. Then (33) becomes

$$\sum_{\sigma \in \{1, \frac{5}{8}\}} \frac{X^{k+\sigma}}{\sigma(\sigma+1)\cdots(\sigma+k)} \cdot \text{Res}_{s=\sigma} \xi(s, \Phi_m) + \frac{X^k}{k!} \cdot \xi(0, \Phi_m) + m^{4k+4} \sum_{n \geq 1} \frac{a(\widehat{\Phi}_m, n)}{n^{k+1}} I_k \left(\frac{nX}{m^4} \right),$$

where

$$I_k(t) := \pm 3 \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Delta^\pm(s)}{\Delta^\pm(1-s)} \cdot \frac{\Gamma(1-s)}{\Gamma(k+2-s)} t^{k+1-s} ds$$

for $c = \frac{9}{8}$, say. A straightforward argument shows that $\xi(0, \Phi_m) \ll \widehat{\delta}_1(\Phi_m)$ (see [LDTT, Section 3]). The integral $I_k(t)$ and its derivatives are treated by approximating them by Bessel functions, for which classical estimates are available.

We therefore obtain estimates for the smoothed partial sums on the left side of (33), for each of the two zeta functions $\sqrt{3}\xi^+(s, \Phi_m) \pm \xi^-(s, \Phi_m)$. The last step in [LDTT] is to consider (33) with X replaced by $X + iY$ for $0 \leq i \leq k$ for a parameter Y , and recover a formula for $\sum_{n < X} a(\Phi_m, n)$ by finite differencing, within an error term depending on Y (see [LDTT, Lemma 8]).

As described in [LDTT, (27)], part of the error term may be eliminated if the coefficients $a(\Phi_m, n)$ are all nonnegative. We therefore undo the diagonalization first, obtaining partial sum estimates $\Gamma(k+1)^{-1} \sum_{n < X} a^\pm(\Phi_m, n)(X-n)^k$ for the two zeta functions $\xi^\pm(s, \Phi_m)$ with the same error terms, up to an implied constant. These $a^\pm(\Phi_m, n)$ are nonnegative, since Φ_m is assumed to be so, and we then proceed with the finite differencing.

The error term is simplified exactly as in the proof of [LDTT, Theorem 2], with the factor of m^4 in (29) arising from our normalization of the dual zeta function above. The condition (30) ensures that the error term arising from applying finite differencing to the residual term at $s = \frac{5}{6}$ may be subsumed into the other error terms.

In conclusion, as long as $Y \ll X$, we have

$$N^\pm(X, \Phi_m) = \sum_{\sigma \in \{1, \frac{5}{6}\}} \frac{X^\sigma}{\sigma} \cdot \text{Res}_{s=\sigma} \xi^\pm(s, \Phi_m) + O\left(\delta_1 Y + \widehat{\delta}_1 X^{3/2} Y^{-3/2}\right). \quad (34)$$

The choice $Y = X^{3/5} \delta_1^{-2/5} (\widehat{\delta}_1)^{2/5}$ equalizes the two error terms, with the condition $Y \ll X$ being equivalent to (31), and we obtain the desired result. \square

We also have the following ‘‘average version’’:

Theorem 3.2. *Keeping the notation of Theorem 3.1, consider an arbitrary finite set of pairs $(\Phi_{m_i}, X_i)_{i \in \mathcal{I}}$, where each Φ_{m_i} is a function satisfying the hypotheses of Theorem 3.1 other than (31), and each X_i is a positive real number with $X_i \asymp X$ for some fixed X . Then*

$$\sum_{i \in \mathcal{I}} \left| N^\pm(X_i, \Phi_{m_i}) - \sum_{\sigma \in \{1, \frac{5}{6}\}} \frac{X_i^\sigma}{\sigma} \cdot \text{Res}_{s=\sigma} \xi^\pm(s, \Phi_{m_i}) \right| \ll X^{3/5} \left(\sum_{i \in \mathcal{I}} \delta_1(\Phi_{m_i}) \right)^{3/5} \left(\sum_{i \in \mathcal{I}} \widehat{\delta}_1(\Phi_{m_i}) \right)^{2/5} \quad (35)$$

provided that, in place of (31), we have

$$\sum_{i \in \mathcal{I}} \widehat{\delta}_1(\Phi_{m_i}) \ll X \sum_{i \in \mathcal{I}} \delta_1(\Phi_{m_i}). \quad (36)$$

Proof. The proof is identical to that of Theorem 3.1, except that in (34) we make the choice

$$Y = X^{3/5} \left(\sum_{i \in \mathcal{I}} \delta_1(\Phi_{m_i}) \right)^{-2/5} \left(\sum_{i \in \mathcal{I}} \widehat{\delta}_1(\Phi_{m_i}) \right)^{2/5}$$

for all $i \in \mathcal{I}$ simultaneously. \square

Remark 3.3. The condition (30) is immediate from the residue formulas in Theorem 2.4, for $m \ll X^{1-\epsilon}$ and any Φ_m whose residues are described there. Outside of Remark 6.3, this covers all Shintani zeta functions introduced in this paper.

4. ADDITIONAL PRELIMINARIES ON REDUCIBLE RINGS AND UNIFORMITY ESTIMATES

4.1. Reducible rings and their weighting. By the Levi–Delone–Faddeev correspondence, the Shintani zeta functions may also be written as

$$\xi^\pm(s) := \sum_{\pm \text{Disc}(R) > 0} \frac{1}{|\text{Aut}(R)|} |\text{Disc}(R)|^{-s}, \quad (37)$$

where the sum is over all isomorphism classes of cubic rings. The main work (here, and also in previous papers on Davenport–Heilbronn) is to sieve for maximality. After doing so, we will obtain a count for the quantity

$$N_{\leq 3}^\pm(X) := \sum_{0 < \pm \text{Disc}(F) < X} |\text{Aut}(F)|^{-1}, \quad (38)$$

where the sum is over étale algebras F of degree 3, i.e., direct products of number fields whose degrees sum to 3. Note that these are in bijection with number fields of degrees at most 3, and that the automorphism group of such an algebra is naturally isomorphic to that of its maximal order. For any collection of cubic local specifications Σ , we define $N_{\leq 3}^\pm(X, \Sigma)$

analogously. In Proposition 4.3, we relate these quantities to the quantities $N_3^\pm(X)$ and $N_3^\pm(X, \Sigma)$ introduced in Theorems 1.1 and 1.3.

As in [Coh54], the following lemma follows from Hasse’s characterization [Has48] of cyclic cubic field discriminants:

Lemma 4.1. *Let t be a squarefree integer. Then the number of cyclic cubic fields F with $|\text{Disc}(F)| < X$ and $t \mid \text{Disc}(F)$ is $O(X^{1/2}/t^{1-\epsilon})$.*

We also require an estimate for the contribution of quadratic fields, optionally with local conditions. Suppose, as before, that $\Sigma = (\Sigma_p)_p$ is a set of cubic local specifications that are ordinary away from an integer M , the conductor of Σ . We say that a quadratic field F satisfies the cubic local specifications of Σ_p if the cubic étale algebra $F \times \mathbb{Q}$ does.

Except for \mathbb{Q}^3 , reducible cubic étale algebras take the form $F \times \mathbb{Q}$ where F is a quadratic field. For the reducible algebra $F \times \mathbb{Q}$, the only nonempty conditions that Σ imposes on F are those at primes dividing M . We write U , s , and r for the product of those prime factors p of M for which, respectively: Σ_p contains at least one unramified algebra; Σ_p consists of a single algebra $F \times \mathbb{Q}_p$ where F is ramified; Σ_p consists of all algebras $F \times \mathbb{Q}_p$ where F is ramified.

The following proposition counting quadratic fields satisfying a collection of local specifications is a slight generalization of [TT13b, Lemma 6.1] and [EPW17, Proposition 8.1], and with an improved error term. We will give a complete and self-contained proof in Section 8.

Proposition 4.2. *Write $N_2^\pm(X, \Sigma)$ for the number of quadratic fields F with $0 < \pm \text{Disc}(F) < X$ that satisfy a set of local specifications Σ with M , U , s , and r as above. Then*

$$N_2^\pm(X, \Sigma) = \prod_p c_p(\Sigma_p) \cdot \frac{3}{\pi^2} X + O\left(X^{1/2} \frac{U^{1/4}}{s^{1/4} r^{1/2}} (U s r)^\epsilon\right), \quad (39)$$

where

$$c_p(\Sigma_p) = \frac{\sum_{F: F \times \mathbb{Q}_p \in \Sigma_p} \frac{1}{\text{Disc}_p(F)} \frac{1}{|\text{Aut}(F)|}}{\sum_{F: F \times \mathbb{Q}_p \in A_p} \frac{1}{\text{Disc}_p(F)} \frac{1}{|\text{Aut}(F)|}}. \quad (40)$$

In Proposition 4.2, we may explicitly evaluate $c_p(\Sigma_p)$ for specific Σ_p as follows:

- If $p \nmid M$, then $c_p(\Sigma_p) = 1$ and so it may be omitted from the product.
- If Σ_p consists of a single unramified and reducible algebra, then $c_p(\Sigma_p) := \frac{1}{2}(1+p^{-1})^{-1}$.
- If $p \neq 2$, then there are two ramified quadratic extensions F of \mathbb{Q}_p . If Σ_p consists of $F \times \mathbb{Q}_p$ for one of these F , then $c_p(\Sigma_p) = \frac{1}{2}(p+1)^{-1}$.
- If Σ_p consists of all algebras $F \times \mathbb{Q}_p$ where F is ramified, then $c_p(\Sigma_p) := (p+1)^{-1}$.
- There are six ramified quadratic extensions of \mathbb{Q}_2 ; if Σ_2 consists one of these six extensions, then the associated values of $c_2(\Sigma_2)$ are given in Section 8.
- If Σ_p contains more than one algebra, then $c_p(\Sigma_p)$ is the sum of the values for the individual algebras.

The following formula relates $N_3^\pm(X, \Sigma)$ to $N_{\leq 3}^\pm(X, \Sigma)$.

Proposition 4.3. *We have*

$$N_3^\pm(X, \Sigma) = N_{\leq 3}^\pm(X, \Sigma) - \frac{1}{2} N_2^\pm(X, \Sigma) + O(X^{1/2}/t^{1-\epsilon}), \quad (41)$$

where t is the product of those primes p such that Σ_p consists of only totally ramified cubic extensions of \mathbb{Q}_p .

Proof. This is immediate from Lemma 4.1 and the formula

$$|\mathrm{Aut}(F)| = \begin{cases} 1 & \text{if } F \text{ is a non-Galois cubic field,} \\ 3 & \text{if } F \text{ is a Galois cubic field,} \\ 2 & \text{if } F = F_2 \times \mathbb{Q} \text{ with } F_2 \text{ a quadratic field,} \\ 6 & \text{if } F = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}. \end{cases} \quad (42)$$

□

Remark 4.4. Note that the error term in Proposition 4.2 is always bounded above by that of our main theorems (Theorem 1.3 and 1.4).

4.2. Uniformity/tail estimates. Our basic uniformity/tail estimate is a variation of [DH71, Proposition 1, p. 410] and [BBP10, Lemma 3.4], which bounds the number of cubic rings having absolute discriminant less than X and divisible by q^2 , where q is squarefree; the exact statement can be found as [BBP10, Lemma 3.4] (see also [TT13b, Lemma 3.4]).

Proposition 4.5. *For each squarefree integer q , the number of cubic rings R with $q^2 \mid \mathrm{Disc}(R)$ and $0 < \pm \mathrm{Disc}(R) < X$ is $\ll 6^{\omega(q)} X/q^2$.*

Here (and elsewhere), $\omega(q)$ is the number of prime divisors of q .

For nonmaximal rings, the following stronger estimate can be proved in essentially the same way as [BBP10, Lemmas 2.7 and 3.3] (see also [TT13b, Lemma 3.4]):

Proposition 4.6. *For each squarefree integer q , the number of cubic rings R that are non-maximal at q and satisfy $0 < \pm \mathrm{Disc}(R) < X$ is $\ll 3^{\omega(q)} X/q^2$.*

Finally, we require the following uniformity estimate on the number of cubic rings having discriminant divisible by a given cubefree integer q :

Proposition 4.7. *For each cubefree integer $q < X^{1/4-\epsilon}$, the number of cubic rings R with $0 < \pm \mathrm{Disc}(R) < X$ for which $q \mid \mathrm{Disc}(R)$ is $\ll X/q$.*

Proof. Let $\Phi_q : V(\mathbb{Z}/q\mathbb{Z}) \rightarrow \{0, 1\}$ be the characteristic function of those forms whose discriminants are divisible by q . Then, we have

$$\sum_{n < X} a^\pm(\Phi_q, n) \leq e \cdot \sum_n a^\pm(\Phi_q, n) \exp(-n/X) = \frac{e}{2\pi i} \int_{2-i\infty}^{2+i\infty} \xi^\pm(\Phi_q, s) X^s \Gamma(s) ds. \quad (43)$$

Shift the contour to $\Re(s) = -c$ for some small $c > 0$, obtaining a ‘main term’ of $O(X/q)$ from the pole of $\xi^\pm(\Phi_q, s)$ at $s = 1$. The functional equation with the trivial bound $|\widehat{\Phi}_q(x)| \leq 1$ establishes a bound of $O(q^{4+4c})$ on the line $\Re(s) = -c$, and the factor of $\Gamma(s)$ guarantees the absolute convergence of the integral. The contribution of the other residues may be either estimated directly, or bounded using the Phragmen-Lindelöf principle, and this completes the proof. □

Remark 4.8. A similar result also appeared as [TT13b, Lemma 4.4]. We will actually only need this result for $q < X^\delta$ for *any* arbitrarily small $\delta > 0$. This result is used in the penultimate paragraph of Section 5 (and analogously in Section 6) where, after applying the functional equation to the relevant Shintani zeta functions, we must bound a ‘dual sum’ on

$V^*(\mathbb{Z})$ that shares some characteristics with our original counting problem. A result of this form could also be immediately deduced from the main results of this paper or from [TT13b]. The above proof is independent of our other results and thus allows us to avoid any circular reasoning.

5. DIRECT PROOF

We first prove Theorems 1.1 and 1.2 directly using Landau’s method as presented in Theorems 3.1 and 3.2. We obtain error terms of $O(X^{2/3+\epsilon})$ in both results. (For the improvement to $O(X^{2/3}(\log X)^{2.09})$ in Theorem 1.1, see the alternative proof using discriminant reduction in Section 7.)

We fix a collection of local specifications $\Sigma = (\Sigma_p)_p$, ordinary away from its conductor M , such that the conditions (mod M) correspond to a $\mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$ -invariant function $\Phi_M : V(\mathbb{Z}/M\mathbb{Z}) \rightarrow \mathbb{C}$. (For Theorems 1.1 and 1.2 we take $M = 1$; the same setup will also be used in Section 6, when proving the more general Theorem 1.4.) As before, we may assume that Φ_M is supported only on those $x \in V(\mathbb{Z}/M\mathbb{Z})$ satisfying the Davenport–Heilbronn maximality condition of Proposition 2.2 for each prime $p \mid M$.

To carry out the sieve, for each prime $p \nmid M$ we write $\Psi_{p^2} : V(\mathbb{Z}/p^2\mathbb{Z}) \rightarrow \{0, 1\}$ for the characteristic function of one of the following two sets, in accordance with whether $\Sigma_p = A_p$ or $\Sigma_p = A'_p$:

- Those $x \in V(\mathbb{Z}/p^2\mathbb{Z})$ that are nonmaximal at p in the sense of Proposition 2.2.
- Those $x \in V(\mathbb{Z}/p^2\mathbb{Z})$ that are nonmaximal at p or have a triple root (mod p). When $p > 2$, this is equivalent to requiring that $p^2 \mid \mathrm{Disc}(x)$.

For each squarefree q coprime to M , we view $\Psi_{q^2} := \otimes_{p \mid q} \Psi_{p^2}$ as a function $V(\mathbb{Z}/q^2\mathbb{Z}) \rightarrow \{0, 1\}$.

The quantity $N_3^\pm(X, \Sigma)$ of interest is related to $N_{\leq 3}^\pm(X, \Sigma)$ as discussed in Proposition 4.3, so that it suffices to estimate the latter. The Levi–Delone–Faddeev correspondence (Theorem 2.1), the Davenport–Heilbronn correspondence (Proposition 2.2), and inclusion-exclusion give

$$N_{\leq 3}^\pm(X, \Sigma) = \sum_q \mu(q) N^\pm(X, \Phi_M \Psi_{q^2}). \quad (44)$$

We now split the sum into two parts in accordance with whether $q \leq Q$ or $q > Q$, and apply Landau’s method (Theorem 3.2) for the former. We obtain

$$N_{\leq 3}^\pm(X, \Sigma) = \sum_{\sigma \in \{1, \frac{5}{6}\}} \frac{X^\sigma}{\sigma} \sum_{\substack{q=1 \\ (q, M)=1}}^{\infty} \mu(q) \cdot \mathrm{Res}_{s=\sigma} \xi^\pm(s, \Phi_M \Psi_{q^2}) + O(E_1 + E_2 + E_3), \quad (45)$$

with

$$\begin{aligned} E_1 &:= \sum_{\sigma \in \{1, \frac{5}{6}\}} X^\sigma \sum_{q > Q} |\mathrm{Res}_{s=\sigma} \xi^\pm(s, \Phi_M \Psi_{q^2})|, \\ E_2 &:= X^{\frac{3}{5}} \sum_{Q_1} \left(\sum_{q \in [Q_1, 2Q_1]} \delta_1(\Phi_M \Psi_{q^2}) \right)^{\frac{3}{5}} \left(\sum_{q \in [Q_1, 2Q_1]} \widehat{\delta}_1(\Phi_M \Psi_{q^2}) \right)^{\frac{2}{5}}, \\ E_3 &:= \sum_{q > Q} N^\pm(X, \Phi_M \Psi_{q^2}), \end{aligned}$$

provided that (36) is satisfied for our choice of the parameter Q . Here Q_1 ranges over all integer powers of 2 less than Q . Throughout, all summations over q are over *squarefree* integers coprime to M .

In this section, we specialize to the case $M = 1$, i.e., when $\Sigma_p = A_p$ or $\Sigma_p = A'_p$ for every prime p . (Making the same choice for all primes p leads to Theorem 1.1 or Theorem 1.2 respectively.) In this case, we have $\Phi_M \Psi_{q^2} = \Psi_{q^2}$, and our residue formulas (22) and (23) then imply that

$$E_1 \ll X \sum_{q>Q} q^{-2+\epsilon} + X^{5/6} \sum_{q>Q} q^{-5/3+\epsilon} \ll \frac{X}{Q^{1-\epsilon}}$$

for $Q < X^{1/2}$, and that (30) is satisfied in the same range. By the tail estimate of Proposition 4.5, we also have

$$E_3 \ll X \sum_{q>Q} 6^{\omega(q)} q^{-2} \ll \frac{X}{Q^{1-\epsilon}}.$$

To bound E_2 , note that $\delta_1(\Psi_{q^2}) \ll q^{-2+\epsilon}$ for each q by the residue formulas already quoted, and we claim the following average bound on $\widehat{\delta}_1(\Psi_{q^2})$:

Proposition 5.1. *For either definition of Ψ_{q^2} , we have*

$$\sum_{q \in [Q, 2Q]} \widehat{\delta}_1(\Psi_{q^2}) \ll Q^{2+\epsilon}. \quad (46)$$

Granting this for now, we have

$$E_1 + E_2 + E_3 \ll \frac{X}{Q^{1-\epsilon}} + X^{3/5} \sum_{Q_1} Q_1^{-\frac{3}{5}+\epsilon} \cdot Q_1^{\frac{4}{5}+\epsilon} \ll \frac{X}{Q^{1-\epsilon}} + X^{3/5} Q^{1/5+\epsilon}.$$

Choosing $Q = X^{1/3-\epsilon}$, which is acceptable in (36), we obtain error terms of $O(X^{2/3+\epsilon})$ in Theorem 1.1 and in (3), which is equivalent to Theorem 1.2. The main terms in counting $N_{\leq 3}^{\pm}(X)$ are given by the infinite sums in (45), namely

$$C^{\pm} \cdot \frac{\pi^2}{72} \cdot X \prod_p (1 - \mathcal{A}(\Psi_{p^2})) + \frac{\pi^2}{24} \cdot X \prod_p (1 - \mathcal{B}(\Psi_{p^2})) + K^{\pm} \cdot \frac{\pi^2 \zeta(1/3)}{9\Gamma(2/3)^3} \cdot \frac{6}{5} X^{5/6} \cdot \prod_p (1 - \mathcal{C}(\Psi_{p^2})). \quad (47)$$

The products are evaluated using the formulas in (22) and (23) respectively, and then an application of Propositions 4.2 and 4.3 finishes the proof.

It remains only to prove Proposition 5.1. Expanding the definition (29) of $\widehat{\delta}_1$, the bound to be proved is

$$\sum_{\alpha \in \{\pm\}} \sum_{q \in [Q, 2Q]} \sum_{n < N} a^{\alpha}(|\widehat{\Psi}_{q^2}|, n) \ll NQ^{-6+\epsilon}. \quad (48)$$

By the definition of the coefficients, this is equivalent to showing that

$$\sum_{q \in [Q, 2Q]} \sum_{\substack{x \in \text{GL}_2(\mathbb{Z}) \setminus V(\mathbb{Z}) \\ 0 \neq |\text{Disc}(x)| < N}} |\widehat{\Psi}_{q^2}(x)| \ll NQ^{-6+\epsilon}. \quad (49)$$

Our analysis will bound the sum over $q \in [Q, 2Q]$ as a whole, and avoid the need for sharp bounds on the inner sum for each q individually.

The first step is an evaluation of the Fourier transforms $|\widehat{\Psi}_{q^2}(x)|$; the following combination of [TT13b, Lemmas 3.3 and 4.3] summarizes¹ the results we need:

Proposition 5.2. *For either definition of Ψ_{q^2} , the function $\widehat{\Psi}_{q^2}(x)$ is multiplicative in q , and satisfies the following bounds (where R is the cubic ring corresponding to x):*

- (Content p^2) $|\widehat{\Psi}_{p^2}(x)| = O(p^{-2})$ if p^2 divides the content of R .
- (Content p) $|\widehat{\Psi}_{p^2}(x)| = O(p^{-3})$ if p divides the content of R but p^2 does not.
- (Divisible by p^4) $|\widehat{\Psi}_{p^2}(x)| = O(p^{-3})$ if R is nonmaximal at p , p does not divide the content of R , and $p^4 \mid \text{Disc}(R)$.
- (Divisible by p^3) $|\widehat{\Psi}_{p^2}(x)| = O(p^{-4})$ if R is nonmaximal at p and $p^3 \parallel \text{Disc}(R)$.
- (Divisible by p^2) $|\widehat{\Psi}_{p^2}(x)| = O(p^{-5})$ if $p^2 \parallel \text{Disc}(R)$.
- Otherwise, and in particular if $p^2 \nmid \text{Disc}(R)$, we have $\widehat{\Psi}_{p^2}(x) = 0$.

For each squarefree $q \in [Q, 2Q]$, we consider the contribution to (49) from every factorization

$$q = c_2 c_1 d_4 d_3 d_2 \quad (50)$$

and from those x such that $x \pmod{p^2}$ satisfies the first five conditions of Proposition 5.2 for p dividing c_2 , c_1 , d_4 , d_3 , and d_2 , respectively. We begin by replacing each form x with $x/c_2^2 c_1$, using the natural bijection between forms x with $|\text{Disc}(x)| < N$ and content divisible by $c_2^2 c_1$, and forms x with $|\text{Disc}(x)| < N c_2^{-8} c_1^{-4}$.

By Lemma 2.3 (i), each ring R with $d_4 d_3 > 1$ is contained in an overring R' of index $d_4 d_3$. The content of R' may be divisible by prime factors of d_4 ; we write $d_4 = d_{4c} d_{4n}$, where d_{4c} is the gcd of d_4 and the content of R' , and refine the factorization of (50) to

$$q = c_2 c_1 d_{4c} d_{4n} d_3 d_2. \quad (51)$$

We count our rings R by counting these overrings R' with multiplicity given by the number of R thus contained in any such R' , which is $\ll q^\epsilon d_{4c}$ by Lemma 2.3 (ii). We then replace the form x' corresponding to each such R' by x'/d_{4c} .

The total contribution to (49) is therefore bounded above by

$$Q^\epsilon \sum_{c_2, c_1, d_{4c}, d_{4n}, d_3, d_2} \frac{1}{c_2^2 c_1^3 d_{4c}^2 d_{4n}^3 d_3^4 d_2^5} \sum_{\substack{|\text{Disc}(x')| < \frac{N}{c_2^8 c_1^4 d_{4c}^6 d_{4n}^2 d_3^2} \\ d_{4n}^2 d_3 d_2^2 \mid \text{Disc}(x')}} 1, \quad (52)$$

where the outer sum is over all choices of the six variables whose product (51) is in $[Q, 2Q]$.

When N is not too large—when $N \leq Q^{100}$, say— we first sum the variable d_3 over dyadic intervals $[D_3, 2D_3]$. For each fixed x' in the inner sum, there are at most $O(Q^\epsilon)$ such d_3 with $d_3 \mid \text{Disc}(x')$, and so the sum in (52) is bounded above by

$$Q^\epsilon \sum_{D_3} \sum_{c_2, c_1, d_{4c}, d_{4n}, d_2} \frac{1}{c_2^2 c_1^3 d_{4c}^2 d_{4n}^3 D_3^4 d_2^5} \sum_{\substack{|\text{Disc}(x')| < \frac{N}{c_2^8 c_1^4 d_{4c}^6 d_{4n}^2 D_3^2} \\ d_{4n}^2 d_2^2 \mid \text{Disc}(x')}} 1. \quad (53)$$

¹We point out a minor mistake in Lemmas 3.3 and 4.3 of [TT13b]: exact formulas are claimed for the values of Ψ_{q^2} for those x which are nonmaximal at p , have content coprime to p , and for which $p^4 \mid \text{Disc}(x)$. These formulas hold as equalities only for some such x , but as upper bounds they hold for all x .

By Proposition 4.5, the inner sum is $\ll Q^\epsilon \frac{N}{c_2^8 c_1^4 d_{4c}^6 d_{4n}^2 D_3^2} \cdot \frac{1}{d_{4n}^2 d_2^2}$, so that the above simplifies to

$$Q^\epsilon \sum_{D_3} \sum_{c_2, c_1, d_{4c}, d_{4n}, d_2} \frac{N}{c_2^{10} c_1^7 d_{4c}^8 d_{4n}^7 D_3^6 d_2^7}. \quad (54)$$

The double sum is over choices of the six variables whose product is $> Q$, so that the sum is $\ll NQ^{-6+\epsilon}$.

For $N > Q^{100}$, Proposition 4.7 more than suffices to bound the inner sum in (52), again yielding a bound $\ll NQ^{-6+\epsilon}$.

We have thus proven (49) in both cases, and therefore Proposition 5.1. This yields Theorem 1.1 (with $(\log X)^{2.09}$ replaced with X^ϵ) and 1.2.

6. PROOF OF THEOREM 1.4: LOCAL CONDITIONS

Let U , r , and t be squarefree and coprime integers, and let $\Sigma^{r,t}$ be a collection of local specifications satisfying the hypotheses described immediately before Theorem 1.4. For simplicity, we assume for each $p \mid U$ (and we know a priori for each $p \mid rt$) that Σ_p consists of all extensions having one of the five cubic splitting types; by summation, this will imply the result when Σ_p is a union of splitting types. In Remark 6.3, we describe how to amend the proof to accommodate Σ_p that are not a union of splitting types.

We always assume that r and t are coprime to 6, incorporating any 2- and 3-adic conditions into the U component instead. Conversely, for each $p > 3$ dividing U , we assume that Σ_p corresponds to one of the three *unramified* splitting types; ramified splitting types may be incorporated using the variables r and t instead.

As described in Section 2.3, there is an integer M , called the *conductor* of $\Sigma^{r,t}$ and divisible by precisely the primes dividing Urt , such that the partial collection $(\Sigma_p)_{p \mid Urt}$ corresponds to a function $\Phi_M : V(\mathbb{Z}/M\mathbb{Z}) \rightarrow \{0, 1\}$ of the form $\otimes_{p^a \parallel M} \Phi_{p^a}$, where Φ_{p^a} is any of the following functions:

- $\Theta_{(111),p}, \Theta_{(21),p}, \Theta_{(3),p} : V(\mathbb{Z}/p\mathbb{Z}) \rightarrow \{0, 1\}$, the characteristic functions of the three unramified splitting types (totally split, partially split, or inert), respectively. These functions are automatically supported only on forms that are maximal at p .
- $\Theta_{(1^2 1),p^2} : V(\mathbb{Z}/p^2\mathbb{Z}) \rightarrow \{0, 1\}$, the characteristic function of the partially ramified splitting type, corresponding to binary cubic forms that are maximal at p and have exactly two roots in $\mathbb{P}^1(\mathbb{F}_p)$, one of which is a double root.
- $\Theta_{(1^3),p^2} : V(\mathbb{Z}/p^2\mathbb{Z}) \rightarrow \{0, 1\}$, the characteristic function of the totally ramified splitting type, corresponding to binary cubic forms that are maximal at p and have a triple root in $\mathbb{P}^1(\mathbb{F}_p)$.

To accommodate our analysis, we also introduce the following additional functions:

- $\Theta_{\text{div},p} : V(\mathbb{Z}/p\mathbb{Z}) \rightarrow \{0, 1\}$, the characteristic function of binary cubic forms $x \pmod{p}$ for which $p \mid \text{Disc}(x)$, but without any maximality condition.
- $\Theta_{\text{div}^2,p^2} : V(\mathbb{Z}/p^2\mathbb{Z}) \rightarrow \{0, 1\}$, the characteristic function of forms x for which $p^2 \mid \text{Disc}(x)$, but without any maximality condition.
- $\Theta_{\text{nonmax},p^2} : V(\mathbb{Z}/p^2\mathbb{Z}) \rightarrow \{0, 1\}$, the characteristic function of those forms that are nonmaximal at p .

The last two functions were introduced with the notation Ψ_{q^2} in the previous section, and Fourier transform bounds for these functions were obtained in Proposition 5.2. For $p \neq 2, 3$, we have the relation

$$\Theta_{(1^2 1), p^2} = \Theta_{\text{div}, p} - \Theta_{\text{div}^2, p^2} \quad (55)$$

when one views $\Theta_{\text{div}, p}$ as a function on $V(\mathbb{Z}/p^2\mathbb{Z})$. We also have $\Theta_{(1^3), p^2} = \Theta_{\text{div}^2, p^2} - \Theta_{\text{nmax}, p^2}$, implying that $\Theta_{(1^3), p^2}$ satisfies the same Fourier transform bounds in Proposition 5.2.

We use the relation (55) to write

$$N_{\leq 3}^{\pm}(X, \Sigma^{r,t}) = \sum_{d|r} \mu(d) \sum_{\substack{q=1 \\ (q,M)=1}}^{\infty} \mu(q) N^{\pm}(X, \Phi_U \Theta_{\text{div}, \frac{r}{d}} \Theta_{\text{div}^2, d^2} \Theta_{(1^3), t^2} \Psi_{q^2}), \quad (56)$$

where $\Phi_U = \otimes_{p^a | U} \Phi_{p^a}$, with Φ_{p^a} denoting any of $\Theta_{(111), p}$, $\Theta_{(21), p}$, or $\Theta_{(3), p}$ for $p \nmid 6$, and $\Psi_{q^2} = \prod_{p|q} \Psi_{p^2}$, with Ψ_{p^2} denoting one of the two functions according as $\Sigma_p = A_p$ or $\Sigma_p = A'_p$ as in Section 5. We let

$$E(X, \Phi_m) := \sum_{\substack{q=1 \\ (q,m)=1}}^{\infty} \left| N^{\pm}(X, \Phi_m \Psi_{q^2}) - \sum_{\sigma \in \{1, \frac{5}{6}\}} \frac{X^{\sigma}}{\sigma} \cdot \text{Res}_{s=\sigma} \xi^{\pm}(s, \Phi_m \Psi_{q^2}) \right|, \quad (57)$$

and we will sum this error term over those functions $\Phi_m = \Phi_U \Theta_{\text{div}, \frac{r}{d}} \Theta_{\text{div}^2, d^2} \Theta_{(1^3), t^2}$ which occur in (56) subject to the following conditions:

- r and t range over squarefree integers coprime to U and to each other, and d ranges over divisors of r ;
- given parameters R' and T' , we require that $r' := r/d \in [R', 2R']$ and $t' := dt \in [T', 2T']$;
- X_{Φ_m} may vary arbitrarily within a dyadic interval $\frac{X}{2} \leq X_{\Phi_m} \leq X$.

With this setup, Theorem 3.2 implies for each R' and T' that

$$\sum_{\Phi_m} |E(X_{\Phi_m}, \Phi_m)| \ll E'_1 + E'_2 + E'_3, \quad (58)$$

with

$$\begin{aligned} E'_1 &:= \sum_{\sigma=1, 5/6} X^{\sigma} \sum_{\Phi_m} \sum_{q>Q} |\text{Res}_{s=\sigma} \xi^{\pm}(s, \Phi_m \Psi_{q^2})|, \\ E'_2 &:= X^{\frac{3}{5}} \sum_{Q_1} \left(\sum_{\Phi_m} \sum_{q \in [Q_1, 2Q_1]} \delta_1(\Phi_m \Psi_{q^2}) \right)^{\frac{3}{5}} \left(\sum_{\Phi_m} \sum_{q \in [Q_1, 2Q_1]} \widehat{\delta}_1(\Phi_m \Psi_{q^2}) \right)^{\frac{2}{5}} \\ &= X^{\frac{3}{5}} \sum_{Q_1} E''_2(Q_1)^{3/5} \widehat{E}''_2(Q_1)^{2/5}, \\ E'_3 &:= \sum_{\Phi_m} \sum_{q>Q} N^{\pm}(X, \Phi_m \Psi_{q^2}), \end{aligned}$$

where again Q_1 ranges over integer powers of 2 less than Q , subject to the condition that (36) holds, i.e., for each $Q_1 < Q$ that we have

$$\widehat{E}''_2(Q_1) \ll E''_2(Q_1)X, \quad (59)$$

which we will check later. We prove the bound

$$E'_1 + E'_2 + E'_3 \ll X^{2/3+\epsilon} U^{2/3} (R')^{2/3} \quad (60)$$

for a suitable choice of $Q < X$. Given (60), we extend the result and then prove Theorem 1.4 as follows:

- In our analysis we may assume that $U^2 R'^2 T'^3 < X^{1-\nu}$ for some small $\nu > 0$; otherwise, we obtain the same bound in (58) as a consequence of the “trivial bound” of $O(X^{1+\epsilon} T'^{-1})$ that follows directly from Proposition 4.5.
- We extend the ranges of r' and t' in (58) to the set of pairs (r', t') with $r' \leq R$ and $r't' \leq RT$ by summing the results of (60) over $O(X^\epsilon)$ dyadic intervals. The resulting bound in (58) is again $O(X^{2/3+\epsilon} U^{2/3} R^{2/3})$.
- We therefore obtain

$$\sum_{r \leq R} \sum_{t \leq T} \sum_{\Phi_m} |E(X_{r,t}, \Phi_m)| \ll X^{2/3+\epsilon} U^{2/3} R^{2/3}. \quad (61)$$

provided that $\frac{X}{2} \leq X_{r,t} \leq X$ for each r and t . The inner sum is over the set of all Φ_m appearing in (56) (depending on r and t), and by (57) is therefore a bound on the error made in approximating $N_{\leq 3}^\pm(X_{r,t}, \Sigma^{r,t})$ in terms of residues of Shintani zeta functions.

- We write out the analogue of (47), with the quantities $C_p(\Sigma_p)$ and $K_p(\Sigma_p)$ of Table 1 coming from our residue formula (25). We use Propositions 4.2 and 4.3 to relate $N_{\leq 3}^\pm(X_{r,t}, \Sigma^{r,t})$ to $N_3^\pm(X_{r,t}, \Sigma^{r,t})$, and note that the error terms in Proposition 4.2 are less than those in Theorem 1.4. This completes the proof of Theorem 1.4, provided that $\frac{X}{2} \leq X_{r,t} \leq X$ for each r and t .
- Finally, we extend Theorem 1.4 to allow arbitrary $X_{r,t} \leq X$ by dividing the interval $[1, X]$ into $O(X^\epsilon)$ dyadic intervals, considering the contribution of those (r, t) with $X_{r,t}$ in each interval, and then summing the resulting error terms.

It remains to prove (60), and we begin with $\delta_1(\Phi_m \Psi_{q^2})$. For ease of reading we replace r', t', R', T' by r, t, R, T respectively. The residue formulas in Theorem 2.4 imply that

$$\delta_1(\Phi_m \Psi_{q^2}) \ll X^\epsilon R^{-1} T^{-2} q^{-2},$$

so that

$$E'_1 \ll X^{1+\epsilon} Q^{-1} T^{-1}, \quad E''_2(Q_1) \ll X^\epsilon Q_1^{-1} T^{-1}.$$

We can also use the tail estimate of Proposition 4.5 to bound E'_3 : we have

$$\begin{aligned} E'_3 &= \sum_{r \in [R, 2R]} \sum_{t \in [T, 2T]} \sum_{d|t} \sum_{q > Q} N^\pm(X, \Phi_m \Psi_{q^2}) \\ &\ll X^\epsilon \sum_{t \in [T, 2T]} \sum_{d|t} \sum_{q > Q} N^\pm(X, \Phi_{Ut^2} \Psi_{q^2}) \\ &\ll X^\epsilon \sum_{t \in [T, 2T]} \sum_{d|t} \sum_{q > Q} X t^{-2+\epsilon} q^{-2+\epsilon} \\ &\ll X^{1+\epsilon} Q^{-1} T^{-1}, \end{aligned}$$

where in the second line Φ_{Ut^2} is the product of the local conditions modulo Ut^2 . Here we use the fact that any R with $0 < |\text{Disc}(R)| < X$ satisfies $p \mid \text{Disc}(R)$ for at most $O(X^\epsilon)$ primes p .

We now prove the bound

$$\widehat{E}_2''(Q_1) \ll (RTUQ_1)^2 X^\epsilon, \quad (62)$$

still assuming that $U^2 R^2 T^3 < X^{1-\epsilon}$. By the definition (29), the claim to be proved is

$$\sum_{q \in [Q, 2Q]} \sum_{r \in [R, 2R]} \sum_{t \in [T, 2T]} \sum_{d|t} \sum_{|\text{Disc}(x)| < N} |\widehat{\Phi}_U(x) \widehat{\Phi}_r(x) \widehat{\Theta}_{\text{div}^2, d^2}(x) \widehat{\Theta}_{(1^3), \frac{t^2}{d^2}} \widehat{\Psi}_{q^2}(x)| \ll NQ^{-6} R^{-2} T^{-6} U^{-2} X^\epsilon. \quad (63)$$

Here, and in what follows, the sum ranges over $q, r, t \in \mathbb{Z}$ such that $qrtU$ is squarefree.

We expand the definition of Ψ_{p^2} to be any of the functions $\Theta_{(1^3), p^2}$, $\Theta_{\text{nmax}, p^2}$ or $\Theta_{\text{div}^2, p^2}$, all of which satisfy the Fourier transform bounds of Proposition 5.1. We may simplify (63) by combining the T and Q variables. Since any product qt with $q \in [Q, 2Q]$ and $t \in [T, 2T]$ has $O((QT)^\epsilon)$ such factorizations, and any t in the sum has at most $O(X^\epsilon)$ divisors, it suffices to prove that

$$\sum_{q \in [Q, 2Q]} \sum_{r \in [R, 2R]} \sum_{|\text{Disc}(x)| < N} |\widehat{\Phi}_U(x) \widehat{\Phi}_r(x) \widehat{\Psi}_{q^2}(x)| \ll NQ^{-6} R^{-2} U^{-2} X^\epsilon. \quad (64)$$

(Our claim is initially reduced to proving a variation of (64) with a sum over $q \in [QT, 4QT]$ on the left and $(QT)^{-6}$ in place of Q^{-6} on the right; a change of variables and two applications of (64) yield such a bound.)

The Fourier transform $\widehat{\Phi}_m$ is multiplicative. We require the following bounds on the Fourier transforms of the constituent functions $\widehat{\Phi}_p$ for $p \mid m$ due to Mori [Mor10, Theorem 1]; see also [TT13a, Proposition 6.1] and [TT20, Theorem 11 and Corollary 12].

Proposition 6.1. *Let p be a prime, and let $\Phi_p : V(\mathbb{Z}/p\mathbb{Z}) \rightarrow \{0, 1\}$ be any of $\Phi_{(111), p}$, $\Phi_{(21), p}$, or $\Phi_{(3), p}$. Then*

$$|\widehat{\Phi}_p(x)| \ll \begin{cases} 1 & x = 0, \\ p^{-1} & x \text{ has a triple root modulo } p, \\ p^{-2} & \text{otherwise.} \end{cases} \quad (65)$$

Proposition 6.2. *Let p be a prime. The function $\Phi_{\text{div}, p} : V(\mathbb{Z}/p\mathbb{Z}) \rightarrow \{0, 1\}$ satisfies*

$$|\widehat{\Phi}_{\text{div}, p}(x)| \ll \begin{cases} p^{-1} & x = 0, \\ p^{-2} & x \neq 0, \quad p \mid \text{Disc}(x), \\ p^{-3} & \text{otherwise.} \end{cases} \quad (66)$$

Analogously to (52), we therefore obtain that (64) is bounded above by

$$X^\epsilon \sum_{c_2, c_1, d_{4c}, d_{4n}, d_3, d_2} \frac{1}{c_2^2 c_1^3 d_{4c}^2 d_{4n}^3 d_3^4 d_2^5} \sum_{r \in [R, 2R]} \sum_{\substack{|\text{Disc}(x')| < \frac{N}{c_2^8 c_1^4 d_{4c}^6 d_{4n}^2 d_3^2} \\ d_{4n}^2 d_3 d_2^2 |\text{Disc}(x')}} |\widehat{\Phi}_U(x') \widehat{\Phi}_r(x')|, \quad (67)$$

where the notation is as in (52), and in particular the outer sum is over all choices of the variables, squarefree and coprime to each other and to Ur , and the product is in $[Q, 2Q]$.

We now unravel the inner sum by summing over all factorizations $U = u_1 u_2 u_3$ and $r = r_1 r_2 r_3$, where $u_1 r_1 \mid x$, and $u_2 r_2 \nmid x$ but $u_2^2 r_2 \mid \text{Disc}(x)$, and then replacing x' with $x'' := \frac{x'}{u_1 r_1}$.

We thus see that the innermost sum is bounded above by

$$X^\epsilon \sum_{u_1 u_2 u_3 = U} \sum_{r_1 r_2 r_3 = r} \sum_{\substack{|\text{Disc}(x'')| < \frac{N}{c_2^8 c_1^4 d_{4c}^6 d_{4n}^2 d_3^2 u_1^4 r_1^4} \\ d_{4n}^2 d_3 d_2^2 u_2^2 r_2}} (u_2 r_1)^{-1} (u_3 r_2)^{-2} r_3^{-3},$$

so that the total sum is bounded above by

$$X^\epsilon \sum_{\substack{c_2, c_1, d_{4c}, d_{4n}, d_3, d_2 \\ u_1, u_2, u_3, r_1, r_2, r_3}} \frac{1}{c_2^2 c_1^3 d_{4c}^2 d_{4n}^3 d_3^4 d_2^5 u_2 r_1 u_3^2 r_2^3} \sum_{\substack{|\text{Disc}(x'')| < \frac{N}{c_2^8 c_1^4 d_{4c}^6 d_{4n}^2 d_3^2 u_1^4 r_1^4} \\ d_{4n}^2 d_3 d_2^2 u_2^2 r_2}} 1. \quad (68)$$

As before, for small N we sum the variable d_3 over dyadic intervals $[D_3, 2D_3]$, and we similarly sum r_2 over dyadic intervals $[R_2, 2R_2]$, and use the fact that there are at most $O(N^\epsilon) = O(X^\epsilon)$ pairs d_3 and r_2 with $d_3 r_2 \mid \text{Disc}(x'')$ for any fixed x'' with $|\text{Disc}(x'')| \ll N$. The above sum therefore reduces to

$$\begin{aligned} & X^\epsilon \sum_{D_3, R_2} \sum_{\substack{c_2, c_1, d_{4c}, d_{4n}, d_2 \\ u_1, u_2, u_3, r_1, r_3}} \frac{1}{c_2^2 c_1^3 d_{4c}^2 d_{4n}^3 D_3^4 d_2^5 u_2 r_1 u_3^2 R_2^3 r_3^3} \cdot \frac{N}{c_2^8 c_1^4 d_{4c}^6 d_{4n}^2 D_3^2 u_1^4 r_1^4} \cdot \frac{1}{d_{4n}^2 d_2^2 u_2^2} \\ &= X^\epsilon \sum_{D_3, R_2} \sum_{\substack{c_2, c_1, d_{4c}, d_{4n}, d_2 \\ u_1, u_2, u_3, r_1, r_3}} \frac{1}{c_2^{10} c_1^7 d_{4c}^8 d_{4n}^7 D_3^6 d_2^7} \cdot \frac{1}{u_1^4 u_2^3 u_3^2 r_1^5 R_2^2 r_3^3} \cdot N \\ &\ll X^\epsilon U^{-2} R^{-2} \sum_{D_3} \sum_{c_2, c_1, d_{4c}, d_{4n}, d_2} \frac{1}{c_2^{10} c_1^7 d_{4c}^8 d_{4n}^7 D_3^6 d_2^7} \cdot N \\ &\ll X^\epsilon Q^{-6} U^{-2} R^{-2} N, \end{aligned}$$

as desired, establishing (62). For large N , as before we apply Proposition 4.7 to obtain the same bound.

Collecting all of our error terms, we therefore have

$$\begin{aligned} E'_1 + E'_2 + E'_3 &\ll X^\epsilon \left(XQ^{-1}T^{-1} + X^{3/5} \sum_{Q_1} (Q_1^{-1}T^{-1})^{3/5} (R^2 T^2 U^2 Q_1^2)^{2/5} + XQ^{-1}T^{-1} \right) \quad (69) \\ &\ll X^\epsilon (X^{3/5} Q^{1/5} T^{1/5} R^{4/5} U^{4/5} + XQ^{-1}T^{-1}). \end{aligned}$$

We choose $Q = X^{1/3-\epsilon} T^{-1} R^{-2/3} U^{-2/3}$. We see that (36) is satisfied and that

$$E'_1 + E'_2 + E'_3 \ll X^{2/3+\epsilon} U^{2/3} R^{2/3}$$

as claimed in (60), completing the proof.

Remark 6.3. As stated in Theorem 1.3, our method can handle essentially arbitrary local conditions. Given any $\text{GL}_2(\mathbb{Z}/M'\mathbb{Z})$ -invariant function $\Phi_{M'} : V(\mathbb{Z}/M'\mathbb{Z}) \rightarrow \mathbb{C}$ with $(M', M) = 1$ and $|\Phi_{M'}(x)| \leq 1$ for all x , we may replace Φ_M and Φ_m by $\Phi_M \Phi_{M'}$ and $\Phi_m \Phi_{M'}$ throughout the proof.

In particular, for $p^a \parallel M'$ one may choose $\Phi_{p^a} : V(\mathbb{Z}/p^a\mathbb{Z}) \rightarrow \{0, 1\}$ corresponding to a fixed ramified cubic algebra over \mathbb{Q}_p , the possibilities for which were described below Theorem 1.4. Again insisting on maximality at p , we may always take $a = 2$ when $p > 3$.

The assumption that $U^2 R^2 T^3 < X^{1-\epsilon}$ may be strengthened to $U^2 R^2 T^3 M'^4 < X^{1-\epsilon}$. One must then obtain residue formulas for the Shintani zeta function (carried out in some cases in [TT13a, Section 8]), check condition (30), and prove an analogue of Proposition 4.2, showing

that the contribution of reducible rings is as expected. The remaining arguments are then the same; in (69), one multiplies the expression $R^2 T^2 U^2 Q_1^2$ by M'^4 , or by a smaller power of M' given suitable bounds on $\widehat{\Phi}_{M'}(x)$.

One thus obtains the counting function for all cubic orders satisfying the local conditions prescribed by Φ_M and $\Phi_{M'}$, and which are maximal except as prescribed by $\Phi_{M'}$.

7. PROOF OF THEOREM 1.1 USING A DISCRIMINANT-REDUCING IDENTITY

We now describe a “discriminant-reducing identity” for nonmaximal cubic rings. It is essentially equivalent to an identity in [BST13, Section 9.1], but we give a formulation and proof in the language of Shintani zeta functions. We then apply our identity to give a second proof of Theorem 1.1, but with the $O(X^{2/3+\epsilon})$ error term of Section 5 improved to $O(X^{2/3}(\log X)^\alpha)$ with any $\alpha > -\frac{1}{2} + \frac{5}{33/5}$. Note that our identity allows for the presence of unrelated local conditions, so that we could likely also obtain another proof of Theorems 1.3 and 1.4 in the case where $\Sigma_p = A_p$ at every ordinary prime.

In this section, for each squarefree integer q , we write $\Psi_{q^2} : V(\mathbb{Z}/q^2\mathbb{Z}) \rightarrow \{0, 1\}$ for the characteristic function of those $x \in V(\mathbb{Z}/q^2\mathbb{Z})$ that are nonmaximal at each prime divisor of q . We also write $\eta_q : V(\mathbb{Z}/q\mathbb{Z}) \rightarrow \mathbb{Z}$ for the function counting the number of roots of $x \in V(\mathbb{Z}/q\mathbb{Z})$ in $\mathbb{P}^1(\mathbb{Z}/q\mathbb{Z})$.

Proposition 7.1. *For any $\mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$ -invariant function $\Phi_M : V(\mathbb{Z}/M\mathbb{Z}) \rightarrow \mathbb{C}$ with $(M, q) = 1$, we have*

$$\xi^\pm(s, \Psi_{q^2} \otimes \Phi_M) = q^{-4s} \sum_{klm=q} \mu(l) k^{2s} \xi^\pm(s, \eta_{kl} \otimes \Phi_M). \quad (70)$$

Proof. By induction on the number of prime factors of q , it suffices to assume that $q = p$ is prime. For convenience write $\Gamma := \mathrm{GL}_2(\mathbb{Z})$, and define

$$\begin{aligned} X_p &= \{(x_1, x_2, x_3, x_4) \in V(\mathbb{Z}) : x_3 \equiv 0 \pmod{p}, x_4 \equiv 0 \pmod{p^2}\}, \\ Y^p &= \{(x_1, x_2, x_3, x_4) \in V(\mathbb{Z}) : x_1 \equiv 0 \pmod{p}\}, \\ B_p &= \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \Gamma : \gamma \equiv 0 \pmod{p} \right\}, \\ B^p &= \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \Gamma : \beta \equiv 0 \pmod{p} \right\}. \end{aligned}$$

Then B_p and B^p preserve X_p and Y^p respectively. The action of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1/p \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ induces a bijection $\phi : X_p \rightarrow Y^p$, such that if $gx = x'$ in X_p with $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in B_p$, then $\tilde{g}\phi(x) = \phi(x')$ with $\tilde{g} = \begin{bmatrix} \alpha & \beta p \\ \gamma/p & \delta \end{bmatrix} \in B^p$. Writing $B_{p,x}$ and B_x^p for $\mathrm{Stab}_{B_p}(x)$ and $\mathrm{Stab}_{B^p}(x)$ respectively, ϕ thus induces the equality

$$\sum_{\substack{x \in B_p \setminus X_p \\ \pm \mathrm{Disc}(x) > 0}} \frac{1}{|B_{p,x}|} \Phi_M(x) |\mathrm{Disc}(x)|^{-s} = p^{-2s} \sum_{\substack{x \in B^p \setminus Y^p \\ \pm \mathrm{Disc}(x) > 0}} \frac{1}{|B_x^p|} \Phi_M(x) |\mathrm{Disc}(x)|^{-s}. \quad (71)$$

We claim that the right side of (71) is $p^{-2s} \xi^\pm(s, \eta_p \otimes \Phi_M)$; it suffices to show that for each $x \in V(\mathbb{Z})$ that

$$\eta_p(x) = \sum_{y \in B^p \setminus (\Gamma x \cap Y^p)} [\Gamma_y : B_y^p]. \quad (72)$$

Recall that there is a transitive right action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$, given by matrix multiplication when we represent a point $(t_0 : t_1) \in \mathbb{P}^1$ as a row vector. With this action, B^p is

the stabilizer of $(1 : 0) \in \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$. We identify $B^p \backslash \Gamma$ with $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$ via $[g] \mapsto (1 : 0)g$, so that

$$\eta_p(x) = \#\{[g] \in B^p \backslash \Gamma \mid gx \in Y^p\}.$$

Consider the map

$$\{[g] \in B^p \backslash \Gamma \mid gx \in Y^p\} \longrightarrow B^p \backslash (\Gamma x \cap Y^p), \quad [g] \longmapsto [gx].$$

It is surjective by construction. Writing $y = gx$, we have $B^p y = B^p g'x$ if and only if $y = hg'g^{-1}y$ for some $h \in B^p$, i.e., if and only if $g' \in B^p \Gamma_y g$. Therefore, the fibers each have size

$$|B^p \backslash B^p \Gamma_y g| = |B^p \backslash B^p \Gamma_y| = |B^p \cap \Gamma_y \backslash \Gamma_y| = [\Gamma_y : B_y^p],$$

proving (72).

To conclude, we show that the left side of (71) is

$$\xi^\pm(s, \Psi_{p^2} \otimes \Phi_M) - p^{-4s} \xi^\pm(s, \Phi_M) + p^{-4s} \xi^\pm(s, \eta_p \otimes \Phi_M). \quad (73)$$

For those $x \in X_p$ not in $pV(\mathbb{Z})$, it is readily checked that $\Gamma_x = B_{p,x}$, so that the contribution of such x is equal to $\xi^\pm(s, \Psi_{p^2} \otimes \Phi_M) - p^{-4s} \xi^\pm(s, \Phi_M)$ by Proposition 2.2. The contribution from those $x \in X_p \cap pV(\mathbb{Z})$ is

$$p^{-4s} \sum_{\substack{x \in B_p \backslash Y_p \\ \pm \text{Disc}(x) > 0}} \frac{1}{|B_{p,x}|} \Phi_M(x) |\text{Disc}(x)|^{-s}, \quad (74)$$

where $Y_p = \{(x_1, x_2, x_3, x_4) \in V(\mathbb{Z}) : x_4 \equiv 0 \pmod{p}\}$. Let $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \Gamma$. Then since $Y^p = \gamma Y_p$ and $B^p = \gamma B_p \gamma^{-1}$, the bijection $Y_p \ni x \mapsto \gamma x \in Y^p$ induces the bijection $B_p \backslash Y_p \rightarrow B^p \backslash Y^p$, and the sum in (74) coincides with the sum in the right side of (71). Thus the contribution (74) is $p^{-4s} \xi^\pm(s, \eta_p \otimes \Phi_M)$. \square

To give another proof of Theorem 1.1, we again apply Landau's method and treat the error terms E_1 , E_2 , and E_3 from (45). To avoid $O(X^\epsilon)$ factors in the error terms, we recall several standard bounds from analytic number theory. We have

$$\sum_{q \leq Q} \mu^2(q) C^{\omega(q)} \ll Q (\log Q)^{C-1}, \quad (75)$$

which can be proved (for example) as in [IK04, p. 24]. We also have

$$\sum_{q > Q} \mu^2(q) \frac{C^{\omega(q)}}{q^\alpha} \ll \frac{(\log Q)^{C-1}}{Q^{\alpha-1}} \quad (76)$$

for each $\alpha > 1$, as can be seen by subdividing the interval (Q, ∞) into dyadic subintervals and applying (75) to each. Finally, we have

$$\prod_{p|n} \left(1 + \frac{1}{p}\right) \ll \log \log n \quad (77)$$

as a consequence of Gronwall's classical bound $\sigma(n) \ll n \log \log n$ [Gro13].

We now return to (45) with $M = 1$. By (22) and (76), we obtain

$$E_1 \ll X \sum_{q > Q} \prod_{p|q} (2p^{-2}) + X^{5/6} \sum_{q > Q} \prod_{p|q} (2p^{-5/3}) \ll \frac{X(\log Q)}{Q} + \frac{X^{5/6}(\log Q)}{Q^{2/3}}.$$

For E_3 , by using Proposition 4.6 in place of Proposition 4.5, we have

$$E_3 \ll X \sum_{q>Q} 3^{\omega(q)} q^{-2} \ll \frac{X(\log Q)^2}{Q}.$$

To treat the E_2 error term, we apply Theorem 3.1 (instead of Theorem 3.2) to each Shintani zeta function in the summation in (70) with $M = 1$. This gives

$$E_2 := X^{3/5} \sum_{q \leq Q} \sum_{klm=q} (q^{-4} k^2)^{3/5} (\delta_1(\eta_{kl}))^{3/5} (\widehat{\delta}_1(\eta_{kl}))^{2/5}. \quad (78)$$

We have $\delta_1(\eta_{kl}) \ll 2^{\omega(kl)}$ by (26), and by definition we have

$$\widehat{\delta}_1(\eta_{kl}) \leq (kl)^4 \sup_N \frac{1}{N} \sum_{\alpha \in \{\pm\}} \sum_{n < N} a^\alpha(|\widehat{\eta}_{kl}|, n).$$

We will prove, for each squarefree d and $N > 1$, that

$$\sum_{\alpha \in \{\pm\}} \sum_{n < N} a^\alpha(|\widehat{\eta}_d|, n) \ll Nd^{-3} 6^{\omega(d)} \log \log d. \quad (79)$$

Granting this for now, we have

$$E_2 \ll X^{3/5} \sum_{q \leq Q} \sum_{klm=q} (q^{-4} k^2)^{3/5} (2^{\omega(kl)})^{3/5} (kl 6^{\omega(kl)} \log \log(kl))^{2/5} \quad (80)$$

$$\ll X^{3/5} \sum_{q \leq Q} q^{-4/5} 2^{\frac{3}{5}\omega(q)} 6^{\frac{2}{5}\omega(q)} (\log \log q)^{2/5}$$

$$\ll X^{3/5} Q^{1/5} (\log Q)^{2 \cdot 3^{2/5} - 1} (\log \log Q)^{2/5}, \quad (81)$$

so that choosing $Q = X^{1/3} (\log X)^\beta$ with $\beta = \frac{5}{6}(3 - 2 \cdot 3^{2/5}) = -.086\dots$ yields $E_1 + E_2 + E_3 \ll X^{2/3} (\log X)^\alpha$ for any $\alpha > 2 - \beta = 2.086\dots$, which implies Theorem 1.1.

It remains to prove (79). For this, we first recall the following formula for $\widehat{\eta}_d(x)$, first (essentially) proved by Mori [Mor10], with a second simpler proof given by the second and third authors in [TT20, Proposition 1]:

Proposition 7.2. *The function $\widehat{\eta}_d(x)$ is multiplicative in d , and for a prime $p \neq 3$ we have*

$$\widehat{\eta}_p(x) = \begin{cases} 1 + p^{-1} & x = 0, \\ p^{-1} & x \text{ has a triple root modulo } p, \\ 0 & \text{otherwise,} \end{cases} \quad (82)$$

where we regard x as a point in $V_{\mathbb{Z}/p\mathbb{Z}}$.

We now prove (79). Replacing d by $d/3$ if necessary, we may assume that d is coprime to 3. We then sum over all factorizations $d = d_1 d_2$, and consider the contribution of those x which reduce to zero in $\mathbb{Z}/d_1\mathbb{Z}$ and which have a triple root in $\mathbb{Z}/p\mathbb{Z}$ for each $p \mid d_2$.

By Proposition 4.5, the number of such x with $|\text{Disc}(x)| < N$ is $\ll \frac{N6^{\omega(d_2)}}{d_1^4 d_2^2}$, and so

$$\begin{aligned} \sum_{|\text{Disc}(x)| < N} |\widehat{\eta}_d(x)| &\ll \sum_{d=d_1 d_2} \left(\prod_{p|d_1} (1+p^{-1}) \prod_{p|d_2} p^{-1} \right) \frac{N6^{\omega(d_2)}}{d_1^4 d_2^2} \\ &= Nd^{-3} 6^{\omega(d)} \prod_{p|d} \left(1 + \frac{1}{6} \cdot \frac{1+p^{-1}}{p} \right) \\ &\ll Nd^{-3} 6^{\omega(d)} \log \log d, \end{aligned}$$

completing the proof of Theorem 1.1.

8. COUNTING QUADRATIC FIELDS

In this section, we prove Proposition 4.2 which estimates the number of quadratic fields of bounded discriminant satisfying a prescribed set of local conditions. As we explain, this amounts to counting squarefree integers in $[1, X]$ in suitable arithmetic progressions.

We begin by formulating our results precisely. Recall that a *local specification* Σ_p at a prime p is a subset of the cubic étale algebras over \mathbb{Q}_p ; we say that a cubic field F satisfies the local specification Σ_p if $F \otimes_{\mathbb{Q}} \mathbb{Q}_p \in \Sigma_p$, and that a quadratic field F_2 satisfies the local specification Σ_p if the algebra $F_2 \times \mathbb{Q}$ does.

For quadratic fields, we may assume without loss of generality that Σ_p contains algebras only of the form $(F_2)_p \times \mathbb{Q}_p$, where $(F_2)_p$ is either $\mathbb{Q}_p \times \mathbb{Q}_p$ or a quadratic field extension of \mathbb{Q}_p . In this section, we abuse notation and also write Σ_p for the set of $(F_2)_p$ with $(F_2)_p \times \mathbb{Q}_p \in \Sigma_p$.

Let $p \neq 2$ and let l be an arbitrary non-square element in \mathbb{Z}_p^\times . Then it is immediately checked that the left column of the following table enumerates the possibilities for $(F_2)_p$:

$(F_2)_p$	Condition on $d = \text{Disc}(F_2)$	Density c_p
$\mathbb{Q}_p \times \mathbb{Q}_p$	$\left(\frac{d}{p}\right) = 1$	$\frac{1}{2}(1+p^{-1})^{-1}$
$\mathbb{Q}_p(\sqrt{l})$	$\left(\frac{d}{p}\right) = -1$	$\frac{1}{2}(1+p^{-1})^{-1}$
$\mathbb{Q}_p(\sqrt{lp})$	$\left(\frac{d/lp}{p}\right) = 1$	$\frac{1}{2}(p+1)^{-1}$
$\mathbb{Q}_p(\sqrt{l^2 p})$	$\left(\frac{d/l^2 p}{p}\right) = -1$	$\frac{1}{2}(p+1)^{-1}$

For each $(F_2)_p$, we write $\varepsilon_p \in \{\pm 1\}$ for the value of the Legendre symbol in the second column. Note that $F_2 \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is determined by the value of $\text{Disc}(F_2) \pmod{p^2}$, and by $\text{Disc}(F_2) \pmod{p}$ when F_2 is unramified.

If $p = 2$, then there are eight possibilities for $(F_2)_2$; with the exception of the first row, the following table lists a generating polynomial for $(F_2)_2$:

$(F_2)_2$	Condition on $d = \text{Disc}(F_2)$	Density c_2
$\mathbb{Q}_2 \times \mathbb{Q}_2$	$d \equiv 1 \pmod{8}$	$1/3$
$x^2 - x + 1$	$d \equiv 5 \pmod{8}$	$1/3$
$x^2 + 2x \pm 2$	$d \equiv 4 \mp 8 \pmod{32}$	$1/12$
$x^2 + a$ ($a = \pm 2, \pm 6$)	$d \equiv -4a \pmod{64}$	$1/24$

To verify this, one checks that: the polynomials in the left column satisfy the discriminant conditions in the middle column; these conditions are mutually exclusive; any $\alpha \in \mathbb{Q}_2^\times$ is one of the d listed above times a nonzero 2-adic square; and these conditions cover all residue classes $\pmod{64}$ that can be discriminants of quadratic fields.

Thus the quadratic fields enumerated by $N_2^\pm(X, \Sigma)$ are in bijection with fundamental discriminants that satisfy appropriate congruence conditions. To distinguish among the possibilities for $F_2 \otimes_{\mathbb{Q}} \mathbb{Q}_2$, we assume a congruence condition of the form $\text{Disc}(F_2) \equiv a \pmod{64}$. Hereafter we assume that Σ is a collection of local specifications over odd primes only, and write $N_2^\pm(X; a, \Sigma)$ for the count of fields counted by $N_2^\pm(X, \Sigma)$ with $\text{Disc}(F_2) \equiv a \pmod{64}$.

We assume that for sufficiently large primes p , the local specification Σ_p consists of all quadratic étale \mathbb{Q}_p -algebras, and hence imposes no condition on F_2 . At each remaining odd prime p , we assume that either (1) Σ_p consists of a single algebra, or (2) $\Sigma_p = \{\mathbb{Q}_p(\sqrt{p}), \mathbb{Q}_p(\sqrt{lp})\}$, in which case F_2 satisfies the local specification Σ_p if and only if p ramifies in F_2 . For $\Sigma = (\Sigma_p)_p$, we write: u for the product of primes p where Σ_p consists of a single unramified algebra; s for the product of primes p where Σ_p consists of a single ramified algebra; and r for the product of primes p where $\Sigma_p = \{\mathbb{Q}_p(\sqrt{p}), \mathbb{Q}_p(\sqrt{lp})\}$. Then

$$N_2^\pm(X; a, \Sigma) = \# \left\{ d \in \mathbb{Z} \left| \begin{array}{l} 0 < \pm d < X \\ d \equiv a \pmod{64} \\ p \nmid d, \left(\frac{d}{p}\right) = \varepsilon_p \text{ for all } p \mid u \\ p \parallel d, \left(\frac{d/p}{p}\right) = \varepsilon_p \text{ for all } p \mid s \\ p \parallel d \text{ for all } p \mid r \\ d \text{ is not divisible by } p^2 \text{ for any } p \nmid 2usr \end{array} \right. \right\}, \quad (83)$$

where for each $p \mid us$, $\varepsilon_p \in \{\pm 1\}$ is chosen depending on Σ_p . To count this, define

$$N^\pm(X; a, \Sigma; q) := \# \left\{ d \in \mathbb{Z} \left| \begin{array}{l} 0 < \pm d < X \\ d \equiv a \pmod{64} \\ d \text{ is divisible by } srq^2 \text{ and } (d/r, r) = 1 \\ p \nmid d, \left(\frac{d}{p}\right) = \varepsilon_p \text{ for all } p \mid u \\ p \parallel d, \left(\frac{d/p}{p}\right) = \varepsilon_p \text{ for all } p \mid s \end{array} \right. \right\},$$

where $\varepsilon = (\varepsilon_p)_{p \mid us}$, for each squarefree integer q . By inclusion-exclusion, we have

$$N_2^\pm(X; a, \Sigma) = \sum_q \mu(q) N^\pm(X; a, \Sigma; q), \quad (84)$$

where q runs through all squarefree integers coprime to $2usr$.

Lemma 8.1. *We have*

$$N^\pm(X; a, \Sigma; q) = \frac{1}{64} \prod_{p \mid us} \frac{1 - p^{-1}}{2} \prod_{p \mid r} (1 - p^{-1}) \frac{X}{srq^2} + O(2^{\omega(r)} \sqrt{us} \log(us)), \quad (85)$$

$$N^\pm(X; a, \Sigma; q) = O\left(\frac{X}{srq^2}\right). \quad (86)$$

Proof. The estimate (86) is immediate. To prove (85), note that by definition we have

$$N^\pm(X; a, \Sigma; q) = \# \left\{ n \in \mathbb{Z} \left| \begin{array}{l} 0 < \pm n < X/srq^2 \\ n \equiv a' \pmod{64} \\ \left(\frac{n}{p}\right) = \varepsilon'_p \text{ for all } p \mid us \\ (n, r) = 1 \end{array} \right. \right\},$$

where $a' := a(sr)^{-1}q^{-2} \pmod{64}$ and

$$\varepsilon'_p := \begin{cases} \left(\frac{rs}{p}\right)\varepsilon_p & \text{if } p \mid u, \\ \left(\frac{rs/p}{p}\right)\varepsilon_p & \text{if } p \nmid u. \end{cases} \quad (87)$$

Expanding the conditions on n in terms of Dirichlet characters, and writing $a' = bb'$ with $b := (a', 64)$, we obtain

$$\begin{aligned} N^\pm(X; a, \Sigma; q) &= \sum_{\substack{0 < n < \frac{X}{bsrq^2} \\ (n, 2usr) = 1}} \frac{b}{32} \sum_{\chi_2 \pmod{\frac{64}{b}}} \overline{\chi_2}(b') \chi_2(\pm n) \prod_{p \mid us} \left(\frac{1 + \varepsilon'_p \chi_p(\pm nb)}{2} \right) \\ &= \frac{b}{2^{5+\omega(us)}} \sum_{\chi_2 \pmod{\frac{64}{b}}} \overline{\chi_2}(b') \chi_2(\pm 1) \sum_{m \mid us} \varepsilon'_m \chi_m(\pm b) \sum_{0 < n < \frac{X}{bsrq^2}} \chi_2(n) \chi_m(n) \chi_{0,usr/m}(n) \end{aligned}$$

where $\varepsilon'_m := \prod_{p \mid m} \varepsilon'_p$, $\chi_m := \prod_{p \mid m} \left(\frac{\cdot}{p}\right)$, χ_2 runs over all characters $\pmod{\frac{64}{b}}$, and $\chi_{0,usr/m}$ is the principal character $\pmod{usr/m}$.

By the Pólya-Vinogradov inequality (for imprimitive characters; see, e.g., [Dav80, Chapter 23]), the inner sum is $\ll 2^{\omega(usr)} \sqrt{us} \log(us)$ except when χ_2 is principal and $m = 1$. Therefore, the expression above simplifies to

$$\begin{aligned} N^\pm(X; a, \Sigma; q) &= O(2^{\omega(r)} \sqrt{us} \log(us)) + \frac{b}{2^{5+\omega(us)}} \sum_{\substack{0 < n < X/bsrq^2 \\ (n, 2usr) = 1}} 1 \\ &= O(2^{\omega(r)} \sqrt{us} \log(us)) + \frac{b}{2^{5+\omega(us)}} \cdot \frac{\phi(2usr)}{2usr} \cdot \frac{X}{bsrq^2} \\ &= O(2^{\omega(r)} \sqrt{us} \log(us)) + \frac{1}{64} \prod_{p \mid us} \frac{1-p^{-1}}{2} \prod_{p \mid r} (1-p^{-1}) \frac{X}{srq^2}. \end{aligned}$$

Note that $\#\{0 < n < X \mid (n, r) = 1\} = \frac{\phi(r)}{r} X + O(2^{\omega(r)})$. \square

Proof of Proposition 4.2. We decompose $N_2^\pm(X, \Sigma)$ into $O(U^\epsilon)$ counts $N_2^\pm(X; a, \Sigma)$, where Σ satisfies the assumptions stated before (83). By (84), we then have for any Q that

$$\begin{aligned} N_2^\pm(X; a, \Sigma) &= \frac{1}{64} \prod_{p \mid us} \frac{1-p^{-1}}{2} \prod_{p \mid r} (1-p^{-1}) \sum_{\substack{q \leq Q \\ (q, 2usr) = 1}} \frac{\mu(q)}{q^2} \frac{X}{sr} + O(Q \cdot 2^{\omega(r)} \sqrt{us} \log(us)) + \sum_{q > Q} O\left(\frac{X}{srq^2}\right) \\ &= \frac{1}{64} \prod_{p \mid us} \frac{1-p^{-1}}{2} \prod_{p \mid r} (1-p^{-1}) \prod_{p \nmid 2usr} (1-p^{-2}) \frac{X}{sr} + O\left(Q \cdot 2^{\omega(r)} \sqrt{us} \log(us) + \frac{X}{srQ}\right). \end{aligned}$$

Taking $Q = X^{1/2} u^{-1/4} s^{-3/4} r^{-1/2}$, we obtain an error term of $O(X^{1/2} u^{1/4} s^{-1/4} r^{-1/2+\epsilon} \log(us))$. This yields Proposition 4.2, with the densities c_p corresponding to the main term above. \square

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REFERENCES

- [BBP10] Karim Belabas, Manjul Bhargava, and Carl Pomerance. Error estimates for the Davenport–Heilbronn theorems. *Duke Math. J.*, 153(1):173–210, 2010.
- [Bel97] K. Belabas. A fast algorithm to compute cubic fields. *Math. Comp.*, 66(219):1213–1237, 1997.
- [BF10] Karim Belabas and Étienne Fouvry. Discriminants cubiques et progressions arithmétiques. *Int. J. Number Theory*, 6(7):1491–1529, 2010.
- [BST13] Manjul Bhargava, Arul Shankar, and Jacob Tsimerman. On the Davenport–Heilbronn theorems and second order terms. *Invent. Math.*, 193(2):439–499, 2013.
- [BW08] Manjul Bhargava and Melanie Matchett Wood. The density of discriminants of S_3 -sextic number fields. *Proc. Amer. Math. Soc.*, 136(5):1581–1587, 2008.
- [CFLS21] Peter J. Cho, Daniel Fiorilli, Yoonbok Lee, and Anders Södergren. Omega results for cubic field counts via lower-order terms in the one-level density. *Preprint available at <https://arxiv.org/abs/2102.08077>*, 2021.
- [CFZ08] Brian Conrey, David W. Farmer, and Martin R. Zirnbauer. Autocorrelation of ratios of L -functions. *Commun. Number Theory Phys.*, 2(3):593–636, 2008.
- [CK15] Peter J. Cho and Henry H. Kim. Low lying zeros of Artin L -functions. *Math. Z.*, 279(3-4):669–688, 2015.
- [CN62] K. Chandrasekharan and Raghavan Narasimhan. Functional equations with multiple gamma factors and the average order of arithmetical functions. *Ann. of Math. (2)*, 76:93–136, 1962.
- [Coh54] Harvey Cohn. The density of abelian cubic fields. *Proc. Amer. Math. Soc.*, 5:476–477, 1954.
- [Dav80] Harold Davenport. *Multiplicative number theory*, volume 74 of *Graduate Texts in Mathematics*. Springer-Verlag, New York–Berlin, second edition, 1980. Revised by Hugh L. Montgomery.
- [DF64] B. N. Delone and D. K. Faddeev. *The theory of irrationalities of the third degree*. Translations of Mathematical Monographs, Vol. 10. American Mathematical Society, Providence, R.I., 1964.
- [DH71] H. Davenport and H. Heilbronn. On the density of discriminants of cubic fields. II. *Proc. Roy. Soc. London Ser. A*, 322(1551):405–420, 1971.
- [DW86] Boris Datskovsky and David J. Wright. The adelic zeta function associated to the space of binary cubic forms. II. Local theory. *J. Reine Angew. Math.*, 367:27–75, 1986.
- [DW88] Boris Datskovsky and David J. Wright. Density of discriminants of cubic extensions. *J. Reine Angew. Math.*, 386:116–138, 1988.
- [EPW17] Jordan Ellenberg, Lillian B. Pierce, and Melanie Matchett Wood. On ℓ -torsion in class groups of number fields. *Algebra Number Theory*, 11(8):1739–1778, 2017.
- [GG02] Wee Teck Gan, Benedict Gross, and Gordan Savin. Fourier coefficients of modular forms on G_2 . *Duke Math. J.*, 115(1):105–169, 2002.
- [Gro13] T. H. Gronwall. Some asymptotic expressions in the theory of numbers. *Trans. Amer. Math. Soc.*, 14(1):113–122, 1913.

- [Has48] Helmut Hasse. Arithmetische Bestimmung von Grundeinheit und Klassenzahl in zyklischen kubischen und biquadratischen Zahlkörpern. Abh. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl., 1948(2):95 pp. (1950), 1948.
- [Hou19] Bob Hough. Equidistribution of bounded torsion CM points. J. Anal. Math., 138(2):765–797, 2019.
- [IK04] Henryk Iwaniec and Emmanuel Kowalski. Analytic number theory, volume 53 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004.
- [Lan12] Edmund Landau. Über die Anzahl der Gitterpunkte in gewissen Bereichen. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, pages 687–770, 1912.
- [Lan15] Edmund Landau. Über die Anzahl der Gitterpunkte in gewissen Bereichen. Zweite abhandlung. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, pages 209–243, 1915.
- [LDTT] David Lowry-Duda, Takashi Taniguchi, and Frank Thorne. Uniform bounds for lattice point counting and partial sums of zeta functions. Preprint (2017), available at <https://arxiv.org/abs/1710.02190>.
- [Lev14] F. Levi. Kubische Zahlkörper und binäre kubische Formenklassen. Leipz. Ber., 66:26–37, 1914.
- [Mor10] Shingo Mori. Orbital Gauss sums associated with the space of binary cubic forms over a finite field. RIMS Kōkyūroku, 1715:32–36, 2010.
- [MT] Kevin McGown and Amanda Tucker. The statistics of the genus numbers of cubic fields. Preprint (2016), available at <https://arxiv.org/abs/1611.07088>.
- [Nak98] Jin Nakagawa. On the relations among the class numbers of binary cubic forms. Invent. Math., 134(1):101–138, 1998.
- [O’D17] Evan O’Dorney. On a remarkable identity in class numbers of cubic rings. J. Number Theory, 176:302–332, 2017.
- [Ohn97] Yasuo Ohno. A conjecture on coincidence among the zeta functions associated with the space of binary cubic forms. Amer. J. Math., 119(5):1083–1094, 1997.
- [Rob01] David P. Roberts. Density of cubic field discriminants. Math. Comp., 70(236):1699–1705, 2001.
- [Sat89] Fumihito Satō. On functional equations of zeta distributions. In Automorphic forms and geometry of arithmetic varieties, volume 15 of Adv. Stud. Pure Math., pages 465–508. Academic Press, Boston, MA, 1989.
- [Shi72] Takuro Shintani. On Dirichlet series whose coefficients are class numbers of integral binary cubic forms. J. Math. Soc. Japan, 24:132–188, 1972.
- [SS74] Mikio Sato and Takuro Shintani. On zeta functions associated with prehomogeneous vector spaces. Ann. of Math. (2), 100:131–170, 1974.
- [SST19] Arul Shankar, Anders Södergren, and Nicolas Templier. Sato-Tate equidistribution of certain families of Artin L -functions. Forum Math. Sigma, 7:Paper No. e23, 62, 2019.
- [ST14] Arul Shankar and Jacob Tsimerman. Counting S_5 -fields with a power saving error term. Forum Math. Sigma, 2:e13, 8, 2014.
- [TT13a] Takashi Taniguchi and Frank Thorne. Orbital L -functions for the space of binary cubic forms. Canad. J. Math., 65(6):1320–1383, 2013.
- [TT13b] Takashi Taniguchi and Frank Thorne. Secondary terms in counting functions for cubic fields. Duke Math. J., 162(13):2451–2508, 2013.
- [TT14] Takashi Taniguchi and Frank Thorne. An error estimate for counting S_3 -sextic number fields. Int. J. Number Theory, 10(4):935–948, 2014.
- [TT20] Takashi Taniguchi and Frank Thorne. Orbital exponential sums for prehomogeneous vector spaces. Amer. J. Math., 142(1):177–213, 2020.
- [Wan] Jiuya Wang. Secondary term of asymptotic distribution of $S_3 \times A$ extensions over \mathbb{Q} . Preprint (2017), available at <https://arxiv.org/abs/1710.10693>.
- [Wri85] David J. Wright. The adelic zeta function associated to the space of binary cubic forms. I. Global theory. Math. Ann., 270(4):503–534, 1985.
- [Yan09] Andrew Yang. Distribution problems associated to zeta functions and invariant theory. 2009. Thesis (Ph.D.)—Princeton University.

- [Zha13] Yongqiang Zhao. On sieve methods for varieties over finite fields. 2013. Thesis (Ph.D.)–University of Wisconsin-Madison.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ, USA

E-mail address: `bhargava@math.princeton.edu`

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, KOBE UNIVERSITY, KOBE, JAPAN

E-mail address: `tani@math.kobe-u.ac.jp`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SC, USA

E-mail address: `thorne@math.sc.edu`