SECONDARY TERMS IN COUNTING FUNCTIONS FOR CUBIC FIELDS

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Abstract. We prove the existence of secondary terms of order $X^{5/6}$ in the Davenport-Heilbronn theorems on cubic fields and 3-torsion in class groups of quadratic fields. For cubic fields this confirms a conjecture of Datskovsky-Wright and Roberts. We also prove a variety of generalizations, including to arithmetic progressions, where we discover a curious bias in the secondary term.

Roberts' conjecture has also been proved independently by Bhargava, Shankar, and Tsimerman. In contrast to their work, our proof uses the analytic theory of zeta functions associated to the space of binary cubic forms, developed by Shintani and Datskovsky-Wright.

1. Introduction

Let $N_3^{\pm}(X)$ count the number of cubic fields K with $0 < \pm\mathrm{Disc}(K) < X$. In [30], Roberts conjectured that

$$N_3^{\pm}(X) = C^{\pm}\frac{1}{12\zeta(3)}X + K^{\pm}\frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + o(X^{5/6}), \tag{1.1}$$

where $C^- = 3$, $C^+ = 1$, $K^- = \sqrt{3}$, and $K^+ = 1$. This conjecture also appeared implicitly in an earlier paper of Datskovsky and Wright [11]. It was based on a combination of numerical and theoretical evidence, the latter of which will be described in due course.

In this paper we will prove the conjecture, with an additional power savings in the error term:

Theorem 1.1. *Roberts' conjecture is true. Indeed, we have*

$$N_3^{\pm}(X) = C^{\pm}\frac{1}{12\zeta(3)}X + K^{\pm}\frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O(X^{7/9+\epsilon}). \tag{1.2}$$

The main term is due to Davenport and Heilbronn [13], and (1.2) improves upon a result of Belabas, Bhargava, and Pomerance [3], who obtained an error term of $O(X^{7/8+\epsilon})$. Although their methods were not designed to extract the secondary term in (1.2), our approach nevertheless owes a great deal to theirs.

Remark. Roberts' conjecture has also been proved, with an error term of $O(X^{13/16+\epsilon})$, in recent and independent work of Bhargava, Shankar, and Tsimerman [7]. They also obtain versions of Theorems 1.2, 1.3, and 1.4.

Our methods extend to the related problem of counting 3-torsion in quadratic fields. For any quadratic field with discriminant D, let $\mathrm{Cl}_3(D)$ denote the 3-torsion subgroup of the ideal class group $\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))$. We will prove the following result.

Theorem 1.2. *We have*

$$\sum_{0<\pm D<X} \#\mathrm{Cl}_3(D) = \frac{3+C^{\pm}}{\pi^2}X + K^{\pm}\frac{8\zeta(1/3)}{5\Gamma(2/3)^3}\prod_p\left(1-\frac{p^{1/3}+1}{p(p+1)}\right)X^{5/6} + O(X^{18/23+\epsilon}), \tag{1.3}$$

and where the sum ranges over fundamental discriminants D, the product is over all primes, and the constants are as before.

As with (1.2), the main term is due to Davenport and Heilbronn [13], and an error term of $O(X^{7/8+\epsilon})$ was proved by Belabas, Bhargava, and Pomerance [3]. Our error term is slightly higher than that in (1.2) due to an additional technical complication which appears in the proof.

Remark. Extensive computational results for the 3-parts of class groups appear in [19, 2]. For smaller X it is now practical to check (1.3) numerically using PARI/GP [29]. For example, for $D > 0$ and $X = 10^6$, the left side of (1.3) is 381071, and the main terms on the right sum to $381337.24\cdots$. For $D < 0$ these values are 566398 and $566448.83\cdots$ respectively.

As conjectured by Roberts, our results also extend to counting problems where various local restrictions are imposed. This is perhaps most interesting in the case of counting fields. Let $\mathcal{S} = (\mathcal{S}_p)$ be a finite set of *local specifications*. In particular, we may require that K be inert, partially ramified, totally ramified, partially split, or totally split at p. More generally, we may specify the p-adic completion K_p.

We will prove the following quantitative version of Roberts' extended conjecture:

Theorem 1.3. *With the notation above, the number of cubic fields K satisfying $\mathcal{S}$ with $0 < \pm\mathrm{Disc}(K) < X$ is*

$$N_3^{\pm}(X, \mathcal{S}) = C^{\pm}(\mathcal{S}) \frac{1}{12\zeta(3)} X + K^{\pm}(\mathcal{S}) \frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} X^{5/6} + O\Big(X^{7/9+\epsilon} \prod_p p^{8e_p/9}\Big), \tag{1.4}$$

where $e_p = 0$ if there is no specification at p, $e_p = 1$ if we only count fields unramified at p, and $e_p = 2$ otherwise, and the constants $C^{\pm}(\mathcal{S})$ and $K^{\pm}(\mathcal{S})$ are computed explicitly in Section 6.

We obtain a similar generalization of Theorem 1.2. In this case, we may restrict our sum to D for which finitely many primes p are inert, split, or ramified. In the ramified case we may also specify the completion of $\mathbb{Q}(\sqrt{D})$ at p. We will prove the following result:

Theorem 1.4. *With the notation above, we have*

$$\sum_{0<\pm D<X}{}' \#\mathrm{Cl}_3(D) = \frac{3 + C^{\pm}}{\pi^2} C'(\mathcal{S}) X + K'^{\pm}(\mathcal{S}) \frac{8\zeta(1/3)}{5\Gamma(2/3)^3} \prod_p \left(1 - \frac{p^{1/3}+1}{p(p+1)}\right) X^{5/6} + O\Big(X^{18/23+\epsilon} \prod_p p^{20e_p/23}\Big), \tag{1.5}$$

where the sums are restricted to discriminants meeting the conditions specified by $\mathcal{S}$, and the constants $C'(\mathcal{S})$ and $K'^{\pm}(\mathcal{S})$ are computed explicitly in Section 6.

Finally, our results allow us to count discriminants in arithmetic progressions. Here we encounter a curious phenomenon. Consider the following table of discriminants of cubic fields K with $0 < \mathrm{Disc}(K) < 2 \cdot 10^6$, arranged by their residue class modulo 7:

Discriminant modulo 7	Count
0	15330
1	17229
2	14327
3	15323
4	17027
5	18058
6	15150

This data shows a striking lack of equidistribution, and as related experiments confirm, this is not a fluke. The primary term in the counting function is the same for each residue class other than 0, but we will prove that the secondary term in the counting function is different for each residue class modulo 7. More generally, when there exist cubic Dirichlet characters modulo m (equivalently, when m is divisible by 9 or any prime $\equiv 1 \pmod 6$), the secondary term depends on these characters in a subtle way, and we obtain biases in the distribution of fields in progressions modulo m.

Our general result (Theorem 6.2) also allows local specifications and is rather complicated to state; the following is a special case.

Theorem 1.5. *Suppose that* $(6a, m) = 1$. *Then the number of cubic fields* K *with* $0 < \pm\mathrm{Disc}(K) < X$ *and* $\mathrm{Disc}(K) \equiv a \pmod m$ *is*

$$N_3^{\pm}(X; m, a) = \frac{C^{\pm}}{12\zeta(3)m} \prod_{p|m} \frac{1}{1 - p^{-3}} X + K_1(m, a) \frac{4K^{\pm}}{5\Gamma(2/3)^3} X^{5/6} + O(X^{7/9+\epsilon} m^{8/9}), \tag{1.6}$$

where

$$K_1(m, a) := \frac{1}{m} \prod_{p|m} \frac{1}{1 - p^{-2}} \sideset{}{'}\sum_{\chi^6 = 1} \overline{\chi}(a) \frac{L(1/3, \chi^{-2})}{L(5/3, \chi^2)} \prod_{\substack{p|m \\ p\nmid \mathrm{cond}(\chi)}} \frac{1 - \chi(p)^{-2} p^{-4/3}}{1 - \chi(p)^2 p^{-5/3}} \prod_{\substack{p|m \\ p|\mathrm{cond}(\chi)}} \frac{\tau_p(\chi_p^2)^3}{p^2}. \tag{1.7}$$

Here $\tau_p(\chi_p^2) = \sum_{t \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi_p^2(t) e^{2\pi i t/p}$, *and the sum is over primitive characters* χ *to moduli dividing* m *(including the trivial character modulo* 1*), such that if we write* $\chi = \prod_{p|\mathrm{cond}(\chi)} \chi_p$ *with each* χ_p *of conductor* p, *then each* χ_p *has exact order* 6.

We illustrate our result with numerical data for $m = 5$ and $m = 7$. We consider the number of fields K with $0 < \mathrm{Disc}(K) < 2 \cdot 10^6$ and $\mathrm{Disc}(K) \equiv a \pmod m$, and for each modulus we list the sum of the two main terms of (1.6) (after rounding) as well as the actual numerical data. For $a = 0$, the two main terms come from (1.4) instead of (1.6).

Discriminant modulo 5	0	1	2	3	4
Result from (1.6)	21307	22757	22757	22757	22757
Actual count	21277	22887	22751	22748	22781
Difference	30	130	6	9	24

Discriminant modulo 7	0	1	2	3	4	5	6
Result from (1.6)	15316	17209	14277	15316	17024	18063	15131
Actual count	15330	17229	14327	15323	17027	18058	15150
Difference	14	20	50	7	3	5	19

Remark. As one might guess from the shape of (1.7), we obtain results on arithmetic progressions by first obtaining estimates for

$$N_3^{\pm}(X, \chi) := \sum_{\substack{[K:\mathbb{Q}]=3 \\ 0<\pm\mathrm{Disc}(K)<X}} \chi(\mathrm{Disc}(K)), \tag{1.8}$$

and then using the orthogonality relations for Dirichlet characters. Our results have their origins in work of Datskovsky and Wright, who proved that certain related L-functions (see (6.29)) may have a pole if $\chi^6 = 1$ but are otherwise entire.

Our most general theorem is Theorem 6.2. This yields estimates for $N_3^{\pm}(X; m, a)$ for arbitrary values of a and m, which are unfortunately complicated to state. Note in particular that such results are interesting (and nontrivial!) when $(a, m) > 1$. In this case, and in particular in progressions $\equiv ap \pmod{p^2}$, we find a similar (but not identical) bias in the secondary term.

Moreover, Theorem 6.2 also allows us to simultaneously incorporate local specifications. For example, we can count the number of fields which split completely at a prime p and have discriminant

$\equiv a \pmod p$, for any quadratic residue a modulo p.

Our final result concerns 3-torsion in class groups in arithmetic progressions. Our most general result (Theorem 6.5) is again complicated to state, so we state the following analogue of Theorem 1.5:

Theorem 1.6. *Suppose that* $(6a, m) = 1$. *Then,*

(1.9)
$$\sum_{\substack{0<\pm D<X \\ D\equiv a \pmod m}} \#\mathrm{Cl}_3(D) = \frac{3+C^{\pm}}{\pi^2 m}\left(\prod_{p|m}\frac{1}{1-p^{-2}}\right)X + \frac{8K^{\pm}}{5\Gamma(2/3)^3}K_1'(m,a)X^{5/6} + O(X^{18/23}m^{20/23}),$$

where

$$(1.10)\quad K_1'(m,a) = \frac{1}{m}\prod_{p|m}\frac{1}{1-p^{-2}} \times$$
$$\sum_{\chi^6=1}{}' \overline{\chi}(a)L(1/3,\chi^{-2})\prod_{p\nmid m}\left(1-\frac{\chi(p)^2p^{1/3}+1}{p(p+1)}\right)\prod_{\substack{p|m \\ p\nmid \mathrm{cond}(\chi)}}\left(1-\chi(p)^{-2}p^{-4/3}\right)\prod_{\substack{p|m \\ p|\mathrm{cond}(\chi)}}\frac{\tau_p(\chi_p^2)^3}{p^2}.$$

As in Theorem 1.5, the sum is over primitive sextic characters χ to moduli dividing m, such that χ_p is of exact order 6 for each p.

As we did with Theorem 1.5, we illustrate our result with numerical data modulo 5 and 7. Here we compare the main terms of (1.9) and (1.5) to the actual counts of $\#\mathrm{Cl}_3(D)$ for $0 < D < 2 \cdot 10^6$. Note that these counts include the trivial element of $\mathrm{Cl}(D)$ for each D.

Discriminant modulo 5	0	1	2	3	4
Result from (1.9)	126942	160239	160239	160239	160239
Actual count	126841	160373	160202	160252	160207
Difference	101	134	37	13	32

Discriminant modulo 7	0	1	2	3	4	5	6
Result from (1.9)	95095	113486	109566	110919	113345	114699	110779
Actual count	95138	113407	109506	110955	113232	114741	110898
Difference	43	79	60	36	113	42	119

In this connection, we mention recent work of Hough [18], which provides another proof of the Davenport-Heilbronn theorem for class groups of imaginary quadratic fields (Theorem 1.2). His methods naturally produce the main and secondary terms of Theorem 1.2, albeit with an error term larger than $X^{5/6}$. His methods are notable for avoiding the Delone-Faddeev correspondence (to be described below); he uses a result of Soundararajan [36] which parameterizes nontrivial ideals of $\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))$ in terms of a Diophantine equation, and he then proves his result as a consequence of an equidistribution result for the associated Heegner points.

Although Hough's methods do not extend to counting cubic fields, they do extend to counting k-torsion in class groups of imaginary quadratic fields for odd $k > 3$. He conjectures that a negative secondary term of order $X^{\frac{1}{2}+\frac{1}{k}}$ should appear. Moreover, as he is presently investigating, these methods carry over to arithmetic progressions as well.

We also mention that the main term of (1.9) was previously obtained by Nakagawa and Horie [25], with an application to elliptic curves. In particular, they proved that if $D \equiv 2 \pmod 3$ is negative, and the class group of $\mathbb{Q}(\sqrt{D})$ has no nontrivial 3-torsion, then the elliptic curve $Dy^2 = 4x^3 - 1$

has no rational points. In particular, this gave a family of elliptic curves, a positive proportion of which have Mordell-Weil rank 0. Related ideas were pursued in subsequent works of James [20], Vatsal [41], and Wong [42], among others.

One naturally asks if our secondary terms should be reflected in counting functions of elliptic curves. Some quick numerical experiments suggested a negative answer; for example, of James's quadratic twists $Dy^2 = x^3 - x^2 + 72x + 368$ with $D < 2500$ a fundamental discriminant, 371 of them have[1] rank 0 and 389 have positive rank. Based on (1.3), one might guess that an excess of these curves has rank 0, but this does not seem to happen.

Summary of the proofs. The proofs of all of our results rely on the analytic theory of the *adelic Shintani zeta function*, due to Shintani [34] and Datskovsky-Wright [43, 10, 11] and further developed by the present authors [39]. This contrasts with the geometric approach adopted by Bhargava and his collaborators [3, 5, 7].

To keep the exposition as simple as possible, we have organized our paper around the proof of Theorem 1.1. Except as noted to the contrary (and in Section 4 in particular) our discussion only pertains to Theorem 1.1; the discussion of the proofs of our various generalizations is postponed to Section 6.

1.1. **The Davenport-Heilbronn and Delone-Faddeev correspondences.** Following the original work of Davenport and Heilbronn [13], we begin by relating our problem to the more tractable problem of counting certain integral binary cubic forms. This is accomplished through the well-known correspondence of Davenport-Heilbronn and Delone-Faddeev [14]. An elegant simplified and self-contained account of this work can be found in a paper of Bhargava [5], so we will only briefly summarize it here.

In [13], Davenport and Heilbronn established the main term in (1.2) by first relating *cubic rings* to *integral binary cubic forms*, and then counting those cubic forms which correspond to maximal orders in cubic fields.

A *cubic ring* is a commutative ring which is free of rank 3 as a $\mathbb{Z}$-module. The discriminant of a cubic ring is defined to be the determinant of the trace form $\langle x, y\rangle = \mathrm{Tr}(xy)$, and the discriminant of the maximal order of a cubic field is equal to the discriminant of the field.

The lattice of *integral binary cubic forms* is defined by

$$V_{\mathbb{Z}} := \{au^3 + bu^2v + cuv^2 + dv^3 \ : a, b, c, d \in \mathbb{Z}\}, \tag{1.11}$$

and the *discriminant* of such a form is given by the usual equation

$$\mathrm{Disc}(f) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd. \tag{1.12}$$

There is a natural action of $\mathrm{GL}_2(\mathbb{Z})$ (and also of $\mathrm{SL}_2(\mathbb{Z})$) on $V_{\mathbb{Z}}$, given by

$$(\gamma \cdot f)(u, v) = \frac{1}{\det \gamma} f((u, v) \cdot \gamma). \tag{1.13}$$

We call a cubic form f *irreducible* if $f(u, v)$ is irreducible as a polynomial over $\mathbb{Q}$, and *nondegenerate* if $\mathrm{Disc}(f) \neq 0$.

The Delone-Faddeev correspondence, which extends that of Davenport-Heilbronn and which was further extended by Gan, Gross, and Savin [17] to include the degenerate case, is as follows:

Theorem 1.7 ([14, 17])**.** *There is a natural, discriminant-preserving bijection between the set of* $\mathrm{GL}_2(\mathbb{Z})$*-equivalence classes of integral binary cubic forms and the set of isomorphism classes of*

[1] We performed our computations using Sage [37] with the `EllipticCurve.rank(proof=False)` function, so these counts are not proved correct.

cubic rings. Furthermore, under this correspondence, irreducible cubic forms correspond to orders in cubic fields.

To count cubic fields, Davenport and Heilbronn count their maximal orders, which are exactly those orders which are maximal at all primes p:

Proposition 1.8 ([13, 5])**.** *Under the Delone-Faddeev correspondence, a cubic ring R is maximal if any only if its corresponding cubic form f belongs to the set $U_p \subset V_{\mathbb{Z}}$ for all p, defined by the following equivalent conditions:*

- *The ring R is not contained in any other cubic ring with index divisible by p.*
- *The cubic form f is not a multiple of p, and there is no $\mathrm{GL}_2(\mathbb{Z})$-transformation of $f(u,v) = au^3 + bu^2v + cuv^2 + dv^3$ such that a is a multiple of p^2 and b is a multiple of p.*

In particular, the condition U_p only depends on the coordinates of f modulo p^2.

The proof of the main term in (1.1) goes as follows: One obtains an asymptotic formula for the number of cubic rings of bounded discriminant by counting lattice points in fundamental domains for the action of $\mathrm{GL}_2(\mathbb{Z})$, bounded by the constraint $|\mathrm{Disc}(x)| < X$. The fundamental domains may be chosen so that almost all reducible rings correspond to forms with $a = 0$, and so these may be excluded from the count.

One then multiplies this asymptotic by the product of all the local densities of the sets U_p. This yields a heuristic argument for the main term in (1.1), and one incorporates a simple sieve to convert this heuristic into a proof.

Remark. The Davenport-Heilbronn correspondence also applies to reducible maximal cubic rings. It is readily shown that (up to isomorphism) these are the rings $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ or $\mathcal{O}_K \times \mathbb{Z}$, where $\mathcal{O}_K$ is the ring of integers of a quadratic field. The unit element is given by $(1,1,1)$ or $(1,1)$ respectively, and the discriminant is equal to 1 or $\mathrm{Disc}(K)$ as appropriate.

1.2. **The work of Belabas, Bhargava, and Pomerance.**

In [3], Belabas, Bhargava, and Pomerance (BBP) introduced improvements to Davenport and Heilbronn's method, and obtained an error term of $O(X^{7/8+\epsilon})$ in (1.1) (and also in (1.3)). They begin by observing that

$$N_3^{\pm}(X) = \sum_{q \geq 1} \mu(q) N^{\pm}(q, X), \tag{1.14}$$

where $N^{\pm}(q,X)$ counts the number of cubic orders of discriminant $0 < \pm D < X$ which are nonmaximal at every prime dividing q. For large q, BBP prove that $N^{\pm}(q,X) \ll X 3^{\omega(q)}/q^2$ using reasonably elementary methods. Therefore, one may truncate the sum in (1.14) to $q \leq Q$, with error $\ll X/Q^{1-\epsilon}$. We will use this fact in our proof as well.

For small q, BBP estimate $N^{\pm}(q,X)$ with explicit error terms using geometric methods. These error terms are good enough to allow them to take the sum in (1.14) up to $(X \log X)^{1/8}$, which yields a final error term of $O(X^{7/8+\epsilon})$.

In addition, their methods extend to counting *quartic* fields, where they obtain a main term of $C_4 X$ with error $\ll X^{23/24+\epsilon}$.

1.3. **Shintani zeta functions and the analytic approach.**

The main idea of this paper is to estimate a quantity related to $N^{\pm}(q,X)$ using the analytic theory of Shintani zeta functions. The *Shintani zeta functions* associated to the space of binary cubic forms are defined by the Dirichlet series

$$\xi^{\pm}(s) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}, \tag{1.15}$$

where the lattice $V_{\mathbb{Z}}$ was defined in (1.11), and the sum is over elements of positive or negative discriminant respectively. Here $\mathrm{Stab}(x)$, the stabilizer of x in $\mathrm{SL}_2(\mathbb{Z})$, is always an abelian group of order 1 or 3.

By the Delone-Faddeev correspondence, $\xi^{\pm}(s)$ is *almost* the generating series for cubic rings. There are two differences: The Shintani zeta function counts $\mathrm{SL}_2(\mathbb{Z})$-orbits rather than $\mathrm{GL}_2(\mathbb{Z})$-orbits, and it weights some of them by a factor of $1/3$. As we will see, these discrepancies depend on the Galois group of the splitting field of the cubic form, and we will be able to adjust for them later.

These series converge absolutely for $\Re(s) > 1$, and Shintani proved [34] that that these zeta functions enjoy analytic continuation and a functional equation, to be described later. It therefore follows that we can use Perron's formula and a method of Landau [22] to estimate their partial sums. In particular, we have

(1.16)
$$\sum_{\substack{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_{\mathbb{Z}} \\ \pm \mathrm{Disc}(x) < X}} \frac{1}{|\mathrm{Stab}(x)|} = \int_{2-i\infty}^{2+i\infty} \xi^{\pm}(s) \frac{X^s}{s} ds = \mathrm{Res}_{s=1} \xi^{\pm}(s) X + \frac{6}{5} \mathrm{Res}_{s=5/6} \xi^{\pm}(s) X^{5/6} + O(X^{3/5+\epsilon}).$$

One may also translate this into an estimate for the number of cubic *orders* of discriminant at most X, with main terms of order X and $X^{5/6}$ and error $\ll X^{3/5+\epsilon}$.

To count cubic fields, we introduce the *q-nonmaximal Shintani zeta function* $\xi_q^{\pm}(s)$, which counts only those cubic forms in (1.15) which correspond to orders which are nonmaximal at q. By work of Datskovsky and Wright [43, 10] or F. Sato [31], it follows that these zeta functions have analytic continuation and functional equations, so that their partial sums may be estimated as in (1.16). These partial sums are closely related to $N^{\pm}(q, X)$, and we incorporate estimates for these sums into the sieve (1.14) to obtain our results.

The main technical difficulty is that the error terms in (1.16) now depend on q, and we must explicitly analyze this dependence. The key step is a careful analysis of certain *cubic Gauss sums* appearing in the functional equations for the q-nonmaximal Shintani zeta functions. This is carried out in our companion paper [39]. These Gauss sums are small on average, so we obtain error terms in (1.16) which are not too bad in q-aspect. This fact allows us to take a large cutoff Q in (1.14) and obtain a reasonably good error term in Roberts' conjecture.

We will in fact introduce a smoothing technique to obtain better error terms, but this discussion illustrates the principle of our proof.

Notation. For the most part our choice of notation is standard. Throughout, p will denote a prime and q a squarefree integer. We have referred to $\xi^{\pm}(s)$ as "Shintani zeta functions", which has some precedent in the literature but is not universal. We also remark that the notation $\xi_1(s)$ and $\xi_2(s)$ seems to be common in place of $\xi^{\pm}(s)$, but we did not want to risk confusion with the numerical parameter q.

Throughout, ϵ will denote a positive number which may be taken to be arbitrarily small, not necessarily the same at each occurrence. Any implied constants will always be allowed to depend on ϵ.

As is familiar in analytic number theory, we write $\omega(n)$ and $\Omega(n)$ for the number of prime divisors of a positive integer n, counted without and with multiplicity respectively. It is not difficult to prove that $\omega(n)$ satisfies the bound $A^{\omega(n)} \ll_{A,\epsilon} n^{\epsilon}$ for any $A > 1$, and we will use this bound frequently.

Remark. We find it convenient to refer to [39] for facts about Shintani zeta functions, but we emphasize that [39] builds on the work of other mathematicians, especially Datskovsky and Wright [43, 10, 11]. Some of the results quoted from [39] are originally due to Datskovsky and Wright and appear in [43, 10]. Please see our companion paper for a more specific discussion of where our work builds upon that of Datskovsky and Wright.

Organization of this paper. On account of the excellent exposition in Bhargava's paper ([5]; see also [7]), we will not say any more about the Davenport-Heilbronn and Delone-Faddeev correspondences. Instead, we begin in Section 2 with the analytic theory of the q-nonmaximal zeta functions. This theory is developed in [39] and we summarize it here. We also describe the original heuristic argument of Datskovsky-Wright and Roberts, which is quite similar to our proof.

In addition, we discuss recent and ongoing work and some open problems at the end of Section 2. We postponed this discussion from the introduction so we could use the language developed in Section 2.

In Section 3 we prove bounds for certain partial sums of the duals to the q-nonmaximal Shintani zeta functions. These will be needed in Section 5, and their proofs use corresponding bounds on the cubic Gauss sums, proved in [39].

In Section 4 we carry out the analysis in Section 3 for the 3-torsion problem. (This section may be skipped by readers only interested in the proof of Roberts' conjecture.) This is the only part of the proof that is substantially different for this problem, and indeed a new technical difficulty appears which must be resolved.

In Section 5 we present our proof of Roberts' conjecture. We first reduce Roberts' conjecture to a statement about partial sums of Shintani zeta functions. We then estimate these sums using a variation of (1.16), due essentially to Chandrasekharan and Narasimhan [8].

We conclude in Section 6 by proving our more general results, including the extension to 3-torsion in quadratic fields. As we describe, almost all of our arguments carry over to the general case.

Acknowledgments

To be added later, after the referee report is received.

2. q-nonmaximal Shintani zeta functions and their duals

We recall that the *Shintani zeta functions associated to the space of binary cubic forms* are defined by the Dirichlet series

$$\xi^{\pm}(s) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}, \tag{2.1}$$

where $V_{\mathbb{Z}}$ was defined in (1.11), the sum ranges over points of positive or negative discriminant respectively, and $\mathrm{Stab}(x)$, the stabilizer of a point x in $\mathrm{SL}_2(\mathbb{Z})$, is an abelian group of order 1 or 3. We now introduce *q-nonmaximal* analogues of these zeta functions. Throughout, q will be a squarefree integer.

Definition 2.1. *The* q-nonmaximal Shintani zeta functions $\xi_q^{\pm}(s)$ *are defined by the formula* (2.1)*, with the sum restricted to those x not belonging to U_p (defined in Proposition 1.8) for any $p|q$.*

In this section we describe the analytic theory of these functions, following Shintani [34], Datskovsky-Wright [43, 10], and our companion paper [39].

Shintani's original theorem relates the zeta functions $\xi^{\pm}(s)$ to dual zeta functions

$$\widehat{\xi}^{\pm}(s) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash \widehat{V}_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}, \tag{2.2}$$

where $\widehat{V}_{\mathbb{Z}}$, the dual lattice to $V_{\mathbb{Z}}$, is defined by

$$\widehat{V}_{\mathbb{Z}} := \{au^3 + bu^2v + cuv^2 + dv^3 \ : a, d \in \mathbb{Z};\quad b, c \in 3\mathbb{Z}\}. \tag{2.3}$$

To describe the analogue for the q-nonmaximal zeta functions $\xi_q^{\pm}(s)$, we must introduce the cubic Gauss sums. We define $\Phi_q(x)$ to be the characteristic function of those x not in U_p for any $p|q$, defined on either $V_{\mathbb{Z}}$ or $V_{\mathbb{Z}/q^2\mathbb{Z}}$. The *cubic Gauss sum* is the dual to $\Phi_q(x)$, defined as the following function on $\widehat{V}_{\mathbb{Z}}$:

$$\widehat{\Phi}_q(x) := \frac{1}{q^8} \sum_{y \in V_{\mathbb{Z}/q^2\mathbb{Z}}} \Phi_q(y) \exp(2\pi i [x, y]/q^2). \tag{2.4}$$

Here

$$[x, y] = x_4 y_1 - \frac{1}{3} x_3 y_2 + \frac{1}{3} x_2 y_3 - x_1 y_4 \tag{2.5}$$

is the alternating bilinear form used to identify V with $\widehat{V}$, where x_i and y_j are the coordinates of x and y respectively.

Remark. The dual $\widehat{\Phi}_q(x)$ might be thought of as a sum over $\frac{1}{q^2}\mathbb{Z}/\mathbb{Z}$, as it arises as a product of p-adic Fourier transforms of the function Φ_q. This integral reduces naturally to a finite sum over $V_{\frac{1}{q^2}\mathbb{Z}/\mathbb{Z}}$, which is equivalent to the sum given above.

We observe that $\widehat{\Phi}_q$ satisfies the multiplicativity property

$$\widehat{\Phi}_q(x)\widehat{\Phi}_{q'}(x) = \widehat{\Phi}_{qq'}(x) \tag{2.6}$$

for all $(q, q') = 1$. We also note that if x corresponds to a cubic ring R under the Delone-Faddeev correspondence, then $\widehat{\Phi}_p(x)$ depends only on $R \otimes_{\mathbb{Z}} \mathbb{Z}_p$. This implies that $\widehat{\Phi}_p(x) = \widehat{\Phi}_p(x')$ if x and x' correspond to R and R', where R' is contained in R with index coprime to p.

We are now prepared to describe the analytic properties of $\xi_q^{\pm}(s)$.

Theorem 2.2 (Shintani [34]; Datskovsky and Wright [10]; [39])**.** *The q-nonmaximal Shintani zeta functions $\xi_q^{\pm}(s)$ converge absolutely for $\Re(s) > 1$, have analytic continuation to all of $\mathbb{C}$, holomorphic except for poles at $s = 1$ and $s = 5/6$, and satisfy the functional equation*

$$\begin{pmatrix} \xi_q^+(1-s) \\ \xi_q^-(1-s) \end{pmatrix} = \Gamma\left(s - \frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s + \frac{1}{6}\right) 2^{-1} 3^{6s-2} \pi^{-4s} \begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3 \sin \pi s & \sin 2\pi s \end{pmatrix} \begin{pmatrix} \widehat{\xi}_q^+(s) \\ \widehat{\xi}_q^-(s) \end{pmatrix}, \tag{2.7}$$

where the dual q-nonmaximal Shintani zeta functions are given by

$$\widehat{\xi}_q^{\pm}(s) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash \widehat{V}_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|} \widehat{\Phi}_q(x) \big(|\mathrm{Disc}(x)|/q^8\big)^{-s}. \tag{2.8}$$

The residues are given by

$$\mathrm{Res}_{s=1} \xi_q^{\pm}(s) = \alpha^{\pm} \prod_{p|q} \left(\frac{1}{p^2} + \frac{1}{p^3} - \frac{1}{p^5}\right) + \beta \prod_{p|q} \left(\frac{2}{p^2} - \frac{1}{p^4}\right), \tag{2.9}$$

and

$$\operatorname{Res}_{s=5/6}\xi_q^{\pm}(s) = \gamma^{\pm}\zeta(1/3)\prod_{p|q}\left(\frac{1}{p^{5/3}}+\frac{1}{p^2}-\frac{1}{p^{11/3}}\right), \tag{2.10}$$

where $\alpha^+ = \pi^2/36$*,* $\alpha^- = \pi^2/12$*,* $\beta = \pi^2/12$*,* $\gamma^+ = \frac{\Gamma(1/3)^3}{4\sqrt{3}\pi} = \frac{2\pi^2}{9\Gamma(2/3)^3}$*, and* $\gamma^- = \sqrt{3}\gamma^+$*.*

We introduce the notation

$$\widehat{\xi}_q^{\pm}(s) =: \sum_{\mu_n} b_q^{\pm}(\mu_n){\mu_n}^{-s}$$

for the dual Shintani zeta function, where $\mu_n \in \frac{1}{q^8}\mathbb{Z}$ refers to the quantity $|\mathrm{Disc}(x)|/q^8$ in (2.8). Note that in [39] the factor of q^{8s} appears in (2.7) instead of (2.8). We chose the normalization here to get a uniform shape for the functional equation for all q, and to be consistent with our analytic reference [8], which we will describe later.

We can now present the heuristic argument of Datskovsky-Wright and Roberts. For any set of primes $\mathcal{P}$, let $N_{3,\mathcal{P}}^{\pm}(X)$ denote the number of cubic orders $\mathcal{O}$ with $\pm\mathrm{Disc}(\mathcal{O}) < X$ which are maximal at all primes in $\mathcal{P}$. Assuming[2] that we can separately estimate and subtract the contribution from reducible rings, the equations above imply that

$$N_{3,\mathcal{P}}^{\pm}(X) = \frac{1}{2}\alpha^{\pm}X\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p^3}\right)+\frac{3}{5}\gamma^{\pm}\zeta(1/3)X^{5/6}\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^{5/3}}\right)\left(1-\frac{1}{p^2}\right)+O_{\mathcal{P}}(X^{3/5+\epsilon}). \tag{2.11}$$

Formally taking a limit as $\mathcal{P}$ tends to the set of all primes, we obtain (1.1).

We also see that the inclusion-exclusion sieve (1.14) introduced by Belabas, Bhargava, and Pomerance allows us *in principle* to prove Roberts' conjecture. However, without an analysis of the $\mathcal{P}$-dependence of the error term in (2.11), it is unclear that we can obtain an error term smaller than $X^{5/6}$. Indeed, our initial attempts yielded "proofs" of Roberts' conjecture with error terms that were too large.

To analyze the $\mathcal{P}$-dependence of the error terms in (2.11), we must study the cubic Gauss sum in (2.4).This sum is studied in [39], and Section 3 we will use these results to prove bounds on appropriate partial sums of the dual zeta functions $\widehat{\xi}_q^{\pm}(s)$.

Before proceeding, we will simplify the functional equations by using a diagonalization argument of Datskovsky and Wright. At the end of this section, we will also describe some related work involving Shintani zeta functions.

2.1. **Datskovsky and Wright's diagonalization.** To simplify our analysis we apply an observation of Datskovsky and Wright [10]. The functional equations above have a curious matrix form, such that the negative and positive discriminant Shintani zeta functions are interdependent. By diagonalizing this matrix, we can greatly simplify the form of the functional equation.

We define, for each q, diagonalized Shintani zeta functions

$$\xi_q^{\mathrm{add}}(s) := 3^{1/2}\xi_q^{+}(s) + \xi_q^{-}(s), \tag{2.12}$$

$$\xi_q^{\mathrm{sub}}(s) := 3^{1/2}\xi_q^{+}(s) - \xi_q^{-}(s). \tag{2.13}$$

We diagonalize the dual zeta functions in exactly the same way.

[2]In fact our assumption is a bit rash, but for $\mathcal{P} = \emptyset$ see [35] for a proof with an error term of $O(X^{2/3+\epsilon})$.

The diagonalizations then take the following shape. Define

$$\Lambda_q^{\mathrm{add}}(s) := \left(\frac{2^4 \cdot 3^6}{\pi^4}\right)^{s/2} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{12}\right)\Gamma\left(\frac{s}{2}-\frac{1}{12}\right)\xi_q^{\mathrm{add}}(s),$$

$$\Lambda_q^{\mathrm{sub}}(s) := \left(\frac{2^4 \cdot 3^6}{\pi^4}\right)^{s/2} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{5}{12}\right)\Gamma\left(\frac{s}{2}+\frac{7}{12}\right)\xi_q^{\mathrm{sub}}(s),$$

and define $\widehat{\Lambda}_q^{\mathrm{add}}(s)$ and $\widehat{\Lambda}_q^{\mathrm{sub}}(s)$ in the same way. Then, the functional equations take the shape

$$\Lambda_q^{\mathrm{add}}(1-s) = 3\widehat{\Lambda}_q^{\mathrm{add}}(s), \tag{2.14}$$

$$\Lambda_q^{\mathrm{sub}}(1-s) = -3\widehat{\Lambda}_q^{\mathrm{sub}}(s). \tag{2.15}$$

This is the classical shape for functional equations of zeta functions, apart from the interesting factors of ± 3, and it will be a convenient one to work with. We also note the interesting fact that only Λ_{add}, and not Λ_{sub}, retains the pole at $s = 5/6$.

2.2. **Some related work.** We conclude this section by describing some recent and ongoing related work. These results will not be needed elsewhere in this paper, but we hope that they may prove useful in addressing related problems.

We first mention a striking result, conjectured by Ohno [26] and then proved by Nakagawa [24]. They established that the dual Shintani zeta functions are related to the original Shintani zeta functions by the simple formulas

$$\widehat{\xi}^{+}(s) = 3^{-3s}\xi^{-}(s), \tag{2.16}$$

$$\widehat{\xi}^{-}(s) = 3^{1-3s}\xi^{+}(s). \tag{2.17}$$

One can incorporate these formulas into Datskovsky and Wright's diagonalization, and therefore put the classical Shintani zeta functions into a self-dual form, with functional equations related to the ones above.

More recently, Ohno, the first author, and Wakatsuki [28, 27] classified all of the $\mathrm{SL}_2(\mathbb{Z})$-invariant sublattices of $V_{\mathbb{Z}}$, and proved that the Shintani zeta functions associated to these lattices share the nice properties above.

There is also the work of Yukie [45], who has initiated the study of *quartic* Shintani zeta functions, which are associated to a certain 12-dimensional prehomogeneous vector space. These zeta functions have not yet been studied as thoroughly as their cubic analogues, but it seems that one may be able to prove estimates for quartic fields with power saving error terms, and perhaps improve the result of Belabas, Bhargava, and Pomerance [6, 3]. Moreover, if any secondary terms are present, this method seems likely to yield them, at least in principle. However, this approach comes with substantial technical difficulties, and so far it has yet to even yield the main term.

We may also study extensions of base fields other than $\mathbb{Q}$. In [11], Datskovsky and Wright proved the analogue of the Davenport-Heilbronn theorem for any global field of characteristic not equal to 2 or 3. They also suggest that secondary terms should appear in this case as well. Moreover, Morra [23] has designed and implemented an algorithm to compute cubic extensions of imaginary quadratic fields of class number 1. At present we have verified that Morra's calculations closely match the Datskovsky-Wright heuristics for extensions of $\mathbb{Q}(i)$.

In principle we expect to be able to prove an analogue of Roberts' conjecture in this general setting. However, we expect that our error terms would be larger than $X^{5/6}$, even for cubic extensions of quadratic fields. However, one may be able to establish secondary terms for *smoothed* sums, such as

$$\sum_K |\mathrm{Disc}(K)| \exp^{-|\mathrm{Disc}(K)|/X},$$

where K ranges over cubic extensions of a fixed number field. We look forward to investigating this in the near future.

There is also a much more general theory of prehomogeneous vector spaces and their zeta functions, developed in the seminal works of Sato-Kimura [32], Sato-Shintani [33], and Wright-Yukie [44], among many others. Many authors have applied this theory to obtain a variety of interesting arithmetic density results, and it is possible that the methods of this paper might be applied to further refine some of these results. As one example we mention work of the first author [38], studying the zeta functions associated to some non-split forms of representations of $\mathrm{GL}_2(k) \times \mathrm{GL}_n(k)^2$ on the space $k^2 \otimes k^n \otimes k^n$ for $n = 2$ or 3. These zeta function are proved ([38], Theorem 4.24) to be meromorphic with two simple poles. In the case $n = 2$, this yields an arithmetic density result for the average size of $(h_F R_F)^2$, where F ranges over quadratic extensions of k, and h_F and R_F denote the class number and regulator respectively. For $n = 3$, this work is incomplete, but similar methods should yield a result for the average size of $h_F R_F$, where F now ranges over cubic extensions of k. Moreover, in the cubic case it can be shown that the secondary pole of the zeta function does not vanish when twisted by cubic characters. This suggests that an analogue of Theorem 1.5 might hold for this case as well.

Finally, the methods of this paper may be used to prove statements about prime and almost-prime discriminants of cubic fields. When one replaces the q-nonmaximality condition with a divisibility condition on the discriminant, the methods of this paper yield estimates for the number of discriminants divisible by q, and combining these estimates with different sieve methods allows us to prove a variety of results. However, we were unable to improve upon results of Belabas and Fouvry [4], and so we did not pursue this further.

3. Bounds for duals of the q-nonmaximal Shintani zeta function

Let

$$\widehat{\xi}_q^{\pm}(s) =: \sum_{\mu_n} b_q^{\pm}(\mu_n) {\mu_n}^{-s}$$

be the dual q-nonmaximal Shintani zeta functions, defined in (2.8). Throughout, we will fix a choice of sign and drop the $\pm$ from our notation. We also recall that the sum is over $\mu_n \in \frac{1}{q^8}\mathbb{Z}$.

Our later analytic estimates will require bounds for partial sums of the $b_q(\mu_n)$. The primary goal of this section will be to prove the following bound.

Theorem 3.1. *We have the bound*

$$\sum_{\mu_n < X} |b_q(\mu_n)| \ll q^{1+\epsilon} X, \tag{3.1}$$

uniformly for all q and X.

The proof essentially involves two steps. The first is an analysis of the Gauss sums $\widehat{\Phi}_q(x)$, carried out in [39]. Our analysis (see Lemma 3.3) shows that the Gauss sums are only supported on certain $\mathrm{GL}_2(\mathbb{Z}/q^2\mathbb{Z})$-orbits of $V_{\mathbb{Z}/q^2\mathbb{Z}}$, and in particular that cubic rings contributing to (3.1) must be either nonmaximal or totally ramified at each prime dividing q.

In the second step, we use a counting argument to bound the contribution of each orbit type, largely following work of Belabas, Bhargava, and Pomerance [3].

Before presenting the details, we derive the bounds that we will need later.

Proposition 3.2. *For any z and any $\delta > 1$, we have the bound*

$$\sum_{\mu_n > z} |b_q(\mu_n)| {\mu_n}^{-\delta} \ll_\delta q^{1+\epsilon} z^{-\delta+1}. \tag{3.2}$$

Furthermore, for any $\delta \in (0,1)$, we have the bound

$$\sum_{\mu_n < z} |b_q(\mu_n)| {\mu_n}^{-\delta} \ll_\delta q^{1+\epsilon} z^{-\delta+1}. \tag{3.3}$$

Both bounds are uniform in q.

Proof. To prove these bounds, we divide the respective intervals into dyadic subintervals of the form $[y, 2y]$. By (3.1), the contribution of each such interval is $\ll q^{1+\epsilon} y^{-\delta+1}$. Both bounds now follow by summing over y. □

To prepare for the proof of Theorem 3.1, we restate (3.1) in the form

$$\sum_{|\mathrm{Disc}(x)| < Y} |\widehat{\Phi}_q(x)| \ll q^{-7+\epsilon} Y, \tag{3.4}$$

where the sum is over integral binary cubic forms up to $\mathrm{GL}_2(\mathbb{Z})$-equivalence. In light of the Delone-Faddeev correspondence, we may (and do) refer to the x as either cubic forms or cubic rings. We will find it convenient to talk about divisibility (i.e., content) in terms of forms, and maximality properties in terms of rings.

Exact formulas for $\widehat{\Phi}_q(x)$ are proved in [39], and the following lemma extracts the results we need:

Lemma 3.3. [39] *The function $\widehat{\Phi}_q(x)$ is multiplicative in q. Moreover, for a prime $p > 3$, the value of $\widehat{\Phi}_p(x)$ is given by the following table (where R is the cubic ring corresponding to x):*

- (Content p^2:) $\widehat{\Phi}_p(x) = p^{-2} + p^{-3} - p^{-5}$ *if p^2 divides the content of R.*
- (Content p:) $|\widehat{\Phi}_p(x)| < p^{-3}$ *if p divides the content of R, but p^2 does not.*
- (Divisible by p^4:) $\widehat{\Phi}_p(x) = p^{-3} - p^{-5}$ *if R is nonmaximal at p, has content coprime to p, and $p^4|\mathrm{Disc}(R)$.*
- (Divisible by p^2:) $|\widehat{\Phi}_p(x)| = p^{-5}$ *for certain other rings for which $p^2|\mathrm{Disc}(R)$. (In particular, whenever $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is totally ramified at p and R does not belong to any of the first three categories.)*
- *Otherwise, and in particular if $p^2 \nmid \mathrm{Disc}(R)$, we have $\widehat{\Phi}_p(x) = 0$.*

A few remarks are in order. We have excluded $p = 2, 3$, but for these two primes we may incorporate the trivial estimate $|\widehat{\Phi}_p(x)| \leq 1$ into our implied constants. Also, we recall (2.6) and the note afterwards, which will be used in our proof. Finally, we note that any R with discriminant divisible by p^4 ($p > 3$) is in fact automatically nonmaximal at p, so that there is some redundancy in the conditions described above.

The basic idea of the proof is to separate the contributions to (3.1) according to the list above, and then count the number of each type of ring. If we could prove that the number of cubic rings R for which $d|\mathrm{Disc}(R)$ and $|\mathrm{Disc}(R)| < X$ was $\ll X/d^{1-\epsilon}$, uniformly in X and d, then the theorem would quickly follow. This seems to be difficult in general, but we will be able to prove an adequate substitute. We begin with the case where $d = r^2$ for squarefree r, where Belabas, Bhargava, and Pomerance [3] proved the inequality described above:

Lemma 3.4. *For squarefree r, we have the bound*

$$\sum_{\substack{|\mathrm{Disc}(x)|<Y \\ r^2|\mathrm{Disc}(x)}} 1 < M6^{\omega(r)}Y/r^2, \tag{3.5}$$

for an absolute constant M.

Proof. For those x corresponding to cubic orders, this is Lemma 3.4 of [3], and for general x it may be proved in the same manner. We briefly recall the details.[3]

For any factorization $r = ab$, we count the number of maximal cubic rings R with discriminant divisible by b^2, and then the number of cubic rings R' contained in R with index divisible by a. We then obtain (3.5) by summing over all such factorizations.

The count of maximal cubic rings is $\ll Y3^{\omega(b)}/b^2$ by Lemma 3.3 of [3] in the irreducible case, and the same bound follows trivially for the reducible case (as there is at most one such ring of any given discriminant).

Now for any fixed maximal cubic ring R, write η_R for the generating series counting the subrings of index n contained in R. By work of Datskovsky-Wright ([10], Theorem 6.1), we have the coefficientwise bound

$$\eta_R(s) \preceq \zeta(2s)\zeta(3s-1)\zeta(s)^3. \tag{3.6}$$

Arguing exactly as in [3], the number of cubic rings being counted is

$$\ll Y\frac{3^{\omega(ab)}}{(ab)^2}\sum_{j\geq 1}\frac{3^{\Omega(j)}}{j^2}\sum_{v,w:\ vw^2|j} w. \tag{3.7}$$

The sum over j converges, and one concludes the argument by summing over the $2^{\omega(r)}$ choices of a and b. □

We would like an analogue of (3.5) where r is not required to be squarefree. The methods of [3] do not extend to this case, but Lemma 3.3 gives us the additional information that any such rings occurring in (3.4) are nonmaximal. This allows us to apply the following lemma:

Lemma 3.5. *Let R be a cubic ring which is nonmaximal at p, and whose content is coprime to p. Then R is contained in an overring R' with index p. There are at most 3 rings R so contained in any R' whose content is coprime to p, and $p+1$ rings otherwise.*

Proof. Suppose that $x = (a, b, c, d) \in V_{\mathbb{Z}}$ is an element of the $\mathrm{GL}_2(\mathbb{Z})$-orbit corresponding to R. By Proposition 1.8, we may assume that $p^2|a$ and $p|b$. Then, by the explicit form of the Delone-Faddeev correspondence (see, e.g., [17]) we can write R as $R = \mathbb{Z}1 \times \mathbb{Z}\omega \times \mathbb{Z}\theta$, where

$$\omega^2 = -ac - b\omega + a\theta, \quad \theta^2 = -bd - d\omega + c\theta, \quad \omega\theta = -ad. \tag{3.8}$$

Let $R' := \mathbb{Z}1 \times \mathbb{Z}(\omega/p) \times \mathbb{Z}\theta \subseteq R \otimes_{\mathbb{Z}} \mathbb{Q}$. Then, R' is closed under multiplication because

$$(\omega/p)^2 = -(a/p^2)c - (b/p)(\omega/p) + (a/p^2)\theta, \quad \theta^2 = -bd - dp(\omega/p) + c\theta, \quad (\omega/p)\theta = -(a/p)d. \tag{3.9}$$

It follows that R' is a cubic ring which contains R with index p, and that both are contained in the same maximal ring.

The bound on the number of such R is proved in Lemma 2.4 of [3]; although this lemma is stated for cubic orders, its proof remains valid for any cubic ring. □

[3] We refer to the published version of [3], which offers a different proof than some preprints of [3].

Remark. In terms of cubic forms, this construction sends (p^3a, p^2b, pc, d) to (pa, pb, pc, pd), which illustrates that a ring with trivial content can be contained in a ring with content p.

Proof of Theorem 3.1. For each factorization $q = abcd$, consider the contribution to (3.4) from those x satisfying the following:

- If $p|a$, then $p^2|\mathrm{ct}(x)$;
- if $p|b$, then $p^2 \nmid \mathrm{ct}(x)$ but $p|\mathrm{ct}(x)$;
- if $p|c$, then $p \nmid \mathrm{ct}(x)$ but $p^4|\mathrm{Disc}(x)$;
- if $p|d$, then $p^4 \nmid \mathrm{Disc}(x)$ but $p^2|\mathrm{Disc}(x)$.

(Here $\mathrm{ct}(x)$ denotes the content of x.) Lemma 3.3 implies that for each x we have

$$|\widehat{\Phi}_q(x)| \leq \frac{2^{\omega(a)}}{a^2b^3c^3d^5}. \tag{3.10}$$

We use Lemma 3.5 to replace each x by an overring x' of index c. For each x', we define $c'|c$ by $c' = \gcd(\mathrm{ct}(x'), c)$, such that there are at most $3^{\omega(c)}c'$ rings x corresponding to each overring x'. We note also that the discriminant of each x' is divisible by $(\frac{cd}{c'})^2$. (It is also divisible by factors which divide the content.)

For each choice of a, b, c, c', d, the contribution to (3.4) is therefore

$$\leq \frac{2^{\omega(a)}3^{\omega(c)}c'}{a^2b^3c^3d^5} \sum_{\substack{|\mathrm{Disc}(x')|<Y/c^2 \\ a^2bc'|\mathrm{ct}(x') \\ (\frac{cd}{c'})^2|\mathrm{Disc}(x')}} 1 = \frac{2^{\omega(a)}3^{\omega(c)}c'}{a^2b^3c^3d^5} \sum_{\substack{|\mathrm{Disc}(x')|<\frac{Y}{a^8b^4c'^4c^2} \\ (\frac{cd}{c'})^2|\mathrm{Disc}(x')}} 1. \tag{3.11}$$

By Lemma 3.4, this is

$$\ll \frac{2^{\omega(a)}3^{\omega(c)}c'}{a^2b^3c^3d^5} \cdot 6^{\omega(cd)} \frac{Y}{a^8b^4c'^2c^4d^2} \ll 18^{\omega(q)} \frac{Y}{a^{10}b^7c^7c'd^7} \ll q^{-7+\epsilon}Y. \tag{3.12}$$

The theorem follows by summing over the $5^{\omega(q)} \ll q^\epsilon$ factorizations $q = abcd$ and choices for c'. $\square$

4. Bounds for the dual Shintani zeta function in the 3-torsion problem

In this section we will carry out the analysis of Section 3 for the related problem of estimating 3-torsion in class groups. In particular, throught this section, $\Phi_p(x)$ and $b_q(\mu_n)$ will correspond to the (complement of the) set V_p instead of U_p. This set was defined in [13], and we recall the definition in Section 6.

The idea of the proof is very much the same, but one new technical difficulty appears: The Fourier transform $\widehat{\Phi}_p(x)$ will take a form which is more difficult to estimate over $|\mathrm{Disc}(x)| < Y$ when p is large in relation to Y. As a result, we will be limited to proving the following analogue of Theorem 3.1:

Theorem 4.1. *If $\Phi_p(x)$ corresponds to the complement of V_p, then we have the bounds*

$$\sum_{\mu_n<X} |b_q(\mu_n)| \ll q^{2+\epsilon}X \tag{4.1}$$

and

$$\sum_{\mu_n<X} |b_q(\mu_n)| \ll q^{1+\epsilon}X + q^{-1+\epsilon}, \tag{4.2}$$

uniformly for all q and X.

Remark. The bound (4.1) is quite simple to prove (assuming the results of the previous section), and as we show at the end of Section 6.1, this already suffices to obtain Theorem 1.2 with a larger error term of $O(X^{9/11+\epsilon})$. This section describes a "trick" which allows us to obtain (4.2) and thus an error term of $O(X^{18/23+\epsilon})$, and may be skipped without loss of continuity.

We obtain the following corollary in the same way as before.

Proposition 4.2. *For $\delta \in (0,1)$, we have the bounds*

$$\sum_{\mu_n < z} |b_q(\mu_n)| {\mu_n}^{-\delta} \ll_\delta q^{2+\epsilon} z^{-\delta+1}, \tag{4.3}$$

when $z \leq q^{-3}$, and

$$\sum_{q^{-3} < \mu_n < q^{-2}} |b_q(\mu_n)| {\mu_n}^{-\delta} \ll_\delta q^{3\delta-1+\epsilon}, \tag{4.4}$$

$$\sum_{q^{-2} < \mu_n < z} |b_q(\mu_n)| {\mu_n}^{-\delta} \ll_\delta q^{1+\epsilon} z^{-\delta+1}, \tag{4.5}$$

when $z > q^{-2}$. We also obtain, as before, for any $\delta > 1$ and any $z > q^{-2}$,

$$\sum_{\mu_n > z} |b_q(\mu_n)| {\mu_n}^{-\delta} \ll_\delta q^{1+\epsilon} z^{-\delta+1}. \tag{4.6}$$

We now come to the proof of Theorem 4.1. We begin with the following analogue of Lemma 3.3:

Lemma 4.3. [39] *The function $\widehat{\Phi}_q(x)$ (now corresponding to the sets V_p for $p|q$) is multiplicative in q. Moreover, for a prime $p > 3$, the value of $\widehat{\Phi}_p(x)$ is given by the following table (where R is the cubic ring corresponding to x):*

- (Content p^2:) $\widehat{\Phi}_p(x) = 2p^{-2} - p^{-4}$ *if p^2 divides the content of R.*
- (Content p:) $|\widehat{\Phi}_p(x)| < 2p^{-3}$ *if p divides the content of R, but p^2 does not.*
- (Divisible by p^4:) $\widehat{\Phi}_p(x) = p^{-3} - p^{-4}$ *if R is nonmaximal at p, p does not divide the content of R, and $p^4 | \mathrm{Disc}(R)$.*
- (Divisible by p^3:) $|\widehat{\Phi}_p(x)| = p^{-4}$ *for certain other rings which are nonmaximal at p and for which $p^3 | \mathrm{Disc}(R)$.*
- *Otherwise, and in particular if $p^3 \nmid \mathrm{Disc}(R)$, we have $\widehat{\Phi}_p(x) = 0$.*

We recall again the remarks after Lemma 3.3, and observe that if $p^3 | \mathrm{Disc}(R)$ ($p > 3$), R is in fact automatically nonmaximal at p.

In Section 3, we needed to count discriminants which were divisible by p^2 and which contributed $O(p^{-5})$ each to our final estimates. Now, we must count discriminants divisible by p^3 and which contribute $O(p^{-4})$ each. We expect the total contributions to be comparable in size, but we were unable to prove this. In particular, Lemma 3.4 is proved using class field theory (see Lemma 3.3 of [3]), and the proof does not carry over to this case.

We will begin by applying Lemma 3.5 to reduce to counting p-divisible rings. We then need to bound the number of such rings, and we can prove such a bound using the methods of this paper! The associated Dirichlet series are "p-divisible Shintani zeta functions"[4], and so we may estimate their partial sums using contour integration. We thus obtain a statement (Lemma 4.4) whose proof

[4]They are Shintani zeta functions if we weight each ring as described in the proof of Proposition 5.1. The weights are all between 1/3 and 2 and we are only seeking an O-estimate, so this technical point will not affect the proof.

requires (4.1), but which is used in the proof of (4.2). This may seem like circular reasoning; the reason this "circular" argument works is that the proof of Lemma 3.4 exploits similarities in the structure of $\Phi_p(x)$ and $\widehat{\Phi}_p(x)$.

Lemma 4.4. *For $d = rs^2$ with r and s coprime and squarefree, we have the bound*

$$\sum_{\substack{|\mathrm{Disc}(x)|<X \\ d|\mathrm{Disc}(x)}} 1 \ll X/d^{1-\epsilon} + (rs)^{2+\epsilon}. \tag{4.7}$$

We first use this to prove Theorem 4.1, and then we will prove Lemma 4.4.

Proof of Theorem 4.1. The proof of (4.1) follows by comparing Lemma 4.3 to Lemma 3.3. For each x, the bound in Lemma 4.3 is at most q times that in Lemma 3.3, so (4.1) follows from Theorem 3.1.

To prove (4.2), we reformulate our bound in the shape

$$\sum_{|\mathrm{Disc}(x)|<Y} |\widehat{\Phi}_q(x)| \ll q^{-7+\epsilon}Y + q^{-1+\epsilon}. \tag{4.8}$$

As in the proof of Theorem 3.1, for each factorization $q = abcd$ we consider the contribution to (3.4) from those x satisfying the following:

- If $p|a$, then $p^2|\mathrm{ct}(x)$;
- if $p|b$, then $p^2 \nmid \mathrm{ct}(x)$ but $p|\mathrm{ct}(x)$;
- if $p|c$, then $p \nmid \mathrm{ct}(x)$ but $p^4|\mathrm{Disc}(x)$;
- if $p|d$, then $p^4 \nmid \mathrm{Disc}(x)$ but $p^3|\mathrm{Disc}(x)$.

Lemma 3.3 implies that for each x we have

$$|\widehat{\Phi}_q(x)| \leq \frac{2^{\omega(ab)}}{a^2b^3c^3d^4}. \tag{4.9}$$

We use Lemma 3.5 to replace each x by an overring x' of index cd. For each x', we define $c'|c$ and $d'|d$ by $c'd' = \gcd(\mathrm{ct}(x'), cd)$, such that there are at most $3^{\omega(cd)}c'd'$ rings x corresponding to each overring x'. We note also that the discriminant of each x' is divisible by $(\frac{c}{c'})^2(\frac{d}{d'})$.

For each choice of a, b, c, c', d, d', the contribution to (3.4) is therefore

$$\leq \frac{2^{\omega(a)}3^{\omega(cd)}c'd'}{a^2b^3c^3d^4} \sum_{\substack{|\mathrm{Disc}(x')|<Y/(cd)^2 \\ a^2bc'd'|\mathrm{ct}(x') \\ (\frac{c}{c'})^2(\frac{d}{d'})|\mathrm{Disc}(x')}} 1 = \frac{2^{\omega(a)}3^{\omega(cd)}c'd'}{a^2b^3c^3d^4} \sum_{\substack{|\mathrm{Disc}(x')|<\frac{Y}{a^8b^4c'^4c^2d'^4d^2} \\ (\frac{c}{c'})^2(\frac{d}{d'})|\mathrm{Disc}(x')}} 1. \tag{4.10}$$

By Lemma 4.4, this is

$$\ll \frac{q^\epsilon c'd'}{a^2b^3c^3d^4}\left(\frac{Y}{a^8b^4c'^2c^4d'^3d^3} + \left(\frac{cd}{c'd'}\right)^2\right) \ll q^\epsilon \frac{Y}{a^{10}b^7c^7c'd^7d'^2} + \frac{q^\epsilon}{a^2b^3cc'd^2d'} \ll q^{-7+\epsilon}Y + q^{-1+\epsilon}. \tag{4.11}$$

The theorem now follows by summing over the $6^{\omega(q)} \ll q^\epsilon$ possibilities for a, b, c, c', d, d'. $\square$

Proof of Lemma 4.4. The proof follows the methods presented in this paper, but it is simpler. As the proof is very similar, we will omit some of the details.

We begin by defining the *d-divisible Shintani zeta functions*, which count only those discriminants divisible by $d = rs^2$. For prime factors of s (other than 2, 3) the local condition is given by the complement of V_p, and for factors of prime r it is described in [39]. These zeta functions satisfy a close analogue of Theorem 2.2; we omit the details, but the exact form of the functional equation can be readily deduced from [39].

For cubefree $d = rs^2$, let $\xi_d^{\pm}(s) := \sum_{d|n} a^{\pm}(n) n^{-s}$ denote[5] the d-divisible Shintani zeta function, and let $\xi_d(s) := \sum_{d|n} a(n) n^{-s}$ denote either of the diagonalized zeta functions, as in (5.7). Then we have the relation

$$\sum_{d|n} a(n) \exp(-n/X) = \int_{c-i\infty}^{c+i\infty} \xi_d(s) X^s \Gamma(s) ds \tag{4.12}$$

for any $c \in (1, \frac{3}{2})$ (analogously to (5.11)). Note that $\sum_{d|n;\ n<X} a(n) \ll \sum_{d|n} a(n) \exp(-n/X)$; the smoothing factor of $\exp(-n/X)$ is introduced to improve the error terms.

As $\xi_d(s)\Gamma(s)$ is holomorphic for $\Re s > -1/2$, except for poles at $s = 1, \frac{5}{6}, 0$, we may shift the contour and use the functional equation we obtain that

$$\begin{aligned} \sum_{d|n} a(n) \exp(-n/X) = X^{1-c} \sum_{\mu_n} \frac{b_d(\mu_n)}{\mu_n^c} \int_{c-i\infty}^{c+i\infty} \frac{\Delta(s)}{\Delta(1-s)} (X\mu_n)^{-it} \Gamma(1-s) ds + \\ \Big(\Gamma(1) \mathrm{Res}_{s=1} \xi_d(s)\Big) X + \Big(\Gamma(5/6) \mathrm{Res}_{s=5/6} \xi_d(s)\Big) X^{5/6} + \xi_d(0) \end{aligned} \tag{4.13}$$

where $\sum_{\mu_n \in \frac{1}{d^4}\mathbb{Z}} b_d(\mu_n) \mu_n^{-s}$ is the dual zeta function, $\Delta(s)$ is as in (5.16), the residues are $\ll d^{-1+\epsilon}$ and $d^{-5/6+\epsilon}$ respectively, and $\xi_d(0) \ll d$, as follows from [39]. The integral above is absolutely convergent, and we bound it by an absolute constant (which in particular does not depend on μ_n). We must therefore bound $X^{1-c} \sum_{\mu_n} b_d(\mu_n) \mu_n^{-c}$.

We choose $c = 1 + \frac{\epsilon}{4}$ so that $X^{1-c} \leq (d^{-4})^{-\epsilon/4} = d^{\epsilon}$. Arguing as in Proposition 3.2, we see that the sum over μ_n will be $\ll (rs)^{2+\epsilon}$, implying the lemma, provided we can show that

$$\sum_{\mu_n < Y} |b_d(\mu_n)| \ll (rs)^{2+\epsilon} Y. \tag{4.14}$$

Note that when $d = s^2$ and $(d, 6) = 1$, this is *exactly* (4.1). Crucially, the proof of (4.1) does not depend on this lemma, but our argument does have an interesting circular flavor: the bound (4.1) is an essential ingredient in the proof of the stronger bound (4.2). As we discussed earlier, the idea is that an important piece of the Fourier transform $\widehat{\Phi}_q(x)$ resembles $\Phi_q(x)$ itself.

To prove (4.14), note that as before, it suffices to prove that

$$\sum_{|\mathrm{Disc}(x)| < Y} |\widehat{\Phi}_d(x)| \ll d^{-4} (rs)^{2+\epsilon} Y, \tag{4.15}$$

where $\widehat{\Phi}_d(x)$ is again multiplicative in $d = rs^2$. We extend the proof of (4.1) (which extended the proof of Theorem 3.1) to cover the case $r > 1$. For each factorization $r = ef$, consider the contribution from those x for which $(\mathrm{ct}(x), r) = e$. We divide each such x by e, and the formulas in [39] imply that $|\widehat{\Phi}_f(x)| \leq f^{-2}$, so the total r-contribution to (4.15) is $\leq e^{-4} f^{-2} \leq r^{-2}$ for each factorization $r = ef$. Summing over the $\ll r^{\epsilon}$ such factorizations, we obtain a total r-contribution $\ll r^{-2+\epsilon}$, as claimed in (4.15). This completes the proof. □

[5]This is the same notation that we used for the q-nonmaximal zeta function. This notation will be used only in the proof of this lemma.

5. The proof of Roberts' conjecture

We will prove Roberts' conjecture in three steps. In Section 5.1 we discuss the relationship between the Shintani zeta coefficients and counting functions for cubic rings, and reduce Roberts' conjecture to a statement about partial sums of Shintani zeta functions. In Section 5.2 we incorporate the Datskovsky-Wright diagonalization, and transform our problem into one that can be readily addressed using a contour integration argument of Chandrasekharan and Narasimhan [8]. Finally, in Section 5.3 we do this contour integration. As it would be impractical to reproduce the entire argument in [8], we will refer to their paper for many of the details and call the reader's attention to the few changes we introduce to their argument.

5.1. **Reduction to Shintani zeta coefficients.** We want to obtain estimates for $N_3^{\pm}(X)$, the count of cubic fields of positive or negative discriminant less than X. The first step in our argument is to relate these quantities to partial sums of the coefficients of the Shintani zeta function. Define Dirichlet series $F^{\pm}(s) = \sum_n c^{\pm}(n)n^{-s}$ by

$$F^{\pm}(s) = \sum_{n \geq 1} c^{\pm}(n)n^{-s} := \sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_{\mathbb{Z}}}{}^{'} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}, \tag{5.1}$$

where the prime indicates that the sum is restricted to those x which are maximal at all places (i.e., contained in U_p for all p).

We define partial sums

$$N^{\pm}(X) := \sum_{n \leq X} c^{\pm}(n). \tag{5.2}$$

We will prove the following:

Proposition 5.1. *We have*

$$N_3^{\pm}(X) = \frac{1}{2} N^{\pm}(X) - \frac{3}{\pi^2} X + O(X^{1/2}). \tag{5.3}$$

Proof. By the Delone-Faddeev correspondence (see also Section 2 of [10]), the Dirichlet series in (5.1) counts fields of degree ≤ 3 (or, more properly, their maximal orders), with different weights for different types of fields. Non-Galois cubic fields are counted with weight 2, Galois fields are counted with weight 2/3, quadratic fields are counted with weight 1, and $\mathbb{Q}$ is counted with weight 1/3.

The number of cyclic cubic extensions of discriminant $\leq X$ is $O(X^{1/2})$ [9], the number of quadratic extensions of either positive or negative discriminant $\leq X$ is equal to $\frac{3}{\pi^2}X + O(X^{1/2})$, and of course there is only one trivial extension of $\mathbb{Q}$. The result therefore follows by subtracting and reweighting these contributions as appropriate. □

5.2. **Setup for the contour integration.** In this section we will incorporate the inclusion-exclusion sieve and Datskovsky and Wright's diagonalization, and bring our problem to a form where we can apply contour integration.

By Proposition 5.1, it suffices to count

$$\sum_{\substack{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_{\mathbb{Z}} \\ \pm \mathrm{Disc}(x) \leq X}}{}^{'} \frac{1}{\mathrm{Stab}(x)}, \tag{5.4}$$

where the dash on the sum indicates that we count only those lattice points corresponding to *maximal* cubic rings. A cubic ring is maximal if and only if it is maximal at each prime. By inclusion-exclusion, this sum is equal to

$$\sum_q \mu(q)\bigg(\sum_{n \leq X} a_q^{\pm}(n)\bigg), \tag{5.5}$$

where the $a_q^{\pm}(n)$ are the coefficients of the q-nonmaximal Shintani zeta functions. By Lemma 3.4, the inner sum is $\ll Xq^{-2+\epsilon}$, uniformly in q, and it follows that the total sum is

$$\sum_{q \leq Q} \mu(q)\bigg(\sum_{n \leq X} a_q^{\pm}(n)\bigg) + O\bigg(\frac{X}{Q^{1-\epsilon}}\bigg), \tag{5.6}$$

for any choice of Q. The main term above is what we want to estimate.

Although it is not strictly necessary (see Theorem 3 of [33]), it will simplify our computations to incorporate Datskovsky and Wright's diagonalization, described in Section 2.1. We will write

$$a_q(n) := \sqrt{3}a_q^+(n) \pm a_q^-(n), \tag{5.7}$$

such that the zeta functions $\xi_q(s) := \sum_n a_q(n)n^{-s}$ satisfy the simple functional equation (2.14) or (2.15). As we will prove our results simultaneously for both choices of sign in (5.7), we will not indicate this sign in our notation.

We write $N(X)$ for either of the analogous linear combinations of $N^{\pm}(X)$, and we will prove estimates for

$$N_Q(X) := \sum_{q \leq Q} \mu(q)\bigg(\sum_{n \leq X} a_q(n)\bigg). \tag{5.8}$$

We then take the appropriate linear combinations to recover the analogous estimates for the original Shintani zeta function.

To evaluate (5.8), recall that Perron's formula yields the equality[6]

$$\sum_{n \leq X} a_q(n) = \int_{c-i\infty}^{c+i\infty} \xi_q(s)\frac{X^s}{s}ds \tag{5.9}$$

for any $c > 1$. *In principle*, one evaluates the integral by shifting the contour to the left, obtaining main terms of order X and $X^{5/6}$ from the poles of $\xi_q(s)$, along with an error term. In practice, one runs into convergence issues at infinity and must tweak the method somehow. We adopt the method of Chandrasekharan and Narasimhan [8], which has its origins in work of Landau [22]. In particular, following [8], we will smooth the sum above to obtain an integral with nice convergence properties at infinity, and then use a finite differencing method to recover the sum in (5.9) from the smoothed sum.

As we will see, we may improve our error terms by departing from [8] in one respect. We will smooth the entire sum in (5.8), estimate the smoothed sum over each q separately, and combine the contributions from all q to obtain a smoothed version of the count in (5.8). Recovering the count in (5.8) from the smoothed count involves an error term, and the error made in unsmoothing the combined count is roughly equal to the error made in unsmoothing the contribution from any individual q. Therefore, we will not actually estimate the contribution of any individual q to (5.8).

[6]For strict equality, we must take X not equal to any value of n (any irrational number will do).

5.3. **The contour integration.** We now begin in earnest, closely following [8]. We introduce a smoothing factor $(X-n)^\rho$, and write

$$N_Q^\rho(X) := \frac{1}{\Gamma(\rho+1)} \sum_{q \le Q} \mu(q) \bigg(\sum_{n \le X} (X-n)^\rho a_q(n) \bigg). \tag{5.10}$$

Here ρ is any sufficiently large integer. We may in fact take $\rho = 3$, but to follow the notation of [8][7] we will leave the value undetermined. (Any error terms may depend on ρ.)

For each q, we have

$$\frac{1}{\Gamma(\rho+1)} \sum_{n \le X} (X-n)^\rho a_q(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s(s+1)\cdots(s+\rho)} \xi_q(s) X^{s+\rho} ds, \tag{5.11}$$

for any $c > 1$. We move the integral to the line $\sigma = 1 - c$, choosing $c < \frac{5}{4}$ so that we do not pick up any singularities of the integral left of $s = 0$, and so that the integral (5.15) converges for $\rho \ge 3$. In doing so, we pick up contributions from the residues of $\xi_q(s)$ at $s = 1$ and $s = 5/6$.

Later, we will estimate the integral on the line $\sigma = 1 - c$ using the functional equation. We first explain how $N_Q^\rho(X)$ is related to our unsmoothed count $N_Q(X)$. For a parameter y to be determined later, define a finite differencing operator Δ_y^ρ (on the space of real valued functions F) by

$$\Delta_y^\rho F(x) := \sum_{\nu=0}^{\rho} (-1)^{\rho-\nu} \binom{\rho}{\nu} F(x + \nu y).$$

It is proved in (4.14) of [8] that

$$\Delta_y^\rho [N_Q^\rho(X) - R_Q^\rho(X)] = y^\rho [N_Q(X) - R_Q(X)] + O\bigg(y^{\rho+1} + y^\rho \sum_{X < n \le X + \rho y} \sum_{q \le Q} a_q(n) \bigg), \tag{5.12}$$

where

$$R_Q^\rho(x) = \sum_{q \le Q} \mu(q) \left(\frac{1}{1(1+1)\cdots(1+\rho)} X^{1+\rho} \mathrm{Res}_{s=1} \xi_q(s) + \frac{1}{\frac{5}{6}\left(\frac{5}{6}+1\right)\cdots\left(\frac{5}{6}+\rho\right)} X^{5/6+\rho} \mathrm{Res}_{s=5/6} \xi_q(s) \right)$$

(with an additional residue term at $s = 0$ which we subsume into our error term), and

$$R_Q(X) = \sum_{q \le Q} \mu(q) \bigg(X \mathrm{Res}_{s=1} \xi_q(s) + \frac{6}{5} X^{5/6} \mathrm{Res}_{s=5/6} \xi_q(s) \bigg).$$

The error term in (5.12) is $O(y^{\rho+1+\epsilon})$ if $y > X^{3/5}$; this follows by estimating

$$\sum_{X < n \le X + \rho y} \sum_{q \le Q} a_q(n) \ll y^\epsilon \sum_{X < n \le X + \rho y} a(n) \ll y^{1+\epsilon}.$$

The first estimate follows because $|a_q(n)| \le |a(n)|$ and $a_q(n) = 0$ unless $q^2 | n$, and the latter estimate follows from partial sum estimates for the standard Shintani zeta function.

Therefore, for $y > X^{3/5}$ it follows that

$$N(X) - R_Q(X) \ll y^{-\rho} \Delta_y^\rho [N_Q^\rho(X) - R_Q^\rho(X)] + y^{1+\epsilon} + \frac{X}{Q^{1-\epsilon}}, \tag{5.13}$$

[7] In the notation of [8], we have $A = 2, q = 1, r = 1, \delta = 1$, and $N = 4$, as determined by the structure of our problem.

where

$$N_Q^\rho(X) - R_Q^\rho(X) = \sum_{q \le Q} \left(\frac{1}{2\pi i} \int_{1-c-i\infty}^{1-c+i\infty} \frac{1}{s(s+1)\cdots(s+\rho)} \xi_q(s) X^{s+\rho} ds \right). \tag{5.14}$$

We will study this integral individually for each q. To denote this, we replace the subscript Q with q throughout. Applying the functional equation (2.14) or (2.15), the integral is equal to

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{(1-s)(2-s)\cdots(1+\rho-s)} \frac{\pm\Delta(s)}{3\Delta(1-s)} \widehat{\xi}_q(s) X^{1+\rho-s} ds, \tag{5.15}$$

where

$$\Delta(s) := \left(\frac{2^4 \cdot 3^6}{\pi^4} \right)^{s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2} + \frac{1}{2}\right) \Gamma\left(\frac{s}{2} + a_3\right) \Gamma\left(\frac{s}{2} + a_4\right). \tag{5.16}$$

Here a_3 and a_4 are equal to either $5/12$ and $7/12$ or $\pm 1/12$ as appropriate.

The integral in (5.15) is equal to

$$\sum_{\mu_n \in \frac{1}{q^8}\mathbb{Z}} \frac{b_q(\mu_n)}{{\mu_n}^{1+\rho}} \left(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{(1-s)(2-s)\cdots(1+\rho-s)} \frac{\pm\Delta(s)}{3\Delta(1-s)} (\mu_n X)^{1+\rho-s} ds \right). \tag{5.17}$$

This integral and its finite difference are thoroughly analyzed in [8]. Although one might hope to play the oscillation of the $b_q(\mu_n)$ against oscillation in this integral, our attempts to do this were unsuccessful. However, we still obtain good error terms by taking absolute values of the $b_q(\mu_n)$ and using bounds for the integral proved in [8].

Recall that our error term in (5.13) consists of a sum over q of the operator Δ_y^ρ applied to this integral. Following the argument[8] in [8], and in particular the bounds on p. 109 there, we have

$$\Delta_y^\rho [N_q^\rho(X) - R_q^\rho(X)] \ll y^\rho X^{3/8} \sum_{\mu_n \le z} \frac{|b_q(\mu_n)|}{\mu_n^{5/8}} + X^{3/8+3\rho/4} \sum_{\mu_n > z} \frac{|b_q(\mu_n)|}{\mu_n^{5/8+\rho/4}}, \tag{5.18}$$

where z is a free parameter. We estimate the sums on the right using the bounds given in Proposition 3.2. We conclude that

$$y^{-\rho} \Delta_y^\rho [N_q^\rho(X) - R_q^\rho(X)] \ll q^{1+\epsilon} X^{3/8} z^{3/8} \left(1 + \left(\frac{X^3}{y^4 z} \right)^{\rho/4} \right), \tag{5.19}$$

and therefore, adding over all q,

$$N(X) - R_Q(X) \ll Q^{2+\epsilon} X^{3/8} z^{3/8} \left(1 + \left(\frac{X^3}{y^4 z} \right)^{\rho/4} \right) + y^{1+\epsilon} + \frac{X}{Q^{1-\epsilon}}. \tag{5.20}$$

We choose $y = X/Q$ and $z = X^3/y^4$ to equalize error terms. The above is then

$$\ll Q^{7/2+\epsilon} + X^{1+\epsilon}/Q, \tag{5.21}$$

and choose $Q = X^{2/9}$ to obtain an error term of $O(X^{7/9+\epsilon})$. Note that $y > X^{3/5}$ as required for (5.13).

We now reverse our diagonalizations to obtain estimates for $N^\pm(X)$, with the same error terms. It remains to evaluate $R_Q^\pm(X)$. We see that

$$R_Q^\pm(X) = X \sum_{q \le Q} \mu(q) \mathrm{Res}_{s=1} \xi_q^\pm(s) + \frac{6}{5} X^{5/6} \sum_{q \le Q} \mu(q) \mathrm{Res}_{s=5/6} \xi_q^\pm(s). \tag{5.22}$$

[8] One technical detail arises in applying the arguments of [8], which we address this after the proof.

We now apply the formulas in [39] for the residues, quoted in Theorem 2.2. We have

$$R^{\pm}_Q(X) = X \sum_{q \leq Q} \mu(q) \Big(\alpha^{\pm} \prod_{p|q} \Big(\frac{1}{p^2} + \frac{1}{p^3} - \frac{1}{p^5} \Big) + \beta \prod_{p|q} \Big(\frac{2}{p^2} - \frac{1}{p^4} \Big) \Big) + \frac{6}{5} \gamma^{\pm} \zeta(1/3) X^{5/6} \Big(\sum_{q \leq Q} \mu(q) \prod_{p|q} \Big(\frac{1}{p^{5/3}} + \frac{1}{p^2} - \frac{1}{p^{11/3}} \Big) \Big), \tag{5.23}$$

where $\alpha^+ = \pi^2/36$, $\alpha^- = \pi^2/12$, $\beta = \pi^2/12$, $\gamma^+ = \frac{\Gamma(1/3)^3}{4\sqrt{3}\pi}$, and $\gamma^- = \sqrt{3}\gamma^+$.

We replace the sums over $q \leq Q$ by the appropriate Euler products, with error $\ll XQ^{-1+\epsilon} + X^{5/6}Q^{-2/3+\epsilon} \ll X^{7/9+\epsilon}$, and we see that

$$R^{\pm}_Q(X) = X \Big(\alpha^{\pm} \frac{1}{\zeta(2)\zeta(3)} + \beta \frac{1}{\zeta(2)^2} \Big) + X^{5/6} \Big(\frac{6}{5} \gamma^{\pm} \cdot \frac{\zeta(1/3)}{\zeta(2)\zeta(5/3)} \Big) + O(X^{7/9+\epsilon}). \tag{5.24}$$

Theorem 1.1 now follows by combining (5.20) and (5.24) with Proposition 5.1.

5.4. **Remarks on** [8]**.** Section 4 of Chandasekharan and Narasimhan's work [8] is formulated in terms of a single Dirichlet series with a functional equation. Since we are applying their method to obtain bounds uniform in q, some technical remarks are in order.

Our proof involved the q-nonmaximal zeta function $\xi_q(s) = \sum_n a_q(n) n^{-s}$ and its dual $\widehat{\xi}_q(s) = \sum_{\mu_n} b_q(\mu_n) \mu_n^{-s}$. The functions $\xi_q(s)$ are uniformly bounded above, but their duals are not. Accordingly, we note that Section 4 of [8] (through (4.21)) does not directly use the fact that $\widehat{\xi}_q(s) \ll 1$ for $\Re(s) > 1$. Rather, $\widehat{\xi}_q(s)$ makes its appearance in (5.19), where we applied bounds which are uniform in q.

There is one spot at which the argument in [8] must be amended: the bound (4.16) for $\Delta^{\rho}_y I(\mu_n x)$, and the bounds (4.5) and (4.11) on which it relies, implicitly assume that $\mu_n x \gg 1$. Although this assumption is extremely natural, it is not true in our situation.

We therefore prove (4.5) and (4.11) for $x < 1$. (The variable x of (4.5) and (4.11) corresponds to $\mu_n x$ in (4.16).) The bound (4.5) states that $I(x) = O(x^{3/8+3\rho/4})$. To prove this for $x < 1$, note that there are no poles in the integrand of $I(x)$ for $\Re(s) > 1/6$, the apparent poles coming from $\Gamma(\rho + 1 + \delta - s)$ being cancelled by zeroes of $1/\Delta(\delta - s)$. Therefore, we may shift the contour to $\Re(s) = 5/8$, bound $I(x)$ by $x^{3/8+\rho}$ times an absolute constant, and observe that $x^{3/8+\rho} < x^{3/8+3\rho/4}$ when $x < 1$, giving (4.5).

The bound (4.11) states that $I^{(\rho)}(x) = O(x^{3/8})$, and it is proved for $x < 1$ in similar fashion. We may shift the contour $\mathcal{C}'$ to $\Re(s) = 1/4$, again not passing through any poles, and again with the integral converging absolutely. $I^{(\rho)}(x)$ is thus bounded by $x^{3/4}$ times an absolute constant, which is sharper than (4.11).

6. Generalizations of Roberts' conjecture

The proofs of our generalizations of Roberts' conjecture follow along very similar lines. In this section we will describe these generalizations more explicitly, and explain the new steps required in the proofs.

As we prove a variety of generalizations and work out some explicit examples, this section is rather long. We begin in Section 6.1 with the proof of Theorem 1.2, on 3-torsion in quadratic fields. In Section 6.2 we describe the proof of Theorem 1.3 concerning local specifications, and in Section 6.3 we extend these arguments to the 3-torsion problem (Theorem 1.4).

In Section 6.4 we tackle the problem of arithmetic progressions, and we prove Theorem 6.2, our most general result on cubic fields. In Section 6.5 we use Theorem 6.2 work out some examples in more detail, and in particular we prove Theorem 1.5. We also present the results of some numerical calculations. We conclude in Section 6.6 with a general result on 3-torsion in arithmetic progressions (Theorem 6.5) and the proof of Theorem 1.6.

6.1. **3-torsion in quadratic fields.** As in [13] and [3], we use the folllowing classical result of Hasse, which is proved using class field theory: There is a bijection between pairs of *nontrivial* 3-torsion elements in quadratic fields L with $0 < \pm\mathrm{Disc}(L) < X$, and cubic fields K with $0 < \pm\mathrm{Disc}(K) < X$ which are not totally ramified at any prime. Under this bijection $\mathrm{Disc}(L) = \mathrm{Disc}(K)$, and the condition on K is equivalent to requiring that $\mathrm{Disc}(K)$ be fundamental.

It therefore suffices to count cubic fields which are nowhere totally ramified. Write $M_3^{\pm}(X)$ and $M_3^{\pm}(q, X)$ for the counting functions of such fields. We may count these fields by replacing the q-nonmaximal zeta functions with "q-nonmaximal-or-totally-ramified" zeta functions; in the langauge of Davenport and Heilbronn, we shrink the set U_p, defined in Proposition 1.8, to a new set V_p, which excludes cubic orders which are totally ramified at p. We then adjust Definition 2.1 to incorporate this condition, and we write $a'_q(n)$ for the coefficients of our modified zeta function.

We may estimate $M_3^{\pm}(X)$ using the same proof. By Proposition 5.1, we have

$$M_3^{\pm}(X) = \frac{1}{2}\sum_{q\geq 1}\mu(q)\left(\sum_{n\leq x} a'_q(n)\right) - \frac{3}{\pi^2}X + O(X^{1/2}), \tag{6.1}$$

and Lemma 3.4 establishes that we may again truncate the sum to $q \leq Q$ with error $\ll X/Q^{1-\epsilon}$.

We estimate the sums of the $a'_q(n)$ in the same way as before. The cubic Gauss sum appearing implicitly in the analogue of (5.17) is a little bit different, and we apply the bounds in Proposition 4.2. In place of (5.19), we obtain

$$y^{-\rho}\Delta_y^{\rho}[N_q^{\rho}(X) - R_q^{\rho}(X)] \ll q^{7/8+\epsilon}X^{3/8} + q^{1+\epsilon}X^{3/8}z^{3/8}\left(1 + \left(\frac{X^3}{y^4 z}\right)^{\rho/4}\right), \tag{6.2}$$

as long as $z \geq q^{-2}$.

For each q, we choose $z = z_q = \max(q^{-2}, X^3/y^4)$. When $z_q = q^{-2}$ the second term above is smaller than the first, so that

$$y^{-\rho}\Delta_y^{\rho}[N_q^{\rho}(X) - R_q^{\rho}(X)] \ll q^{7/8+\epsilon}X^{3/8} + q^{1+\epsilon}X^{3/8}(X^3/y^4)^{3/8}. \tag{6.3}$$

Choosing $y = X/Q$ as before, simplifying, and summing over $q \leq Q$, we obtain a contribution to $M(X) - R_Q(X)$ of

$$Q^{15/8+\epsilon}X^{3/8} + Q^{7/2+\epsilon} + X^{1+\epsilon}/Q. \tag{6.4}$$

Because of the new first term, the optimal choice is $Q = X^{5/23}$, which gives an error term of $X^{18/23+\epsilon}$.

The remainder of the machinery of Section 5 works unchanged. We compute the residues of the new Shintani zeta functions using the tables in [39]. We obtain, analogously to (5.23),

$$R_Q^{\pm}(X) = X \sum_{q \le Q} \mu(q) \left(\alpha^{\pm} \prod_{p|q} \left(\frac{2}{p^2} - \frac{1}{p^4} \right) + \beta \prod_{p|q} \left(\frac{2}{p^2} - \frac{1}{p^4} \right) \right) + \frac{6}{5} \gamma^{\pm} \zeta(1/3) X^{5/6} \left(\sum_{q \le Q} \mu(q) \prod_{p|q} \left(\frac{1}{p^{5/3}} + \frac{2}{p^2} - \frac{1}{p^{8/3}} - \frac{1}{p^3} \right) \right), \tag{6.5}$$

and therefore

$$M_3^{\pm}(X) = \frac{1}{2\zeta(2)^2} \alpha^{\pm} X + \frac{3}{5} \gamma^{\pm} \zeta(1/3) X^{5/6} \prod_p \left(1 - \frac{1}{p^{5/3}} - \frac{2}{p^2} + \frac{1}{p^{8/3}} + \frac{1}{p^3} \right) + O(X^{18/23+\epsilon}). \tag{6.6}$$

This is equivalent to (1.3) by the formulas $\zeta(2) = \frac{\pi^2}{6}$ and $\Gamma(1/3)\Gamma(2/3) = \frac{2\pi}{\sqrt{3}}$.

Remark. As we remarked previously, we can obtain an error term of $O(X^{9/11+\epsilon})$ without appealing to the more difficult results of Section 4. To do this, we use only the simple bound (4.1) from Section 4; equivalently, we observe that when we replace U_p with V_p, this multiplies each term in (5.18) by a factor of at most q. (This follows from observing that the bounds in Lemma 4.3 are at most p times those in Lemma 3.3.) This yields, in place of (5.21) and (6.4), the bound

$$M(X) - R_Q(X) \ll Q^{9/2+\epsilon} + X^{1+\epsilon}/Q, \tag{6.7}$$

and we obtain an error term of $X^{9/11+\epsilon}$ by choosing $Q = X^{2/11}$.

6.2. **Generalizations involving local conditions.** In this section we will discuss the proof of Theorem 1.3. By a *local specification* $\mathcal{S}_p$ at p we mean a choice of one or more maximal cubic rings $R/\mathbb{Z}_p$, and we say that a cubic field K satisfies $\mathcal{S}_p$ if $\mathcal{O}_K \otimes \mathbb{Z}_p$ is isomorphic to one of these R. For each p, there are finitely many possibilities for R, and they may be detected by the Delone-Faddeev correspondence modulo 16 (if $p = 2$), 27 ($p = 3$), or p^2 ($p > 3$).

Remark. Our methods also allow us to count nonmaximal cubic orders with various conditions, but for the sake of simplicity we have excluded this possibility.

The possibilities for R are in bijection with extensions of $\mathbb{Q}_p$ of degree at most 3, which have been completely classified. For the classification we refer to the comprehensive paper and database of Jones and Roberts [21]. We also note that the framework we describe here appeared in Roberts' paper [30].

In the tables that follow we list the following information: We list all possibilities for R, and we recall that different choices of R detect the different splitting types of p in K. If p is totally split, partially split, or inert, then R is respectively equal to $\mathbb{Z}_p^3$, $\mathbb{Z}_p \times \mathcal{O}_F$, or $\mathcal{O}_L$, where $\mathcal{O}_F$ and $\mathcal{O}_L$ are the integer rings of the unique unramified quadratic and cubic extensions of $\mathbb{Q}_p$. If p is partially or totally ramified, then R is $\mathbb{Z}_p \times \mathcal{O}_{F'}$ or $\mathcal{O}_{L'}$, where $\mathcal{O}_{F'}$ and $\mathcal{O}_{L'}$ are the integer rings of ramified quadratic and cubic extensions of $\mathbb{Q}_p$. Depending on the value of p, there may be multiple possibilities for F' and L', and we list polynomials generating each possible extension.

We also list the "conductor" p^e for each choice of R. By this we mean the following: Suppose that $x \in V_{\mathbb{Z}}$ corresponds to a cubic ring $\mathcal{O}/\mathbb{Z}$. Then we say that R has conductor p^e if the condition $\mathcal{O} \otimes \mathbb{Z}_p \cong R$ may be detected by reducing x modulo p^e, and if p^e is the minimal integer with this property. We have $e \le 4$ in all cases and $e \le 2$ if $p > 3$, and these quantities naturally appear in our error terms.

Finally, we list the *local densities* at $s = 1$ and $s = 5/6$. The densities in the table are unnormalized, and for each prime p we normalize by dividing by the normalizing factors

$$C_p := 1 + \frac{1}{p} + \frac{1}{p^2}, \quad K_p := \frac{(1 - p^{-5/3})(1 + p^{-1})}{1 - p^{-1/3}} \tag{6.8}$$

for $s = 1$ and $s = 5/6$. (These quantities are simply the sum of the local densities.) The constants $C(\mathcal{S})$ and $K(\mathcal{S})$ appearing in Theorem 1.3 are then given by the products of the normalized local densities at $s = 1$ and $s = 5/6$ respectively. We also include a factor of $C^{\pm}$ or $K^{\pm}$ according to the sign of the discriminants being counted; in light of the adelic origin of our residue formulas, one should consider this choice of sign to be a local specification at the infinite place.

Remark. The normalized local density at $s = 1$ has a simple geometric interpretation. Recall that our count of cubic fields incorporated, for each prime p, a factor of $(1-p^{-2})(1-p^{-3})$ corresponding to the proportion of cubic rings which are maximal at p. The density at p is simply the proportion of maximal cubic rings which have a given splitting type. Under the Delone-Faddeev correspondence, this may then be determined by counting $\mathrm{GL}_2(\mathbb{Z}/p^e\mathbb{Z})$ orbits on $V_{\mathbb{Z}/p^e\mathbb{Z}}$.

To give an example, we compute the density of rings which are totally split at a prime p. Under Delone-Faddeev, these consist of a single $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$-orbit on $V_{\mathbb{Z}/p\mathbb{Z}}$ whose stabilizer has order 6. We then verify that

$$\frac{\frac{1}{6}\#\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})}{p^4} = \frac{1}{6}\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^2}\right) = \frac{1/6}{1 + p^{-1} + p^{-2}} \cdot (1 - p^{-2})(1 - p^{-3}). \tag{6.9}$$

The *unnormalized* density of any cubic ring $R/\mathbb{Z}_p$ is equal to $\frac{1}{|\mathrm{Disc}(R)|_p |\mathrm{Aut}(R)|}$, and the normalization factor has a geometric interpretation which is described in [10] or Proposition 8.8 of [39]. The density at $s = 5/6$ can also be interpreted in a similar but more complicated way.

This brings us now to our tables:

Condition at p	Conductor	Density at $s = 1$	Density at $s = 5/6$
Totally split	p	$1/6$	$(1 + p^{-1/3})^3/6$
Partially split	p	$1/2$	$(1 + p^{-1/3})(1 + p^{-2/3})/2$
Inert	p	$1/3$	$(1 + p^{-1})/3$
Partially ramified	p^2	$1/p$	$(1 + p^{-1/3})^2/p$
$(p \neq 2)$ $x^2 + au^2p$	p^2	$\times\frac{1}{2}$	$\times\frac{1}{2}$
Totally ramified	p^2	$1/p^2$	$(1 + p^{-1/3})/p^2$
$(p \equiv 2 \pmod 3)$	p^2	$\times 1$	$\times 1$
$(p \equiv 1 \pmod 3)$ $x^3 + au^3p$	p^2	$\times\frac{1}{3}$	$\times\frac{1}{3}$

In the ramified case, the fields generated by $x^2 + au^2$ and $x^3 + au^3$ are isomorphic for any $u \in (\mathbb{Z}/p\mathbb{Z})^{\times}$, but are distinct as a ranges over the quadratic or cubic residue classes. The notation $\times\frac{1}{2}$ (for example) means that the local density is halved for each of the two cases.

Remark. The densities at $s = 5/6$ appear in a modified form in Proposition 5.3 of Datskovsky-Wright [10]. All of our subsequent density tables also depend closely on Datskovsky and Wright's work.

At $p = 2$ and $p = 3$ there are additional possibilities, because there are more ramified maximal quadratic rings over $\mathbb{Z}_2$ and cubic rings over $\mathbb{Z}_3$. We list all of the possibilities in the following tables, following the database [21]. We list each ring by giving a generating polynomial over $\mathbb{Z}_2$ or $\mathbb{Z}_3$. (Where a choice of $\pm$ and/or u is listed, each choice generates a different ring.) The values for the conductor were obtained by explicitly calculating the GL_2-orbits on $V_{\mathbb{Z}/16\mathbb{Z}}$ and $V_{\mathbb{Z}/27\mathbb{Z}}$ using PARI/GP.

The densities are given as multiples of the local densities in the table above.[9] To compute these, recall that the local densities at $s=1$ are given by $\frac{1}{|\mathrm{Disc}(R)|_p|\mathrm{Aut}(R)|}$. Note that $|\mathrm{Aut}(R)|=3$ for $R=\mathbb{Z}_3[x]/(x^3-3x^2+3u)$ $(u=1,4,7)$ and $|\mathrm{Aut}(R)|=1$ for the other extensions of $\mathbb{Z}_3$ listed. Moreover, it follows from our work in [39] that the density multipliers at $s=5/6$ are the same as those for $s=1$.

Polynomial over $\mathbb{Z}_2$	Conductor	Density multiplier
$x^2+2x\pm 2$	2^3	$\times\frac{1}{4}$
$x^2\pm 2u$ $(u=1,3)$	2^4	$\times\frac{1}{8}$

Polynomial over $\mathbb{Z}_3$	Conductor	Density multiplier
$x^3\pm 3x+3$	3^2	$\times\frac{1}{3}$
x^3+3x^2+3	3^2	$\times\frac{1}{9}$
x^3-3x^2+3u $(u=1,4,7)$	3^3	$\times\frac{1}{27}$
x^3+3u $(u=1,4,7)$	3^3	$\times\frac{1}{27}$

We are now prepared to prove Theorem 1.3. Consider a set of local specifications $\mathcal{S}_p$ at a finite set of primes $\mathcal{P}$. (We also write $\mathcal{P}=\prod_{p\in\mathcal{P}}p$.) For each q coprime to $\mathcal{P}$, we define zeta functions

$$\xi^{\pm}_{\mathcal{S},q}(s) := \sum_{x\in\mathrm{SL}_2(\mathbb{Z})\backslash V_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|}\Phi_q(x)\Phi_{\mathcal{S}}(x)|\mathrm{Disc}(x)|^{-s}, \tag{6.10}$$

where Φ_q and $\Phi_{\mathcal{S}}$ are the characteristic functions of those x nonmaximal at q and satisfying $\mathcal{S}$, respectively. (Observe that maximality at $\mathcal{P}$ is built into our local specifications.)

As established in [39], and originally proved by Datskovsky and Wright, these zeta functions again have analytic continuations and functional equations of the same shape, with residues

$$\mathrm{Res}_{s=1}\xi^{\pm}_{\mathcal{S},q}(s) = \alpha^{\pm}\mathcal{A}(\mathcal{S})\prod_{p|q}\left(\frac{1}{p^2}+\frac{1}{p^3}-\frac{1}{p^5}\right)+\beta\mathcal{B}(\mathcal{S})\prod_{p|q}\left(\frac{2}{p^2}-\frac{1}{p^4}\right), \tag{6.11}$$

$$\mathrm{Res}_{s=5/6}\xi^{\pm}_{\mathcal{S},q}(s) = \gamma^{\pm}\zeta(1/3)\mathcal{C}(\mathcal{S})\prod_{p|q}\left(\frac{1}{p^{5/3}}+\frac{1}{p^2}-\frac{1}{p^{11/3}}\right). \tag{6.12}$$

The quantities $\mathcal{A}(\mathcal{S}),\mathcal{B}(\mathcal{S}),\mathcal{C}(\mathcal{S})$ are evaluated in [39] in terms of certain adelic integrals. They are naturally multiplicative (e.g., $\mathcal{A}(\mathcal{S})=\prod_{p\in\mathcal{P}}\mathcal{A}(\mathcal{S}_p)$), and when $\mathcal{S}_p$ is the set of all maximal cubic rings over $\mathbb{Z}_p$, we have

$$\mathcal{A}(\mathcal{S}_p)=\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p^3}\right),\quad \mathcal{B}(\mathcal{S}_p)=\left(1-\frac{1}{p^2}\right)^2,\quad \mathcal{C}(\mathcal{S}_p)=\left(1-\frac{1}{p^{5/3}}\right)\left(1-\frac{1}{p^2}\right). \tag{6.13}$$

For a general local specification, $\mathcal{A}(\mathcal{S}_p)$ and $\mathcal{C}(\mathcal{S}_p)$ are equal to the product of (6.13) and the normalized local densities at $s=1$ and at $s=5/6$ given above. Observe that our example in (6.9) exactly computes such an $\mathcal{A}(\mathcal{S}_p)$.

To determine $\mathcal{B}(\mathcal{S}_p)$ in general, we multiply the expression in (6.13) by the *reducible* local density. The unnormalized reducible densities are given by the following table, and we normalize them by dividing by $1+1/p$.

[9] The multipliers of $\frac{1}{27}$ occurring in the $p=3$ table were mistakenly printed as $\frac{1}{81}$ in [30].

Condition at p	Reducible density
Totally split	$1/2$
Partially split	$1/2$
Inert	0
Partially ramified	$1/p$
Totally ramified	0

The partially ramified case is divided into two or (for $p = 2$) six subcases, and the density multipliers are the same as before.

Remark. The reducible density can be described in terms of the geometric interpretation given for the irreducible ($s = 1$) density; the difference is that totally split points are counted triple and inert or totally ramified points are not counted at all. If $\mathcal{S}_p$ counts only one of these types of points, then $\mathcal{B}(\mathcal{S}_p)$ is equal to $3\mathcal{A}(\mathcal{S}_p)$, $\mathcal{A}(\mathcal{S}_p)$, or zero as appropriate. In particular, the ratio of $\mathcal{A}(\mathcal{S}_p)$ and $\mathcal{B}(\mathcal{S}_p)$ in (6.13) is equal to the ratio of the appropriate normalizing factors.

We now ready to prove Theorem 1.3, closely following the proof of Theorem 1.1. Write $N = N(\mathcal{S}) = \prod_{p\in\mathcal{P}} p^{e_p}$, so that $\Phi_{\mathcal{S}}(x)$ is well defined on $V_{\mathbb{Z}/N\mathbb{Z}}$. Then the Fourier transform of $\Phi_{\mathcal{S}}(x)$ is given by

$$\widehat{\Phi}_{\mathcal{S}}(x) = \frac{1}{N^4} \sum_{y\in V_{\mathbb{Z}/N\mathbb{Z}}} \Phi_{\mathcal{S}}(y) \exp(2\pi i[x, y]/N), \tag{6.14}$$

and this Fourier transform appears in the dual zeta function

$$\widehat{\xi}^{\pm}_{\mathcal{S},q}(s) := \sum_{x\in \mathrm{SL}_2(\mathbb{Z})\backslash \widehat{V}_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|} \widehat{\Phi}_q(N^{-1}x)\widehat{\Phi}_{\mathcal{S}}(q^{-2}x)\big(|\mathrm{Disc}(x)|/(q^8N^4)\big)^{-s}. \tag{6.15}$$

Here N^{-1} and q^{-2} are multiplicative inverses of N and q^2 modulo q and N respectively. Observe that $\widehat{\Phi}_q(N^{-1}x) = \widehat{\Phi}_q(x)$. We require a bound on $\widehat{\Phi}_{\mathcal{S}}(q^{-2}x)$, but the trivial bound $|\widehat{\Phi}_{\mathcal{S}}(x)| \leq 1$ suffices for an interesting result. In particular, if we write $\widehat{\xi}^{\pm}_q(s) = \sum b_q(\mu_n)\mu_n^{-s}$ as before, then the series $\widehat{\xi}^{\pm}_{\mathcal{S},q}(s)$ is bounded coefficientwise by $\sum |b_q(\mu_n)|(\mu_n/N^4)^{-s}$. In Proposition 3.2 we now sum over $\mu_n/N^4 > z$ and $\mu_n/N^4 < z$ respectively, so our bounds on the partial sums of (6.15) are equal to N^4 times the bounds of Proposition 3.2.

This factor of N^4 appears in all of our estimates involving the dual zeta function, and the remainder of the analysis is now essentially unchanged. Carrying out the analysis in Section 5, we obtain, in place of (5.21),

$$N(X, \mathcal{S}) - R_Q(X, \mathcal{S}) \ll Q^{7/2+\epsilon} N(\mathcal{S})^4 + X^{1+\epsilon}/Q. \tag{6.16}$$

We optimize our error term by choosing $Q = X^{2/9}N(\mathcal{S})^{-8/9}$. Then the right side of (6.16) is $X^{7/9}N(\mathcal{S})^{8/9}$, and this is the error term appearing in Theorem 1.3.[10] The main terms of Theorem 1.3 are obtained from the residue formulas (6.11) and (6.12).

It remains to prove that the contribution from *reducible* maximal cubic rings matches the contribution of the $\mathcal{B}(\mathcal{S})$ term in (6.11). With the single exception of $\mathbb{Z}^3$, the reducible cubic rings are the rings $\mathbb{Z} \times \mathcal{O}_F$, where $\mathcal{O}_F$ is the ring of integers of a quadratic field. This ring may have any of the splitting types above aside from the inert or totally ramified splitting types, and these conditions depend only on the discriminant modulo $M := 108 \prod_p p^{e_p}$. (Here $108 = 2^2 3^3$ is the gcd of the coefficients in the formula (1.12) for the discriminant; in fact, $M := 4 \prod_p p^{e_p}$ is enough.)

[10] There is a discrepancy between the simple definition of e_p in Theorem 1.3 and the "correct" definition here when $p = 2$ or 3, but this may be absorbed into the implied constant.

The number of such rings is equal to the number of squarefree integers in certain arithmetic progressions modulo M, except for special conditions at 2. To handle these conditions we assume that $2^6|M$, and sum over multiple residue classes modulo 2^6 if necessary. This distinguishes among the eight choices of $\mathcal{O}_F \otimes \mathbb{Z}_2$, and the relevant quadratic fields are counted by the following lemma. The proof is a relatively straightforward generalization of ([40], Chapter I.3.7, Theorem 9), so we omit the detail.

Lemma 6.1. *Assume that* $2^6|m$. *Then the number of quadratic fields* F *with* $0 < \pm\mathrm{Disc}(F) < X$ *and* $\mathrm{Disc}(F) \equiv a \pmod{m}$ *is equal (for each choice of sign) to*

$$\frac{8}{\pi^2 m} X e(a,2) \prod_{\substack{p>2 \\ p^k || m,\ k\geq 1}} e(a,p^k)\left(1-\frac{1}{p^2}\right)^{-1} + O(\sqrt{X}), \tag{6.17}$$

where

$$e(a,p^k) := \begin{cases} 1 & \text{if } p \nmid a, \\ 1 & \text{if } k \geq 2 \text{ and } p^2 \nmid a, \\ 1-1/p & \text{if } k=1 \text{ and } p|a, \\ 0 & \text{otherwise}; \end{cases} \qquad (p \neq 2) \tag{6.18}$$

$$e(a,2) =: \begin{cases} 1 & \text{if } a \equiv 1 \pmod{4}, \\ 1 & \text{if } a \equiv 8, 12 \pmod{16}, \\ 0 & \text{otherwise.} \end{cases} \tag{6.19}$$

We then sum the result in (6.17) over all appropriate residue classes. For example, if we are counting fields split at p, we sum over those a which are quadratic residues modulo p. We then check that this matches the total contribution from $\mathcal{B}(\mathcal{S})$, up to an error $\ll X^{1/2}N(\mathcal{S})$. This is smaller than our previous error term whenever our result is nontrivial, and this completes the proof.

Example. We illustrate our results by computing the expected number of fields K with $0 < \mathrm{Disc}(K) < X := 2 \cdot 10^6$ which are inert at 7 and partially ramified at 5.

Let $\mathcal{S}$ denote the set of these two local specifications. Using the tables above, we compute that

$$C(\mathcal{S}) = \frac{1/3}{1+\frac{1}{7}+\frac{1}{49}} \cdot \frac{1/5}{1+\frac{1}{5}+\frac{1}{25}} = \frac{245}{5301} = .046217\cdots, \tag{6.20}$$

$$K(\mathcal{S}) = \frac{\frac{1}{3}\left(1+\frac{1}{7}\right)\cdot\left(1-\frac{1}{7^{1/3}}\right)}{\left(1-\frac{1}{7^{5/3}}\right)\left(1+\frac{1}{7}\right)} \cdot \frac{\frac{1}{5}\left(1+\frac{1}{5^{1/3}}\right)^2\cdot\left(1-\frac{1}{5^{1/3}}\right)}{\left(1-\frac{1}{5^{5/3}}\right)\left(1+\frac{1}{5}\right)} = .030884\cdots, \tag{6.21}$$

and therefore expect to find

$$\approx .046217 \cdot \frac{1}{12\zeta(3)} X + .030884 \frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} X^{5/6} = 6408.0\cdots - 812.7\cdots \approx 5595 \tag{6.22}$$

fields. Using PARI/GP to analyze the local behavior of the fields in Belabas' tables, we find that there are in fact 5546 such fields.

6.3. **Local conditions for 3-torsion problem in quadratic fields.** We now turn to the proof of Theorem 1.4. In this case, a local specification at p consists of a choice of $\mathbb{Q}(\sqrt{D}) \otimes \mathbb{Q}_p$. In contrast to the case of cubic fields, any local specification at a prime $p > 2$ is determined by the residue class of $D \pmod{p^2}$, and a specification at 2 is determined by $D \pmod{64}$.

As before, if D is a fundamental discriminant, subgroups of $\mathrm{Cl}(D)$ of index 3 are in bijection with cubic fields of discriminant D. Moreover, arguments from algebraic number theory show that local specifications for $\mathbb{Q}(\sqrt{D})$ correspond to local specifications for these cubic fields. In particular, p is inert in $\mathbb{Q}(\sqrt{D})$ if and only if it partially splits in the cubic fields, and p splits if and only if it is either inert or totally split in these cubic fields. If p is ramified and K is a cubic field of discriminant D, then the correspondence is given by the isomorphism $K \otimes \mathbb{Q}_p \simeq (\mathbb{Q}(\sqrt{D}) \otimes \mathbb{Q}_p) \times \mathbb{Q}_p$.

The local densities $C'(\mathcal{S})$ and $K'^{\pm}(\mathcal{S})$ are therefore determined by the tables in Section 6.2.[11] The unnormalized densities are the same, and the normalization factors are given by adding the unnormalized densities for all splitting types other than 'totally ramified':

$$C_p := 1 + \frac{1}{p}, \quad K_p := 1 + \frac{1}{p^{1/3}} + \frac{1}{p^{2/3}} + \frac{2}{p} + \frac{2}{p^{4/3}} + \frac{1}{p^{5/3}}. \tag{6.23}$$

The proof is a straightforward combination of the proofs of Theorems 1.2 and 1.3, and the only new step occurs in our evaluation of the error term. Again $\widehat{\xi}^{\pm}_{\mathcal{S},q}(s)$ is bounded coefficientwise by $\sum |b_q(\mu_n)|(\mu_n/N^4)^{-s}$. This factor of N^4 appears in Proposition 4.2; note that the cutoffs of q^{-3} and q^{-2} for the ranges of μ_n there still apply to μ_n and not μ_n/N^4.

As before we split into small and larger ranges of Q, and in the larger range we obtain, in place of (5.21), (6.4), and (6.16),

$$Q^{15/8+\epsilon} X^{3/8} N^{5/2} + Q^{7/2+\epsilon} N^4 + X^{1+\epsilon}/Q, \tag{6.24}$$

where the first two terms correspond to the ranges $\mu_n < q^{-2}$ and $q^{-2} < \mu_n$ respectively. Our theorem follows by choosing $Q = X^{5/23} N(\mathcal{S})^{-20/23}$.

Remark. To illustrate our method, we remark that we could formally derive Theorem 1.2 as a consequence of Theorem 1.3, by imposing the local condition 'not totally ramified' at every prime. This would not treat the error terms in an acceptable manner, but this would essentially amount to a variation of the same proof.

6.4. **Arithmetic progressions.** This brings us to the problem of counting fields in arithmetic progressions. We wish to simultaneously allow local specifications as in the last section, possibly to the same moduli.

As in prime number theory, we can approach this question by twisting by Dirichlet characters. If χ is a Dirichlet character $\pmod{m}$, write

$$N_3^{\pm}(X, \chi) := \sum_{\substack{[K:\mathbb{Q}]=3 \\ 0<\pm \mathrm{Disc}(K)<X}} \chi(\mathrm{Disc}(K)). \tag{6.25}$$

Then if $(a, m) = 1$, we have the usual orthogonality relation

$$N_3^{\pm}(X; m, a) := \sum_{\substack{[K:\mathbb{Q}]=3 \\ 0<\pm \mathrm{Disc}(K)<X \\ \mathrm{Disc}(K) \equiv a \pmod{m}}} 1 = \frac{1}{\phi(m)} \sum_{\chi \pmod{m}} \overline{\chi}(a) N_3^{\pm}(X, \chi). \tag{6.26}$$

[11]We further check that introducing local specifications also multiplies the contribution of the trivial element of class groups by $C'(\mathcal{S})$. In other words, the local densities of trivial and nontrivial 3-torsion elements of the class group are the same at all finite places, but not at the infinite place. For this reason we don't write $C'^{\pm}(\mathcal{S})$ here.

In addition, estimating $N_3^{\pm}(X; m, a)$ is nontrivial when $(a, m) > 1$. An appropriate choice of local specifications $\mathcal{S}$ allows us to select exactly those cubic fields whose discriminants are divisible by (a, m), and we have

$$N_3^{\pm}(X; m, a) = \frac{1}{\phi(m)} \sum_{\chi \left(\text{mod } \frac{m}{(a,m)}\right)} \overline{\chi}\left(\frac{a}{(a, m)}\right) \sum_{\substack{[K:\mathbb{Q}]=3 \\ 0<\pm\mathrm{Disc}(K)<X \\ K\in\mathcal{S}}} \chi\left(\frac{\mathrm{Disc}(K)}{(a, m)}\right). \tag{6.27}$$

Presuming we can estimate the right hand side, we obtain estimates for $N_3^{\pm}(X; m, a)$ in the same way.

We will obtain estimates for $N_3^{\pm}(X; m, a)$ for any values of m and a, subject to an arbitrary set of local specifications. To do so, we introduce *orbital L-functions*, which are Shintani zeta functions twisted by Dirichlet characters.

Notation and conventions. Although we will work in complete generality, we introduce several simplifying reductions which still allow us to recover results in the general case.

As we are allowing arbitrary local specifications, it suffices to work with *primitive* characters. Assume that we are given a primitive Dirichlet character χ (mod m) and a set of local specifications $\mathcal{S} = (\mathcal{S}_p)_{p\in\mathcal{P}}$. By refining $\mathcal{S}$ if necessary, we assume that $\mathcal{P}$ includes all primes dividing m and that for each $p \in \mathcal{P}$, $\mathcal{S}_p$ consists of a single choice for $\mathcal{O}_K \otimes \mathbb{Z}_p$. We may handle imprimitive characters by introducing local specifications corresponding to the condition $\chi(n) = 0$, and we obtain results for more general local specifications (including the set of all maximal cubic rings) by summing over the $\ll X^{\epsilon}$ possible choices of $\mathcal{S}$.

We define quantities $f_p \geq 0$ by $m = \prod_{p\in\mathcal{P}} p^{f_p}$, and we define characters χ_p (mod p^{f_p}) by the formula $\chi(u) = \prod_p \chi_p(u)$. We always regard the characters χ_p as primitive characters modulo p^{f_p}, except to make sense of this formula (where we regard them as imprimitive characters modulo m). In case $f_p = 0$, χ_p is the trivial character modulo 1.

We define quantities $r_p \geq 0$ to be the p-adic valuations of the discriminants of the cubic rings $R_p/\mathbb{Z}_p$ specified by $\mathcal{S}_p$, and we write $r = \prod_p p^{r_p}$. Then our local specifications $\mathcal{S}$ include a restriction to those fields K whose discriminants satisfy $r|\mathrm{Disc}(K)$ and $(m, \mathrm{Disc}(K)/r) = 1$. In particular, this implies that $\chi(\mathrm{Disc}(K)/r) \neq 0$ for each K being counted.

As before, we define quantities $e_p \geq 0$ such that $\mathcal{S}_p$ may be detected by reducing the Delone-Faddeev correspondence modulo p^{e_p}. Finally, we define an integer N (the "conductor") by $N = \prod_p \mathrm{lcm}(p^{f_p+r_p}, p^{e_p})$. Then all of the conditions described above may be detected by reducing the Delone-Faddeev correspondence modulo N, and apart from possible factors of 2 and 3, N is the minimal integer with this property.

Subject to the assumptions and notation above, we define

$$N_3^{\pm}(X, \mathcal{S}; r, \chi) := \sum_{\substack{0<\pm\mathrm{Disc}(K)\leq X \\ K\in\mathcal{S}}} \chi\left(\frac{\mathrm{Disc}(K)}{r}\right), \tag{6.28}$$

and it is this quantity[12] we will estimate. To do this, we introduce, for each q coprime to N, an orbital L-function

$$L^{\pm}_{\mathcal{S},q}(s, r, \chi) := \sideset{}{'}\sum_{x\in\mathrm{SL}_2(\mathbb{Z})\backslash V_{\mathbb{Z}}} \frac{\chi(\mathrm{Disc}(x)/r)}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}, \tag{6.29}$$

[12]Observe that it is redundant to include r in our notation on the left, but we believe this notation is clearest.

where the sum is restricted to x which are of the correct sign, nonmaximal at q, and satisfy the local specifications given by $\mathcal{S}$. It satisfies the same functional equation as before, with formulas for the residues and the dual zeta function to be described later.

It was proved by Datskovsky and Wright [10] that these L-functions are entire whenever $\chi^6 \neq 1$. In this case, $N_3^{\pm}(X, \mathcal{S}; r, \chi)$ will consist only of an error term. Therefore, we will assume throughout that $\chi^6 = 1$, except as noted in the proof of Theorem 6.2.

We may further reduce to the case where χ is a primitive *cubic* character as follows. If χ is not cubic, then some χ_p is not cubic, so write $\chi_p = \psi_p \phi_p$ where ψ_p is cubic or trivial and ϕ_p is quadratic. Recall that $\mathcal{S}$ specifies a single choice R_p for the cubic ring $\mathcal{O}_K \otimes \mathbb{Z}_p$, and we note that $\mathrm{Disc}(R_p)/p^{r_p}$ is well-defined as an element of $\mathbb{Z}_p^{\times}/(\mathbb{Z}_p^{\times})^2$. It follows that $\phi_p(\mathrm{Disc}(x)/r) = \phi_p(\mathrm{Disc}(R_p)/p^{r_p})\phi_p(r/p^{r_p})$ for any x counted in (6.29), and hence $L^{\pm}_{\mathcal{S},q}(s, r, \chi) = \phi_p(\mathrm{Disc}(R_p)/p^{r_p})\phi_p(r/p^{r_p})L^{\pm}_{\mathcal{S},q}(s, r, \chi\phi_p)$.

This brings us to our most general result on cubic fields:

Theorem 6.2. *Assume the notation and conventions above. Then whenever $\chi^6 \neq 1$, we have*

$$N_3^{\pm}(X, \mathcal{S}; r, \chi) = O(X^{7/9+\epsilon} N^{8/9}). \tag{6.30}$$

When $\chi^6 = 1$, we may reduce to the case $\chi^3 = 1$ as described above, in which case we have

(6.31)
$$N_3^{\pm}(X, \mathcal{S}; r, \chi) = \delta(\chi) C^{\pm}(\mathcal{S}) \frac{1}{12\zeta(3)} X + K^{\pm}(\mathcal{S}, \chi) \frac{4L(1/3, \chi)}{5\Gamma(2/3)^3 L(5/3, \chi^2)} X^{5/6} + O(X^{7/9+\epsilon} N^{8/9}),$$

where $\delta(\chi)$ is 1 if χ is trivial and 0 otherwise, $C^{\pm}(\mathcal{S})$ is as in Section 6.2, and $K^{\pm}(\mathcal{S}, \chi)$ is described below.

The quantity $K^{\pm}(\mathcal{S}, \chi)$ is computed in [39], although many of these computations are originally due to Datskovsky and Wright ([10], Proposition 5.4). We have $K^{\pm}(\mathcal{S}, \chi) = K^{\pm} \prod_{p \in \mathcal{P}} K(\mathcal{S}_p, \chi)$, so it suffices to give the value of $K(\mathcal{S}_p, \chi)$ for each p. Note that $K(\mathcal{S}_p, \chi)$ depends on χ and not only χ_p.

When p does not divide m (the conductor of χ), $K(\mathcal{S}_p, \chi)$ is given by dividing the appropriate value in the table by the normalizing factor

$$K_{p,\chi} := \frac{(1 - \chi(p)^2 p^{-5/3})(1 + p^{-1})}{1 - \chi(p) p^{-1/3}}. \tag{6.32}$$

Condition at p	Density at $s = 5/6$	Value of ϕ_p ($p \neq 2$)
Totally split	$(1 + \chi(p)p^{-1/3})^3/6$	1
Partially split	$(1 + \chi(p)p^{-1/3})(1 + \chi(p)^2 p^{-2/3})/2$	-1
Inert	$(1 + p^{-1})/3$	1
Partially ramified	$(1 + \chi(p)p^{-1/3})^2/p$	–
($p \neq 2$) $x^2 + au^2 p$	$\times\frac{1}{2}$	± 1
Totally ramified	$(1 + \chi(p)p^{-1/3})/p^2$	
($p \equiv 2 \pmod 3$)	$\times 1$	$\phi_p(-3)$
($p \equiv 1 \pmod 3$) $x^3 + au^3 p$	$\times\frac{1}{3}$	$\phi_p(-3)$

For convenience, we have also listed the value of $\phi_p(\mathrm{Disc}(R_p)/p^{r_p})$ for each row, where ϕ_p is the nontrivial quadratic character mod p. Here R_p is any cubic ring over $\mathbb{Z}_p$ of the given splitting type, and r_p is the p-adic valuation of $\mathrm{Disc}(R_p)$.

In the partially or totally ramified cases for $p = 2$ or 3, the density at $s = 5/6$ is given by the relative densities given in our previous tables. The values of ϕ_3 for totally ramified rings are given in the table below.

When p does divide m, then either $p \equiv 1 \pmod 3$ or $p = 3$, and the results are rather different. As each χ_p is primitive cubic, we note that χ_p has conductor p, except χ_3 which has conductor 9.

For $p \neq 3$, our results involve (ordinary) Gauss sums

$$\tau_p(\chi_p) := \sum_{t \in \mathbb{F}_p^\times} \chi_p(t) e^{2\pi i t/p}. \tag{6.33}$$

In either case we divide the values given in the following tables by the normalizing factor

$$K_{p,\chi} := 1 + p^{-1}. \tag{6.34}$$

For $p \neq 3$, we have

Condition at p	Density at $s = 5/6$	Value of ϕ_p ($p \neq 2$)
Totally split	$\tau_p(\chi_p^2)/6p^2$	1
Partially split	$-\tau_p(\chi_p^2)/2p^2$	-1
Inert	$\tau_p(\chi_p^2)/3p^2$	1
Partially ramified	$\chi_p(4)\chi_p'(p)p^{-4/3}$	–
($p \neq 2$) $x^2 + au^2p$	$\times \frac{1}{2}$	± 1
Totally ramified, $x^3 + au^3p$	$\big(\chi_p(a)^2 + \chi_p(a)\chi_p'(p)p^{-1/3}\big)/3p^2$	$\phi_p(-3)$

For $p = 3$, we have the following:

Condition at p	Density at $s = 5/6$	Value of ϕ_3
Totally split	$\chi_p(4)/6p$	1
Partially split	$\chi_p(4)/2p$	-1
Inert	$\chi_p(4)/3p$	1
Partially ramified ($x^2 \pm 3$)	$\pm(1 - \chi_p(2))\chi_p'(p)p^{-7/3}$	∓ 1
$x^3 + 3x + 3$	$(\chi_p(4) - 1)/p^4$	-1
$x^3 + 6x + 3$	$(2\chi_p(4) + 1)/p^4$	1
$x^3 - 3x^2 + 3u$ ($u = 1, 4, 7$)	$\big(\chi_p(u) + \chi_p'(p)p^{-1/3}\big)/p^5$	1
$x^3 + 3x^2 + 3$	$\chi_p(2)\chi_p'(p)p^{-13/3}$	-1
$x^3 + 3u$ ($u = 1, 4, 7$)	$\chi_p(u)\chi_p'(p)p^{-16/3}$	-1

In the tables above, $\chi_p'(n) := \prod_{p' \mid m,\ p' \neq p} \chi_{p'}(n)$.

Proof of Theorem 6.2. We define a test function $\Phi_N(x)$ to be $\chi(\mathrm{Disc}(x)/r)$ when x satisfies $\mathcal{S}$, and zero otherwise. Our assumptions ensure that $\Phi_N(x)$ is well-defined on $V_{\mathbb{Z}}$ and $V_{\mathbb{Z}/N\mathbb{Z}}$, and that $\chi(|\mathrm{Disc}(x)|/r) \neq 0$ for all x being counted. For each q coprime to N, we recall the L-function

$$\xi_{\mathcal{S},q}^{\pm}(s, r, \chi) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|} \Phi_q(x) \Phi_N(x) |\mathrm{Disc}(x)|^{-s}, \tag{6.35}$$

defined in (6.29). The L-functions $L_{\mathcal{S},q}^{\pm}(s, r, \chi)$ again have analytic continuations and satisfy the functional equation (2.7).

We proved the case $\chi = 1$ in Section 6.2, so we now assume that χ is nontrivial. Then $L_{\mathcal{S},q}^{\pm}(s, r, \chi)$ is entire except for a pole at $s = 5/6$ if χ is cubic, in which case the residue is

$$\mathrm{Res}_{s=5/6} L_{\mathcal{S},q}^{\pm}(s, r, \chi) = \frac{2\zeta(2)L(1/3, \chi)}{3\Gamma(2/3)^3} K^{\pm}(\mathcal{S}, \chi) \prod_{p \mid q} \left(\frac{\chi(p)^2}{p^{5/3}} + \frac{1}{p^2} - \frac{\chi(p)^2}{p^{11/3}} \right). \tag{6.36}$$

Moreover, the dual L-function is given by

$$\widehat{L}^{\pm}_{\mathcal{S},q}(s,r,\chi) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z})\backslash \widehat{V}_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|} \widehat{\Phi}_q(x) \widehat{\Phi}_N(q^{-2}x) \big(|\mathrm{Disc}(x)|/(q^8 N^4)\big)^{-s}, \tag{6.37}$$

where

$$\widehat{\Phi}_N(x) = \frac{1}{N^4} \sum_{y \in V_{\mathbb{Z}/N\mathbb{Z}}} \Phi_N(y) \exp(2\pi i [x,y]/N). \tag{6.38}$$

Note in particular that (6.37) does not "see" the Dirichlet character apart from $\widehat{\Phi}_N(q^{-2}x)$.

At this point we argue exactly as we did in Section 6.2, estimating $|\widehat{\Phi}_N(q^{-2}x)| \leq 1$ as before. Everything works in the same way, and we choose $Q = X^{2/9}N^{-8/9}$ in (6.16). When $\chi^3 = 1$ we obtain a $X^{5/6}$ term from the residue at $s = 5/6$, and when $\chi^6 \neq 1$ we obtain only the error term from (6.16).

The values of $K(\mathcal{S}_p, \chi)$ can be readily calculated in terms of the residue formulas appearing in [39]. For example, suppose that χ is unramified at p, and $\mathcal{S}_p$ is "p is totally split". Our formulas in [39] evaluate a quantity $\mathcal{C}_p(a, \chi^2)$, which describes the p-part of the residue of $L^{\pm}_{\mathcal{S},q}(s,r,\chi)$. In this case, $\mathcal{C}_p(a,\chi^2) = \frac{(1-\chi(p)^2 p^{-2/3})(1+\chi(p)p^{-1/3})^2}{1-p^{-2}}$, and the proportion of cubic rings totally split at p is equal to $\frac{1}{6}(1-p^{-1})(1-p^{-2})$. We multiply these together and then divide by $(1-\chi(p)^2 p^{-5/3})(1-p^{-2})$, which was already incorporated into the residue formula (6.36).

Therefore, the normalized value of $K(\mathcal{S}_p, \chi)$ is

$$\frac{(1-\chi(p)^2 p^{-2/3})(1+\chi(p)p^{-1/3})^2}{1-p^{-2}} \cdot \frac{1}{6}(1-p^{-1})(1-p^{-2}) \cdot \frac{1}{(1-\chi(p)^2 p^{-5/3})(1-p^{-2})}, \tag{6.39}$$

which agrees with our table.

We refer to [39] for further explanation and examples.

□

Remark. Datskovsky and Wright state a version of (6.36) ([10], p. 31), but with $L(1/3, \overline{\chi})$ in place of our $L(1/3,\chi)$. Based on [39] and our numerical computations, we believe that $L(1/3,\chi)$ is correct.

6.5. **Examples and computations.** In this section we apply Theorem 6.2 to obtain formulas for the number of cubic field discriminants in arithmetic progressions. We can do this for any arithmetic progression a (mod m), with or without local specifications. Unfortunately, our results are complicated to state in general. Accordingly, we only work out the cases where there are no local specifications beyond those implied by our arithmetic progression, and where either $(a,m) = 1$ or m is a prime power.

We define $N_3^{\pm}(X; m, a)$ and $N_3^{\pm}(X, \chi)$ as in (6.25) and (6.26). Our results imply that for any m and a,

$$N_3^{\pm}(X; m, a) = C_1(m,a) \frac{C^{\pm}}{12\zeta(3)} X + K_1(m,a) \frac{4K^{\pm}}{5\Gamma(2/3)^3} X^{5/6} + O(m^{8/9} X^{7/9+\epsilon}) \tag{6.40}$$

for explicit constants $C_1(m,a)$ and $K_1(m,a)$. This follows from adding the result of Theorem 6.2 for each specification at p, for all p dividing m. It remains to derive explicit formulas for $C_1(m,a)$ and $K_1(m,a)$.

Remark. $C_1(m,a)$ is the density of discriminants congruent to a modulo m, but no similar interpretation exists for $K_1(m,a)$. The quotients $\frac{L(1/3,\chi)}{L(5/3,\chi^2)}$ are part of $K_1(m,a)$, and $K_1(m,a)$ can be positive or negative.

We begin with the case $(a,m)=1$, which we stated in Theorem 1.5. If $4 \nmid m$, we readily deduce that

$$C_1(m,a) = \frac{1}{m}\prod_{p|m}\frac{1}{(1-p^{-3})}, \tag{6.41}$$

and if $4|m$ this is doubled or zero depending on $a \pmod 4$.

We turn now to $K_1(m,a)$. We only obtain contributions to (6.26) from characters χ with $\chi^6=1$, but we must consider imprimitive characters. For any χ with $\chi^6=1$, we write $\chi=\psi\phi$, where ψ is a primitive cubic character to a modulus dividing m, and ϕ is a possibly imprimitive quadratic character modulo m. We note however that the condition $\chi(n)=0$ is built into our local specifications, so the imprimitivity is irrelevant.

We further decompose $\psi=\prod_p \psi_p$ and $\phi=\prod_p \phi_p$ as before, and we have

$$K_1(m,a) = \frac{1}{\phi(m)}\sum_{\chi^6=1}\overline{\chi}(a)\frac{L(1/3,\psi)}{L(5/3,\psi^2)}\prod_{p|m}\sum_{R_p}\phi_p(\mathrm{Disc}(R_p))K(R_p,\psi), \tag{6.42}$$

where for each p, the sum over R_p is over all unramified cubic rings over $\mathbb{Z}_p$.

At this point we refer to our tables for $K(R_p,\psi)$ and make an interesting observation: For each prime $p>3$, the sum over R_p cancels if ϕ_p is trivial and ψ_p is nontrivial, or vice versa. Therefore, the sum in (6.42) is over characters χ such that χ_p is of exact order 1 or 6 for each χ, and for any such χ we have $\psi=\chi^{-2}$.

We now use our tables to evaluate the sum over R_p. For $p\neq 3$, if χ_p is trivial, we have

$$\sum_{R_p}\phi_p(\mathrm{Disc}(R_p))K(R_p,\psi) = \frac{1-\chi(p)^{-2}p^{-4/3}}{(1-\chi(p)^2p^{-5/3})(1+p^{-1})}, \tag{6.43}$$

and if χ_p is sextic then

$$\sum_{R_p}\phi_p(\mathrm{Disc}(R_p))K(R_p,\psi) = \frac{\tau_p(\chi_p^2)^3}{p^2(1+p^{-1})}. \tag{6.44}$$

For $p=3$, a bias appears modulo 9. If ψ_3 is trivial then the sum over R_p is again zero unless ϕ_3 is also trivial, and (6.43) still holds. But if ψ_3 is nontrivial, the sum over R_p is nonzero if and only if ϕ_3 is trivial. In this case, χ_3 has conductor 9 and we have

$$\sum_{R_p}\phi_p(\mathrm{Disc}(R_p))K(R_p,\psi) = \frac{\chi(4)}{4}. \tag{6.45}$$

For $p=2$, there are no cubic characters. There is the nontrivial quadratic character modulo 4, which we denote ϕ_4. In this case we observe that $\phi_4(R_2)=1$ for all unramified cubic rings $R_2/\mathbb{Z}_2$. Therefore $N_3^{\pm}(X,\chi)$ is the same for $\chi=\phi_4$ and for χ the trivial character modulo 4. This reflects the fact that all field discriminants are congruent to 1 modulo 4.

There are two primitive quadratic characters modulo 8; fields K which are totally split or totally inert at 2 have $\mathrm{Disc}(K)\equiv 1 \pmod 8$ and fields which are partially split have $\mathrm{Disc}(K)\equiv 5 \pmod 8$. Since we restrict attention to one of these splitting types at a time, twisting by these characters does not yield any additional information. There are no primitive quadratic characters to moduli which are higher powers of 2, so fields equidistribute in subprogressions modulo 16 and above.

In summary, we have proved the following:

Proposition 6.3. *When m is coprime to a and not divisible by either 3 or 4, we have*

$$K_1(m,a) = \frac{1}{m}\prod_{p|m}\frac{1}{1-p^{-2}}\sum_{\chi^6=1}{}' \,\overline{\chi}(a)\frac{L(1/3,\chi^{-2})}{L(5/3,\chi^2)}\prod_{\substack{p|m\\ p\nmid \mathrm{cond}(\chi)}}\frac{1-\chi(p)^{-2}p^{-4/3}}{1-\chi(p)^2p^{-5/3}}\prod_{\substack{p|m\\ p|\mathrm{cond}(\chi)}}\frac{\tau_p(\chi_p^2)^3}{p^2}, \tag{6.46}$$

where the sum is over primitive sextic characters χ to moduli dividing m, such that χ_p is of exact order 6 for each p.

When m is divisible by 3, the same holds, except that χ_3 must be of exact order 3, and for $p = 3$ we substitute (6.45) for (6.44).

When m is divisible by 4, the above estimate is doubled if $a \equiv 1 \pmod 4$ and zero otherwise.

Remark. In some respects, our formula would be simpler if we summed over imprimitive characters modulo m. However, we find it conceptually clearer to deal only with L-functions associated to primitive characters.

Remark. We used PARI/GP and Dokchitser's ComputeL [15] to compute a variety of values of $K_1(m,a)$. We observed that $K_1(m,a)$ behaves unpredictably with respect to factoring m. As a striking example, there are more cubic field discriminants congruent to 3 than to 2 modulo 7 or 13, but modulo $91 = 7 \cdot 13$ the pattern is reversed.

In fact the K_1 constants for the above progressions are all negative, so if one expects $K_1(m,a)$ to be multiplicative (it is not) then perhaps this result is not surprising. However, we have $K_1(m,2) < K_1(m,3) < K_1(m,4) < 0$ for $m = 7$ and $m = 13$, but $K_1(91,2)$ and $K_1(91,4)$ are very nearly equal (and negative), and $K_1(91,3)$ is much less than either of these.

We further computed that $K_1(91,5) > 0$, which shows that the secondary term can be *positive* when restricted to arithmetic progressions.

We now describe how to handle general arithmetic progressions. This is not difficult, but we do not have a particularly elegant formulation of our results. Any exact statement would involve an enumeration of cases which is essentially equivalent to our previous tables, so we will only give a sketch.

Consider an arithmetic progression $ar \pmod{mr}$, where $(a,m) = 1$ but r may or may not be coprime to m. In this case, apart from the usual behavior at 2, $C_1(m,a)$ is equal to $\frac{1}{\phi(m)}$ times the proportion of cubic fields K such that $(\mathrm{Disc}(K), rm) = r$. This proportion can be written as a product of local proportions over the primes dividing rm, and each local proportion is determined by our previous tables. For example, if $p||r$ and $p||m$, this local proportion is equal to

$$\frac{p^{-1}}{1+p^{-1}+p^{-2}}. \tag{6.47}$$

If we combine this with the factor of $\frac{1}{p-1}$ coming from $\frac{1}{\phi(m)}$, we obtain a local factor of $\frac{1}{p^2(1-p^{-3})}$.

For the secondary term $K_1(m,a)$, (6.42) still holds, except that each sum over R_p is over those rings whose p-adic valuation is compatible with the p-divisibility of r and m. These can again be computed using our previous tables.

To illustrate this, we continue our previous example. Suppose that $p||r$ and $p||m$ with $p > 3$. Consider the contribution of a sextic character χ to (6.42), and write $\chi = \prod_p \chi_p = \prod_p \psi_p\phi_p$ as before. If the quadratic part ϕ_p is nontrivial, then $\sum_{R_p} \phi_p(\mathrm{Disc}(R_p))K(R_p, \psi) = 0$ whether ψ_p is trivial or not.

If ψ_p and ϕ_p are both trivial, then

$$\sum_{R_p} \phi_p(\mathrm{Disc}(R_p))K(R_p, \psi) = \frac{(1 - \psi(p)^2 p^{-2/3})(1 + \psi(p)p^{-1/3})}{(1 - \psi(p)^2 p^{-5/3})(p + 1)}. \tag{6.48}$$

If ϕ_p is trivial and ψ_p is nontrivial, then

$$\sum_{R_p} \phi_p(\mathrm{Disc}(R_p))K(R_p, \psi) = \frac{\psi_p(4)\psi_p'(p)p^{-4/3}}{1 + p^{-1}}. \tag{6.49}$$

We summarize our results in the following proposition. The reader should beware that this result is misleadingly simple, as it obscures the distinction between ψ and ψ_p, but we emphasize that we can obtain results for other progressions in an exactly similar fashion.

Proposition 6.4. *For $p > 3$ and $(a, p) = 1$ we have*

$$C_1(p^2, ap) = \frac{1}{p^2(1 - p^{-3})}, \tag{6.50}$$

$$K_1(p^2, ap) = \frac{1}{p^2 - 1}\left(\frac{\zeta(1/3)}{\zeta(5/3)}\frac{(1 - p^{-2/3})(1 + p^{-1/3})}{1 - p^{-5/3}} + \frac{1}{p^{1/3}}\sideset{}{'}\sum_{\substack{\psi^3=1\\ \psi\neq 1}} \overline{\psi}(2a)\frac{L(1/3, \psi)}{L(5/3, \psi^2)}\right). \tag{6.51}$$

We also obtain in the same manner (again for $p > 3$ and $(a, p) = 1$)

$$C_1(p^3, ap^2) = \frac{1 + \phi_p(-3a)}{p^3(1 - p^{-3})}, \quad K_1(p^3, ap^2) = \frac{(1 + \phi_p(-3a))(1 - p^{-2/3})}{p^3(1 - p^{-2})(1 - p^{-5/3})}\frac{\zeta(1/3)}{\zeta(5/3)}, \tag{6.52}$$

and if we further increase the powers of p in the moduli of any of the previous four equations, then we introduce no new sextic characters and hence we simply divide each term by the appropriate power of p. Moreover, for $p > 3$ there are no cubic fields with discriminants divisible by p^3, and hence we have completely determined the distribution of cubic field discriminants modulo powers of p.

For $p = 2$, there are no cubic characters, and cubic fields equidistribute in subprogressions of the arithmetic progression corresponding to each local specification at 2. For $p = 3$ the analysis is rather lengthy, and discriminants of cubic fields can have 3-adic valuation as large as 5. In the interest of space we will not work out the details here; the idea is that arithmetic progressions $a3^k \pmod{3^{k+1}}$ correlate with local specifications at 3, progressions $a3^k \pmod{3^{k+2}}$ exhibit a bias due to the primitive cubic characters modulo 9, and progressions $a3^k \pmod{3^{k+j}}$ do not exhibit additional bias for $j \geq 3$.

We now illustrate our results with numerical data on the distribution of field discriminants in arithmetic progressions modulo 7 and powers of 7. The "expected" counts are the two main terms of (6.40), and the actual counts were determined from Belabas' tables [1].

$$C_1(7, a) = 0.00993261\ldots,$$

$$K_1(7, a) = \begin{cases} -0.0101147\ldots & a = 5, \\ -0.0149070\ldots & a = 1, \\ -0.0159463\ldots & a = 4, \end{cases} \qquad K_1(7, a) = \begin{cases} -0.0255309\ldots & a = 3, \\ -0.0265702\ldots & a = 6, \\ -0.0313625\ldots & a = 2. \end{cases}$$

a	$N_3^+(2\cdot 10^6, 7, a)$	Expected
1	17229	17209
2	14327	14277
3	15323	15316
4	17027	17024
5	18058	18063
6	15150	15131

a	$N_3^-(10^6, 7, a)$	Expected
1	27281	27216
2	24343	24366
3	25389	25376
4	27035	27036
5	28051	28046
6	25227	25196

$$C_1(49, 7a) = 0.00141894\ldots, \qquad K_1(49, 7a) = \begin{cases} -0.00159849\ldots & a = 3, 4, \\ -0.00382342\ldots & a = 1, 6, \\ -0.00520755\ldots & a = 2, 5. \end{cases}$$

a	$N_3^+(2\cdot 10^6, 49, a)$	Expected
7	2155	2157
14	1920	1910
21	2562	2553
28	2519	2553
35	1921	1910
42	2159	2157

a	$N_3^-(10^6, 49, a)$	Expected
7	3555	3595
14	3362	3355
21	3967	3980
28	3980	3980
35	3345	3355
42	3590	3595

$$C_1(343, 49a) = \begin{cases} 0.000405412\ldots & a = 1, 2, 4, \\ 0 & a = 3, 5, 6, \end{cases} \quad K_1(343, 49a) = \begin{cases} -0.000664801\ldots & a = 1, 2, 4, \\ 0 & a = 3, 5, 6. \end{cases}$$

a	$N_3^+(2\cdot 10^6, 343, a)$	Expected
49	697	692
98	690	692
147	0	0
196	707	692
245	0	0
294	0	0

a	$N_3^-(10^6, 343, a)$	Expected
49	1117	1101
98	1092	1101
147	0	0
196	1083	1101
245	0	0
294	0	0

6.6. **3-torsion in quadratic fields.** We come now to the analogue of Theorem 6.2 for 3-torsion in quadratic fields, and a discussion of 3-torsion in arithmetic progressions. The results are quite similar, so we will keep our discussion brief. Write

$$M_3^{\pm}(X, \mathcal{S}; r, \chi) := \sideset{}{'}\sum_{\substack{0 < \pm \mathrm{Disc}(K) \leq X \\ K \in \mathcal{S}}} \chi\left(\frac{\mathrm{Disc}(K)}{r}\right), \tag{6.53}$$

as in (6.28), but with the restriction to fields K which are nowhere totally ramified. (Recall that these are in bijection with pairs of nontrivial 3-torsion elements in $\mathrm{Cl}(\mathbb{Q}(\sqrt{\mathrm{Disc}(K)}))$. We adopt all of the notation of Section 6.4, and make all of the same assumptions on $\mathcal{S}$, r, and χ. We will prove the following theorem:

Theorem 6.5. *Whenever $\chi^6 \neq 1$, we have*

$$N_3^{\pm}(X, \mathcal{S}; r, \chi) = O(X^{18/23+\epsilon} N^{20/23}). \tag{6.54}$$

When $\chi^6 = 1$, we may reduce to the case $\chi^3 = 1$ as described in Section 6.4, in which case we have

$$M_3^{\pm}(X,\mathcal{S};r,\chi) = \frac{\delta(\chi)C'^{\pm}(\mathcal{S})}{2\pi^2}X + K'^{\pm}(\mathcal{S},\chi)\frac{4L(1/3,\chi)}{5\Gamma(2/3)^3}\prod_{p\nmid\mathrm{cond}(\chi)}\left(1-\frac{\chi(p)^{-1}p^{1/3}+1}{p(p+1)}\right)X^{5/6} + O(X^{18/23+\epsilon}N^{20/23}), \tag{6.55}$$

where $\delta(\chi)$ is 1 if χ is trivial and 0 otherwise, and $C'^{\pm}(\mathcal{S})$ and $K'^{\pm}(\mathcal{S},\chi)$ are described below.

The proof is a straightforward combination of the proofs of Theorems 1.2 and 6.2. The constants $C'^{\pm}(\mathcal{S})$ and $K'^{\pm}(\mathcal{S},\chi)$ are computed in the same way. We normalize $C'^{\pm}(\mathcal{S})$ by dividing by $1+p^{-1}$, and we normalize $K'^{\pm}(\mathcal{S},\chi)$ by dividing by

$$K_{p,\chi} := 1 + \frac{\chi(p)}{p^{1/3}} + \frac{\chi(p)^2}{p^{2/3}} + \frac{2}{p} + \frac{2\chi(p)}{p^{4/3}} + \frac{\chi(p)^2}{p^{5/3}} \tag{6.56}$$

for each p dividing N for which χ_p is trivial (compare with (6.23)), and

$$K_{p,\chi} := 1 + p^{-1} \tag{6.57}$$

for each prime p for which χ_p is nontrivial.

One can now compute as many examples as before, and one finds similar biases in arithmetic progressions to the same moduli. For brevity's sake we will confine ourselves to a discussion of $M_3^{\pm}(X;m,a)$ (defined in the obvious manner) when $(m,6a)=1$. We have, similarly to (6.40),

$$M_3^{\pm}(X;m,a) = C_1'(m,a)\frac{C'^{\pm}}{2\pi^2}X + K_1'(m,a)\frac{4K^{\pm}}{5\Gamma(2/3)^3}X^{5/6} \tag{6.58}$$

for explicit constants $C_1'(m,a)$ and $K_1'(m,a)$. If $(a,6m)=1$, then

$$C_1'(m,a) = \frac{1}{m}\prod_{p\mid m}\frac{1}{(1-p^{-2})}. \tag{6.59}$$

To evaluate $K_1'(m,a)$, we again decompose any nontrivial χ into a primitive cubic character ψ and a quadratic character ϕ, and we have

(6.60)

$$K_1'(m,a) = \frac{1}{\phi(m)}\sum_{\chi^6=1}\overline{\chi}(a)L(1/3,\psi)\prod_{p\nmid\mathrm{cond}(\chi)}\left(1-\frac{\psi(p)^{-1}p^{1/3}+1}{p(p+1)}\right)\prod_{p\mid m}\sum_{R_p}\phi_p(\mathrm{Disc}(R_p))K(R_p,\psi),$$

where the rightmost sum is over all unramified cubic rings over $\mathbb{Z}_p$. This sum is the same as in the problem of counting cubic fields, except for the new normalization factors. When $(m,6a)=1$, this implies (as before) that the outer sum is over characters χ for which each χ_p has exact order 1 or 6, and that $\psi = \chi^{-2}$ for each such χ.

We readily deduce Theorem 1.6, and we could deduce other variations as well.

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