Uniform bounds for lattice point counting and partial sums of zeta functions

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Abstract

We prove uniform versions of two classical results in analytic number theory.

The first is an asymptotic for the number of points of a complete lattice $\Lambda \subseteq \mathbb{R}^d$ inside the d-sphere of radius R. In contrast to previous works, we obtain error terms with implied constants depending only on d.

Secondly, let $\phi(s) = \sum_n a(n)n^{-s}$ be a 'well behaved' zeta function. A classical method of Landau yields asymptotics for the partial sums $\sum_{n<X} a(n)$, with power saving error terms. Following an exposition due to Chandrasekharan and Narasimhan, we obtain a version where the implied constants in the error term will depend only on the 'shape of the functional equation', implying uniform results for families of zeta functions with the same functional equation.

1 Introduction

Let $\Lambda \subseteq \mathbb{R}^d$ be an arbitrary complete lattice (i.e., free $\mathbb{Z}$-module of rank d), and consider the counting function

$$N(\Lambda, R) := \#\{v \in \Lambda \ : \ |v| < R\}.$$

We define $r_{\text{bas}}(\Lambda)$ to be the infimum of all $r \in \mathbb{R}^+$ such that the open ball $B(r)$ of radius r and center 0 contains a $\mathbb{Z}$-basis for Λ.

Theorem 1. *If $R > r_{\text{bas}}(\Lambda)$, then we have*

$$N(\Lambda, R) = \frac{\pi^{d/2}}{\Gamma(d/2+1)} \frac{R^d}{|\det(\Lambda)|} + O_d\left(\frac{1}{|\det \Lambda|} r_{\text{bas}}(\Lambda)^{\frac{2d}{d+1}} \ R^{d \cdot \frac{d-1}{d+1}}\right). \tag{1}$$

Note that $r_{\text{bas}}(\Lambda)$ is $O_d(1)$ times the largest successive minimum of Λ (see [Cas97, Lemma 8, p. 135]), so that this bound could be phrased in terms of successive minima instead.

Many results like Theorem 1 exist in the literature, and we refer to the comprehensive survey article of Ivić, Krätzel, Kühleitner, and Nowak [IKKN06] for an overview and numerous references.

We first note that such results may be proved using the geometry of numbers. One obtains an error term of $O_{d,\Lambda}(R^{d-1})$: see Davenport [Dav51] for the basic principle and Widmer [Wid10, Theorem 5.4] or Ange [Ang14, Proposition 1.5] for versions with a completely explicit error term.

We are interested in the better error terms that come from more analytic techniques. In this context, we could not find any general result where the dependence of the error term on Λ is specified. Such a result (with a different shape, and a slightly better R-dependence of R^{d-2}), was proved by Bentkus and Götze [BG97], but with the dimension d assumed to be at least 9.

Our proof is based on classical work of Landau. It turns out that the Dirichlet series

$$\zeta(s,\Lambda) := \sum_{v \in \Lambda - \{0\}} |v|^{-2s}$$

are *Epstein zeta functions*, enjoying analytic continuation and a functional equation of a uniform shape. Writing $\zeta(s,\Lambda) =: \sum_n a(n)\lambda_n^{-s}$, our question is therefore reduced to obtaining error terms in estimates for the partial sums $\sum_{\lambda_n < X} a(n)$.

This approach was followed in classical work of Landau [Lan12, Lan15], who obtained (1) with the implied constant depending on Λ in an unspecified manner. Landau, and following him Chandrasekharan and Narasimhan [CN62], proceeded by developing general techniques to bound the partial sums of Dirichlet series with analytic continuation and a functional equation. Our second main theorem (of which the first will be a consequence) is a uniform version of this result, valid for a wide class of zeta functions.

We postpone a precise statement to Section 2; the following is a special case.

Theorem 2. *Let $\phi(s) = \sum_n a(n)\lambda_n^{-s}$ be a zeta function with nonnegative coefficients, absolutely convergent for $\mathrm{Re}(s) > 1$, enjoying an analytic continuation to $\mathbb{C}$ which is holomorphic away from a simple pole at $s = 1$, and with a 'well behaved' functional equation of degree d relating $\phi(s)$ to $\widehat{\phi}(1-s)$ for a 'dual zeta function' $\widehat{\phi}(s) = \sum_n b(n)\mu_n^{-s}$.*

Then, we have

$$\sum_{\lambda_n < X} a(n) = \mathrm{Res}_{s=1}\big(\phi(s)\big)X + O\big(X^{\frac{d-1}{d+1}}\delta_1^{\frac{d-1}{d+1}}\widehat{\delta}_1^{\frac{2}{d+1}}\big), \tag{2}$$

provided that the error term is bounded by the main term, and where

$$\delta_1 = \mathrm{Res}_{s=1}\big(\phi(s)\big),$$
$$\widehat{\delta}_1 = \sup_Z \frac{1}{Z}\sum_{\mu_n < Z} |b(n)|.$$

The implied constant depends on the functional equation, but does not depend further on $\phi(s)$ or the $a(n)$.

Here we think of δ_1 as a 'density at $s = 1$', and of $\widehat{\delta}_1$ as the 'density of the dual', even if for technical reasons we cannot formulate the latter in terms of a residue, even if the $b(n)$ are nonnegative. We assume above (as part of being 'well behaved') that $\widehat{\delta}_1$ is finite.

We can now describe how to recognize Theorem 1 as a consequence of Theorem 2. In terms of the Epstein zeta function $\zeta(s,\Lambda)$, we recognize that $N(\Lambda,R) = \sum_{\lambda_n \le R^2} a(n)$. Applying Theorem 2 to $\phi(s) = \zeta(\frac{d}{2}s,\Lambda)$ gives $N(\Lambda,R^{1/d})$ in terms of $\delta_1 = \pi^{d/2}|\det\Lambda|^{-1}\Gamma(\frac{d}{2}+1)^{-1}$ and $\widehat{\delta}_1 = O_d(|\det\Lambda|^{-1}r_{\mathrm{bas}}(\Lambda)^d)$. Renormalizing to get $N(\Lambda,R)$ gives the statement of Theorem 1. We carry out this investigation in more detail in Section 5.

We refer to Section 2 for the precise conditions required of the functional equation in Theorem 2; the definition of 'well behaved' includes (for example) all of the L-functions described in [IK04, Chapter 5.1]. Following [CN62] we stipulate a functional equation (4) without any factors of $\pi^{-s/2}$ or involving the 'conductor'. These factors should instead be incorporated into the definition of $\widehat{\phi}(s)$, so that μ_n will not in general be supported on the integers. This choice of normalization should be kept in mind when bounding $\widehat{\delta}_1$. (See Section 4 for a typical example.)

Results of a similar flavor were proved by Friedlander and Iwaniec [FI05], by an alternative classical method. ('Truncating the contour' instead of 'finite differencing'.) In addition, they explain how their results may be further improved when one can obtain cancellation in certain exponential sums. (It should be possible, at least in principle, to improve the results of this paper by incorporating asymptotic estimates for J-Bessel functions in place of upper bounds.)

Their method assumes more of the zeta function; in particular, they assume that its coefficients $a(n)$ are supported on the positive integers and satisfy the bound $a(n) \ll n^{\epsilon}$. We are especially interested in examples, such as Epstein zeta functions, where these hypotheses fail. Some preliminary work suggests that their method can possibly be made to work without such hypotheses, but that the proofs would not be immediate.

The proof of Theorem 2 consists largely of a careful reading of the analogous proof in [CN62]. Nevertheless, for the convenience of the reader we present a complete proof (closely following [CN62, Theorem 4.1]). (Our result also eliminates a factor of X^{ϵ} from [CN62, Theorem 4.1]; it was mentioned as [CN62, Remark 5.5], and also seen in Landau's earlier work, that this was possible.)

Another application of 'uniform Landau' is the following estimate for the number of ideals of bounded norm in a number field:

Theorem 3. *Let K be a number field of degree $d \geq 1$. Then, the number of integral ideals $\mathfrak{a}$ with $N(\mathfrak{a}) < X$ satisfies the estimate*

$$\#\{\mathfrak{a} \ : \ N(\mathfrak{a}) < X\} = \frac{2^{r_1}(2\pi)^{r_2} hR}{w|\operatorname{Disc}(K)|^{1/2}} X + O\Big(|\operatorname{Disc}(K)|^{\frac{1}{d+1}} X^{\frac{d-1}{d+1}} (\log X)^{d-1}\Big), \tag{3}$$

if the error term is bounded by the main term, and where the implied constant depends on d only.

We prove this theorem for $d \geq 2$ as an application of Theorem 4, our most general version of our main theorem, and we remark that for $d = 1$ the statement is trivial. This is very nearly a direct application of Theorem 2, except that we estimate $\sum_{\mu_n < Z} b(n) \ll Z(\log Z)^{d-1}$, which amounts to formally taking $\widehat{\delta_1} = O\big((\log Z)^{d-1}\big)$. The factor of $(\log X)^{d-1}$ in (3) subsumes both this and a related logarithmic factor in δ_1.

We refer to Ange [Ang14, Corollaire 1.3] and Debaene [Deb16, Corollary 2] for completely explicit bounds, but with error terms growing more rapidly with X. Moreover, [Lan12, (66)] and [CN62, (8.20)] obtain bounds of essentially the same strength, but with the implied constant depending on K. Following the latter reference, we could also obtain an analogous result with the additional condition that $\mathfrak{a}$ represent a fixed element of the ideal class group of K.

There is a further example where Theorem 2 is useful: applied to the *Sato-Shintani zeta functions* [SS74] associated to a *prehomogeneous vector space*. This appeared in the work of the second and third authors [TT13] on counting cubic fields. The zeta functions in question count cubic *rings*, and one can also define zeta functions counting those rings which are 'nonmaximal at q'. A version of Theorem 2 (appearing implicitly in [TT13]), in combination with a sieve, led to good error terms in the counting function for cubic fields. Moreover, these error terms can be further improved — for this, see [BTT], which will apply essentially the version of Theorem 2 stated here, except accounting for secondary poles of the zeta function at $s = \frac{5}{6}$.

Theorem 1 also has potential applications itself. The question came to the third author's attention in the course of his work with Kass [KT], counting rational points of bounded height in the Hilbert scheme of two points in the plane. Some algebraic geometry reduces this to a lattice point counting problem, for which Theorem 1 applies. It turns out that a weaker version of Theorem 1 is equally effective in [KT], but similar lattice point counting problems seem likely to arise in related questions counting points on other vector bundles, and Theorem 1 may prove useful in that context (among others).

Organization of the paper. In Section 2 we state and then prove our most general 'uniform Landau' result (Theorem 4). We follow Chandrasekharan and Narasimhan [CN62] quite closely, albeit with a somewhat different exposition, and while removing factors of X^ϵ in the error terms.

We prove Theorem 2 in Section 3, as a representative (but still fairly general) special case of Theorem 4. We then prove Theorem 3 in Section 4; once the relevant facts about Dedekind zeta functions are recalled, this is also easily deduced from Theorem 4.

Finally, we prove Theorem 1 in Section 5. We must establish a couple of lemmas concerning the geometry of lattices and their duals, and then the results are again immediate from Theorem 4.

2 A uniform version of Landau's method

We now prove a uniform version of Landau's method, which provides estimates for sums of coefficients of a Dirichlet series with functional equation. We will closely follow the version given in [CN62, Theorem 4.1], but indicating the dependence of our estimates on the Dirichlet series itself. In order to give a complete statement of the theorem, we must set up some notation.

2.1 Notation and Statement of Theorem

- (The Dirichlet series) Let $\phi(s)$ and $\psi(s)$ denote two dual Dirichlet series,

$$\phi(s) = \sum_{n\geq 1} \frac{a(n)}{\lambda_n^s}, \quad \psi(s) = \sum_{n\geq 1} \frac{b(n)}{\mu_n^s},$$

  where $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\mu_n\}_{n\in\mathbb{N}}$ are two sequences of strictly increasing positive real numbers tending to ∞. We assume that $\phi(s)$ and $\psi(s)$ each converge absolutely in a certain fixed half-plane.

- (The functional equation and meromorphic continuation) We assume ϕ and ψ satisfy a functional equation of the form

$$\Delta(s)\phi(s) = \Delta(\delta - s)\psi(\delta - s), \tag{4}$$

  where $\delta > 0$ is some real parameter, and

$$\Delta(s) := \prod_{\nu=1}^{N} \Gamma(\alpha_\nu s + \beta_\nu) \qquad (\alpha_\nu > 0, \beta_\nu \in \mathbb{C}) \tag{5}$$

  is a product of $N \geq 1$ Gamma factors where the α_ν are positive. We require $A := \sum_{\nu=1}^{N} \alpha_\nu \geq 1$, and note that $2A$ is frequently called the "degree of the zeta function."

  We also assume that this functional equation provides meromorphic continuation in the following sense: there exists a meromorphic function χ such that $\lim_{|t|\to\infty} \chi(\sigma+it) = 0$ uniformly in every interval $-\infty < \sigma_1 \le \sigma \le \sigma_2 < \infty$, satisfying

$$\chi(s) = \Delta(s)\phi(s), \qquad \text{for } \mathrm{Re}(s) > c_1,$$
$$\chi(s) = \Delta(\delta - s)\psi(\delta - s), \qquad \text{for } \mathrm{Re}(s) < c_2,$$

where c_1 and c_2 are some constants.

Our hypotheses force all the poles of $\phi(s)$ to be contained within a fixed vertical strip, and we assume that $\phi(s)$ has only finitely many poles. This assumption will be necessary for the series in (7) to converge, and so we exclude (for example) Artin L-functions (unless the Artin conjecture is assumed).

- (Polar Data) We define

$$S^0_\phi(X) := \frac{1}{2\pi i}\int_{C_0} \phi(s) X^s \frac{ds}{s} = \sum_\xi X^\xi R_\xi(\log X), \tag{6}$$

where C_0 is any curve enclosing all the singularities of the integrand. In the latter sum over the poles ξ of $\frac{\phi(s)}{s}$, $R_\xi(\log X)$ is a constant for each simple pole ξ, and is generally a polynomial of degree $\mathrm{ord}_\xi\left(\frac{\phi(s)}{s}\right) - 1$.

We also define

$$R_\phi(X) := \sum_\xi X^{\mathrm{Re}(\xi)} R^{\mathrm{abs}}_\xi(\log X), \tag{7}$$

where R^{abs}_ξ is the polynomial obtained from R_ξ by taking absolute values of each of the coefficients.

- (Partial sums) We denote the partial sum by

$$A^0_\phi(X) := \sum_{\lambda_n \le X} a(n).$$

- (Bounds on partial sums) We require a bound on the partial sums of the coefficients of the dual zeta function, which we take to be of the form

$$\sum_{\mu_n \le Z} |b(n)| \le B_\psi(Z) \tag{8}$$

for a function $B_\psi(Z)$ of the form

$$B_\psi(Z) = C_\psi Z^r \log^{r'}(C'_\psi Z) \tag{9}$$

for some $C_\psi, C'_\psi > 0$, $r' \ge 0$ and $r > \frac{\delta}{2} + \frac{1}{4A}$. (We assume $r > \frac{\delta}{2} + \frac{1}{4A}$ for technical reasons; see (24).) This bound is required simultaneously for all Z for which the sum in (8) is nonempty.

With these notations, we prove the following theorem.

Theorem 4. *With the above, we have*

$$A_\phi^0(X) - S_\phi^0(X) \ll X^{-\frac{1}{2A}-\eta} R_\phi(X) + \sum_{X \le \lambda_n \le X+O(y)} |a(n)| + X^{\frac{\delta}{2}-\frac{1}{4A}} z^{-\frac{\delta}{2}-\frac{1}{4A}} B_\psi(z), \tag{10}$$

for every $\eta \ge -\frac{1}{2A}$, and where

$$y = X^{1-\frac{1}{2A}-\eta}, \quad z = X^{2A\eta} = \frac{X^{2A-1}}{y^{2A}}. \tag{11}$$

Moreover, if $a(n) \ge 0$ for all n, then the sum over $|a(n)|$ may be omitted, so that we have simply

$$A_\phi^0(X) - S_\phi^0(X) \ll X^{-\frac{1}{2A}-\eta} R_\phi(X) + X^{\frac{\delta}{2}-\frac{1}{4A}} z^{-\frac{\delta}{2}-\frac{1}{4A}} B_\psi(z). \tag{12}$$

Throughout, and in particular in (10) and (12), the implicit constants depend on: the parameter η, the functional equation (i.e. on δ, N, α_v, and β_v), and on the regions in which ϕ and ψ converge absolutely – but not on other data associated to ϕ or ψ.

This is a variation of Theorem 4.1 in [CN62], with two modifications. First of all, we track the dependence of the error terms on growth estimates for the individual Dirichlet series ϕ and ψ. Secondly, the bound (8) takes the place of a constant β for which

$$\sum_n |b(n)| {\mu_n}^{-\beta} = B'_\psi < \infty, \tag{13}$$

avoiding additional factors of X^ϵ appearing in the error terms in [CN62]. This is not necessarily the only way to do so; indeed, as J. Thorner suggested to the authors, a plausible alternative approach is to choose $\epsilon = o_X(1)$ depending explicitly on X.

Remark 5. *We will only need the full strength of (8) for roughly $Z \asymp z$, and in cases where (8) cannot be proved uniformly for a 'natural' choice of $B_\psi(Z)$, it may be possible to improve the results: see (24) and (25).*

Remark 6. *The bound $\eta \ge -\frac{1}{2A}$ (equivalently, $y \le X$) is essential; without it, Landau's finite differencing method doesn't make sense and counterexamples to the theorem can be constructed.*

As is well known, one can at least obtain upper bounds by smoothing; for example, suppose that the $a(n)$ are nonnegative; then we have

$$A_\phi^0(X) \le \sum_{\mu_n} a(n) e^{1-\lambda_n/X} = \frac{e}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) X^s \Gamma(s) ds.$$

Now shift the contour to the left of the critical strip, apply the functional equation, and bound the value of the dual zeta function.

2.2 Proof

We now prove Theorem 4. We defer some proofs of technical lemmas to after the outline to give a better proof outline.

For each nonnegative integer k, we define the smoothed sums

$$A_\phi^k(X) := \frac{1}{\Gamma(k+1)} \sum_{\lambda_n \le X} a(n)(X-\lambda_n)^k.$$

These smoothed sums are sometimes called *Riesz means*. Typically, it becomes easier to study A_ϕ^k for large k. It is possible to recover asymptotics for the non-weighted sum $A_\phi^0(X)$ from asymptotics for $A_\phi^k(X)$ through Landau's "finite differencing method." Thus the goal is to understand $A_\phi^k(X)$ well.

Recall the notation

$$\frac{1}{2\pi i}\int_{(c)} f(s)ds := \lim_{T\to\infty} \frac{1}{2\pi i}\int_{-T}^{T} f(c+it)dt$$

for $c \in \mathbb{R}$. We recognize $A_\phi^k(X)$ through a classical integral transform (as in [LD17, §2], for example) as

$$A_\phi^k(X) = \frac{1}{2\pi i}\int_{(c)} \phi(s)\frac{\Gamma(s)}{\Gamma(s+k+1)}X^{s+k}ds, \tag{14}$$

where c is large enough so that the Dirichlet series $\phi(s)$ and $\psi(s)$ converge absolutely for $\operatorname{Re} s \ge c$. We take c of the form $c = c(k) = \frac{\delta}{2} + \frac{k}{2A} - \epsilon$ for any ϵ satisfying $0 < \epsilon < \frac{1}{4A}$, where the integer k (labeled ρ in [CN62]) is chosen sufficiently large as to guarantee the following properties:

(i) We have $c > -\operatorname{Re}(\beta_\nu/\alpha_\nu)$ for each ν, guaranteeing that the line $\operatorname{Re} s = c$ is to the right of all poles of $\Delta(s)/\Delta(\delta - s)$, and that the line $\operatorname{Re} s = \delta - c$ is to the left of all poles of $\phi(s)$.

(ii) We have $c > -\operatorname{Re}(\mu/A)$, where $\mu = \frac{1}{2} + \sum_{\nu=1}^{N}(\beta_\nu - \frac{1}{2})$, which we use as a technical prerequisite to satisfy the conditions of Lemma 7.

(iii) We assume that $\frac{\delta}{2} + \frac{1}{4A} + \frac{k}{2A} > r$ (see (24)), and that the fractional part of $\frac{k}{2A} - \epsilon - \frac{\delta}{2}$ is in $(0, \frac{1}{2})$ (see (32)).

(iv) We assume that $c \neq \delta + n$ for any integer n, so that the integrals (15) and (17) do not pass through poles. (In fact, this is implied by (iii), since $c - \delta = \frac{k}{2A} - \epsilon - \frac{\delta}{2}$).

As k may be chosen depending only on 'the shape of the functional equation', implied constants in what follows will be allowed to depend on k.

After shifting the line of integration in (14) to $\operatorname{Re} s = \delta - c$, replacing $\phi(s)$ with $\psi(\delta - s)\Delta(\delta - s)/\Delta(s)$ through the functional equation (4), and performing the change of variables $s \mapsto \delta - s$, we rewrite $A_\phi^k(X)$ as

$$A_\phi^k(X) = S_\phi^k(X) + \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(\delta - s)}{\Gamma(k+1+\delta-s)}\frac{\Delta(s)}{\Delta(\delta-s)}\psi(s)X^{\delta+k-s}ds, \tag{15}$$

where

$$S_\phi^k(X) := \frac{1}{2\pi i}\int_{C_k} \phi(s)\frac{\Gamma(s)}{\Gamma(s+k+1)}X^{s+k}ds, \tag{16}$$

where C_k is a curve enclosing all the singularities of the integrand between $\operatorname{Re}(s) = \delta - c$ and $\operatorname{Res}(s) = c$. (Familiar bounds for the integrand, needed to justify convergence, are recalled in (28).)

We separate the analytic portion of the shifted integral (15) and define

$$I_k(t) := \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(\delta-s)}{\Gamma(k+1+\delta-s)}\frac{\Delta(s)}{\Delta(\delta-s)}t^{\delta+k-s}ds. \tag{17}$$

Then we can rewrite (15) as

$$A_\phi^k(X) - S_\phi^k(X) = W_k(X) := \sum_{n\geq 1}\frac{b(n)}{\mu_n^{\delta+k}}I_k(\mu_n X). \tag{18}$$

In order to study $W_k(X)$, we will need the following properties of $I_k(X)$.

Lemma 7. *Suppose that k is large enough that the line $\operatorname{Re} s = c(k)$ is to the right of all poles of $\Delta(s)/\Delta(\delta-s)$. Let $I_k^{(k)}$ denote the kth derivative of I_k. Then for $t\geq 1$, we have*

$$I_k(t) \ll t^{\frac{\delta}{2}-\frac{1}{4A}+k(1-\frac{1}{2A})}, \qquad I_k^{(k)}(t) \ll t^{\frac{\delta}{2}-\frac{1}{4A}}.$$

As $t\to 0$, we have that

$$I_k(t) \ll t^{\frac{\delta}{2}+k(1-\frac{1}{2A})+\epsilon}, \qquad I_k^{(k)}(t) \ll t^{\frac{\delta}{2}+\epsilon}.$$

Proof. Proved in Section 2.3. □

We are now ready to describe the finite differencing method, which we apply to (18). Define $\Delta_y F(X) := F(X+y) - F(X)$, so that the kth finite difference operator Δ_y^k is given by

$$\Delta_y^k F(X) = \sum_{\nu=0}^{k}(-1)^{k-\nu}\binom{k}{\nu}F(X+\nu y). \tag{19}$$

(See (35) for an alternative formula when F is k times differentiable.)

Lemma 8. *With the same notation as above,*

$$\Delta_y^k A_\phi^k(X) = A_\phi^0(X)y^k + O\Big(y^k\sum_{X\leq\lambda_n\leq X+ky}|a(n)|\Big).$$

Additionally, recalling the definitions of R_ϕ and $S_\phi^k(X)$ from (7) and (16) respectively, we have for $y \ll X$ that

$$\Delta_y^k S_\phi^k(X) = S_\phi^0(X)y^k + O\Big(\frac{y^{k+1}}{X}R_\phi(X)\Big). \tag{20}$$

Proof. Proved in Section 2.3. □

We apply Δ_y^k to (18). For the left hand side of (18), we see from above that

$$\Delta_y^k[A_\phi^k(X) - S_\phi^k(X)] = y^k[A_\phi^0(X) - S_\phi^0(X)] + O\Big(\frac{y^{k+1}}{X}R_\phi(X) + y^k\sum_{X\leq\lambda_n\leq X+ky}|a(n)|\Big). \tag{21}$$

On the other side of (18), we get

$$\Delta_y^k W_k(X) = \sum_{n\geq 1}\frac{b(n)}{\mu_n^{\delta+k}}\Delta_y^k I_k(\mu_n X). \tag{22}$$

Note that the finite difference is taken of $I_k(\mu_n X)$ as a function of X, not of $\mu_n X$. Using the properties of $I_k(X)$ as stated in Lemma 7, one can prove the following lemma.

Lemma 9.

$$\Delta_y^k I_k(\mu_n X) \ll \begin{cases} \max_{t\asymp\mu_n X}|I_k(t)| \ll (\mu_n X)^{\frac{\delta}{2}-\frac{1}{4A}+k(1-\frac{1}{2A})}, \\ (\mu_n y)^k \max_{t\asymp\mu_n X}|I_k^{(k)}(t)| \ll (\mu_n y)^k(\mu_n X)^{\frac{\delta}{2}-\frac{1}{4A}}. \end{cases} \tag{23}$$

Proof. Proved in Section 2.3. □

The first bound in (23) is superior to the second bound when $\mu_n \gg z := X^{2A-1}/y^{2A}$, so that we get the bound

$$\Delta_y^k W_k(X) \ll y^k X^{\frac{\delta}{2}-\frac{1}{4A}} \sum_{\mu_n\le z} b(n)\mu_n^{-\frac{\delta}{2}-\frac{1}{4A}} + X^{\frac{\delta}{2}-\frac{1}{4A}+k(1-\frac{1}{2A})} \sum_{\mu_n>z} b(n)\mu_n^{-\frac{\delta}{2}-\frac{1}{4A}-\frac{k}{2A}} \tag{24}$$

$$\ll y^k X^{\frac{\delta}{2}-\frac{1}{4A}} z^{-\frac{\delta}{2}-\frac{1}{4A}} B_\psi(z) + X^{\frac{\delta}{2}-\frac{1}{4A}+k(1-\frac{1}{2A})} z^{-\frac{\delta}{2}-\frac{1}{4A}-\frac{k}{2A}} B_\psi(z), \tag{25}$$

where in the latter step we deviated from [CN62] by dividing the sums into dyadic intervals $[\frac{Z}{2}, Z]$, bounding the contribution of each by (8), and using (9) to sum the results. Our choice of z equalizes the two terms in (25), so that the second of them may be omitted.

Therefore applying finite difference operators to (18) and inserting the bounds for the left hand side (21), and the right hand side (25), we see that

$$A_\phi^0(X) - S_\phi^0(X) \ll \frac{y}{X} R_\phi(X) + \sum_{X\le\lambda_n\le X+ky} |a(n)| + X^{\frac{\delta}{2}-\frac{1}{4A}} z^{-\frac{\delta}{2}-\frac{1}{4A}} B_\psi(z), \tag{26}$$

which is (10), after the change of variables $y = X^{1-\frac{1}{2A}-\eta}$ for some $\eta \ge 0$.

Suppose further now that $a(n) \ge 0$ for all n. Then, as noted in [CN62, eq. 4.15], $A_\phi^0(X)$ is monotone in X and we have that

$$y^k A_\phi^0(X) \le \Delta_y^k A_\phi^k(X) \le y^k A_\phi^0(X+ky). \tag{27}$$

This may be proved using (35) on $A_\phi^k(X)$. For $i \ge 1$, it's true that $\frac{d}{dX}A_\phi^{i+1}(X) = A_\phi^i(X)$, but one must check that (35) is true of $A_\phi^k(X)$ even though $A_\phi^1(X)$ is not differentiable when X is an integer.

Using the inequalities (27) with (20) gives that

$$A_\phi^0(X) - S_\phi^0(X) \le y^{-k}\Delta_y^k W_k(X) + O\Big(\frac{y}{X}R_\phi(X)\Big),$$

and estimating $\Delta_y^k W_k(X)$ as before we obtain (12) as an upper bound for $A_\phi^0(X) - S_\phi^0(X)$, and similarly as a lower bound for $A_\phi^0(X+ky) - S_\phi^0(X)$. Since $S_\phi^0(X+ky) - S_k^0(X) \ll \frac{y}{X}R_\phi(X)$, we obtain (12) as a lower bound for $A_\phi^0(X+ky) - S_\phi^0(X+ky)$, and correspondingly for $A_\phi^0(X) - S_\phi^0(X)$ after a suitable change of variables.

This completes the proof of Theorem 4.

2.3 Proofs of Technical Lemmas

Proof of Lemma 7. Define

$$G(s) := \frac{\Gamma(\delta-s)}{\Gamma(k+1+\delta-s)} \frac{\Delta(s)}{\Delta(\delta-s)},$$

so that I_k is an inverse Mellin transform of $G(s)$. We will show that $G(s)$ can be compared to a function $H(s)$, whose inverse Mellin transform can be explicitly evaluated in terms of J-Bessel functions. As a consequence of Stirling's approximation, one can show [CN62, 2.12] that for any α,

$$\log \Gamma(z+\alpha) = (z+\alpha-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log 2\pi + O(|z|^{-1})$$

as $|z| \to \infty$, uniformly in regions $|\arg z| < \pi - \delta$ for any fixed $\delta > 0$. Using this expression on $G(s)$, one can show that

$$G(s) \asymp |\operatorname{Im}(s)|^{2A\sigma - A\delta-(k+1)} \tag{28}$$

uniformly on any fixed vertical strip, and further that

$$\log G(s) - \log\Big(\frac{\Gamma(As+\mu)}{\Gamma(\lambda - As)}e^{\Theta s}\Big) = B + O(|s|^{-1}), \tag{29}$$

where

$$\begin{aligned}
\mu &= \frac{1}{2} + \sum_{\nu=1}^{N}(\beta_\nu - \frac{1}{2})\\
\lambda &= \mu + A\delta + k + 1,\\
\Theta &= 2\Big(\sum_{\nu=1}^{N}\alpha_\nu \log\alpha_\nu - A\log A\Big),\\
B &= -\delta\sum_{\nu=1}^{N}\alpha_\nu\log\alpha_\nu + (A\delta+k+1)\log A.
\end{aligned}$$

We therefore have

$$I_k(t) = \frac{1}{2\pi i}\int_{(c)} H(s)t^{\delta+k-s}ds + \frac{1}{2\pi i}\int_{(c)}\big(G(s)-H(s)\big)t^{\delta+k-s}ds, \tag{30}$$

where we define $H(s)$ to be

$$H(s) = \frac{\Gamma(As+\mu)}{\Gamma(\lambda - As)}e^{B+\Theta s}, \tag{31}$$

and we note that it follows from (29) that

$$G(s) - H(s) = H(s)\cdot O(|s|^{-1}).$$

Suppose first that $t \geq 1$. For the second term in (30), we shift the line of integration to $\operatorname{Re} s = c + \frac{1}{2A}$. Our assumption (iii) on k imply that we do not pass through any poles, and the shifted integral converges absolutely by (28), so that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{(c)}\big(G(s)-H(s)\big)t^{\delta+k-s}ds &= \frac{1}{2\pi i}\int_{(c+\frac{1}{2A})} H(s)\cdot O(|s|^{-1})t^{\delta+k-s}ds\\
&\ll t^{\delta+k-c-(1/2A)}\\
&\ll t^{\frac{\delta}{2}+\frac{2A-1}{2A}\cdot k - \frac{1}{4A}}.
\end{aligned} \tag{32}$$

For the first term in (30), we recognize it as a J-Bessel function [Wat95]

$$\frac{1}{2\pi i}\int_{(c)} H(s)t^{\delta+k-s}ds = A_1\,(\tilde{t}^{\,1/2A})^{A\delta+(2A-1)k}J_{2\mu+A\delta+k}(2\tilde{t}^{\,1/2A}) \tag{33}$$

for a positive constant A_1 and where $\tilde{t} = te^{-\Theta}$ is a linear change of variables. Using the classical bound $J_\nu(x) \ll x^{-1/2}$ (as in [CN62, (2.12)] or [Wat95]), we see that (33), and hence also (17), is bounded by

$$\ll (t^{1/2A})^{A\delta+(2A-1)k}t^{-1/4A} = t^{\frac{\delta}{2}+\frac{2A-1}{2A}\cdot k-\frac{1}{4A}}.$$

As $t \to 0$, the bound $I_k(t) \ll t^{\frac{\delta}{2}+\frac{2A-1}{2A}k+\epsilon}$ follows from immediately bounding the integrand in (17) absolutely.

These prove the two bounds for $I_k(t)$. We now prove the corresponding bounds for $I_k^{(k)}(t)$. The argument is largely the same as above. With $c_0 = \frac{\delta}{2}-\epsilon$ (which is c_k when $k = 0$), define a contour C' as follows: from $c_0 - i\infty$ up to $c_0 - iR$, right to $c_0 + r - iR$, up to $c_0 + r + iR$, left to $c_0 + iR$, up to $c_0 + i\infty$. The parameters r and R are chosen as large as necessary so that passing the contour from the line $\operatorname{Re} s = c_k$ to C' does not cross any poles.

Thus shifting the contour, and differentiating under the integral sign, we have

$$I_k^{(k)}(t) = \frac{1}{2\pi i}\int_{C'} h(s)t^{\delta-s}ds + \frac{1}{2\pi i}\int_{C'} (g(s)-h(s))t^{\delta-s}ds, \tag{34}$$

where

$$g(s) = \frac{\Delta(s)}{(\delta-s)\Delta(\delta-s)},$$

and $h(s)$ is defined as in $H(s)$ (in (31)), but with $k = 0$ in the parameter λ. As before

$$g(s) - h(s) = h(s)\cdot O(|s|^{-1}).$$

The second integral is bounded analogously to the integral of $G(s) - H(s)$ above, by shifting to the right, giving for $t \to 0$

$$\begin{aligned}\frac{1}{2\pi i}\int_{C'} (g(s)-h(s))t^{\delta-s}ds &= \frac{1}{2\pi i}\int_{C'+\frac{1}{2A}} h(s)\cdot O(|s|^{-1})t^{\delta-s}ds \\ &\ll t^{\frac{\delta}{2}-\frac{1}{2A}+\epsilon} \ll t^{\frac{\delta}{2}-\frac{1}{4A}}.\end{aligned}$$

The first integral can similarly be explicitly evaluated in terms of a the J-Bessel function. Elementary manipulations as above show

$$\frac{1}{2\pi i}\int_{C'} h(s)t^{\delta+s}ds = A_1\tilde{t}^{\delta/2}J_{2\mu+A\delta}(2\tilde{t}^{1/2A}) \ll t^{\delta/2-\frac{1}{4A}}.$$

Finally, we have $I_k^{(k)}(t) \ll t^{\frac{\delta}{2}+\epsilon}$ as $t \to 0$ by trivially bounding (34) on the initial line of integration. This completes the proof. $\square$

Proof of Lemma 8. Applying the finite differencing operator Δ_y^k directly to $A_\phi^k(X)$ gives that

$$\begin{aligned}\Delta_y^k A_\phi^k(X) &= \sum_{\lambda_n \le X} a(n) \frac{\Delta_y^k (X-\lambda_n)^k}{\Gamma(k+1)} \\ &\quad + \frac{1}{\Gamma(k+1)} \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} \sum_{\lambda_n \in (X, X+\nu y]} a(n)(X+\nu y - \lambda_n)^k \\ &= A_\phi^0(X) y^k + O\big(y^k \sum_{X \le \lambda_n \le X+ky} |a(n)|\big).\end{aligned}$$

We have used the explicit evaluation $\Delta_y^k (X-\lambda_n)^k = y^k \Gamma(k+1)$ to simplify this expression; for a k-times differentiable function F, one can use induction on k to show that

$$\Delta_y^k F(x) = \int_x^{x+y} dt_1 \int_{t_1}^{t_1+y} dt_2 \cdots \int_{t_{k-1}}^{t_{k-1}+y} F^{(k)}(t_k) dt_k. \tag{35}$$

We also use (35) to prove (20): Since the kth derivative of $S_\phi^k(X)$ is exactly $S_\phi^0(X)$ (for any c satisfying the listed hypotheses), we then have that

$$\Delta_y^k S_\phi^k(X) = \int_X^{X+y} dt_1 \int_{t_1}^{t_1+y} dt_2 \cdots \int_{t_{k-1}}^{t_{k-1}+y} S_\phi^0(t_k) dt_k. \tag{36}$$

The result then follows by writing $S_\phi^0(t)$ in terms of the residues of $\phi(s)$, as in (6), and substituting into (36).

□

Proof of Lemma 9. For $y \ll X$, we have the trivial inequality using only the definition of the finite differencing operator Δ_y^k,

$$\Delta_y^k I_k(\mu_n X) \ll \max_{t \asymp \mu_n X} |I_k(t)|.$$

For the second bound, we use (35) to see that

$$\Delta_y^k I(\mu_n X) = \int_X^{X+y} dt_1 \int_{t_1}^{t_1+y} dt_2 \cdots \int_{t_{k-1}}^{t_{k-1}+y} I_k^{(k)}(\mu_n t_k) dt_k \ll y^k \max_{t \asymp \mu_n X} I_k^{(k)}(t),$$

where we have trivially bounded the iterated integrals in the last inequality.

In both cases the lemma now follows from the bounds of Lemma 7. □

3 A simpler version: Proof of Theorem 2

For the reader's convenience, we give the (brief!) explanation of how Theorem 2 follows immediately from Theorem 4. Other variations can be proved in the same way.

We assumed that $\phi(s)$ has a 'well behaved' functional equation. To make this precise, consider the following special case of the conditions described in Section 2.1: Assume that $\delta = 1$, so that the functional equation relates s to $1-s$. We assume that each α_v in (41) equals $\frac{1}{2}$, so that $d = N = 2A$ is the usual degree of the zeta function. We also assume that both ϕ and ψ are holomorphic away from simple poles at $s = 1$. If ψ has nonnegative coefficients, then this implies that there exists a positive constant $\widehat{\delta_1}$ for which we may take $B_\psi(Z) = \widehat{\delta_1} Z$ in (8); in any case, we assume that such a $\widehat{\delta_1}$ exists.

By definition, we have

$$R_\phi(X) = X \cdot \operatorname{Res}_{s=1} \phi(s) + \phi(0). \tag{37}$$

By the functional equation we have $\phi(0) \ll |\operatorname{Res}_{s=1} \psi(s)|$, and

$$|\operatorname{Res}_{s=1} \psi(s)| \leq \limsup_{s \to 1^+} (s-1) \sum_{\mu_n} |b(n)| \mu_n^{-s} \leq \limsup_{s\to 1^+} (s-1) \sum_Z |b(n)| Z^{1-s}, \tag{38}$$

with the last sum over all dyadic intervals $[Z, 2Z]$ on which the μ_n are supported. Writing $Z_{\min}$ for the smallest value of μ_n, this last quantity is bounded by

$$\widehat{\delta_1} \limsup_{s\to 1^+} (s-1) Z_{\min}^{1-s} \frac{1}{1-2^{1-s}} \ll \widehat{\delta_1}.$$

Applying Theorem 4, we thus obtain

$$\sum_{\lambda_n < X} a(n) - \operatorname{Res}_{s=1} \big(\phi(s)\big) X \ll \delta_1 X^{1-\frac{1}{d}-\eta} + \widehat{\delta_1} X^{\frac{1}{2}-\frac{1}{2d}} \cdot (X^{d\eta})^{\frac{1}{2}-\frac{1}{2d}}.$$

We equalize error terms by choosing η so that $\delta_1 X^{\frac{1}{2}-\frac{1}{2d}} = \widehat{\delta_1} X^{\eta} (X^{d\eta})^{\frac{1}{2}-\frac{1}{2d}}$, so that the error is equal to $O(X^{\frac{d-1}{d+1}} \delta_1^{\frac{d-1}{d+1}} \widehat{\delta_1}^{\frac{2}{d+1}})$, as claimed in Theorem 2; the condition $\eta \geq -\frac{1}{2A}$ is equivalent to our demand that the error term be bounded by the main term.

4 Ideals in number fields: Proof of Theorem 3

The proof follows immediately from Theorem 4 upon recalling the properties of the associated Dedekind zeta function. Recall (e.g. from [IK04, Chapter 5.10]) that if $K/\mathbb{Q}$ is a number field of degree d, then its *Dedekind zeta function*

$$\zeta_K(s) = \sum_{\mathfrak{a} \neq 0} (N\mathfrak{a})^{-s} \tag{39}$$

satisfies the functional equation

$$\Delta(s)\zeta_K(s) = \Delta(1-s)\widetilde{\zeta_K}(1-s), \tag{40}$$

with

$$\Delta(s) = \Gamma\Big(\frac{s}{2}\Big)^{r_1+r_2} \Big(\frac{s+1}{2}\Big)^{r_2}, \tag{41}$$

where r_1 is the number of real embeddings of K and r_2 the number of pairs of complex conjugate embeddings (so that $d = r_1 + 2r_2$), $q := |\operatorname{Disc}(K)|$, and

$$\widetilde{\zeta_K}(s) = q^{s-\frac{1}{2}} \pi^{\frac{d}{2}-ds} \zeta_K(s) = \sum_{\mathfrak{a}\neq 0} q^{-\frac{1}{2}} \pi^{\frac{d}{2}} \Big(N\mathfrak{a} \cdot \frac{\pi^d}{q}\Big)^{-s}. \tag{42}$$

The zeta function $\zeta_K(s)$ is entire, away from a simple pole at $s=1$ with residue

$$\operatorname{Res}_{s=1}\zeta_K(s)=\frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{q}}\ll_d(\log q)^{d-1} \tag{43}$$

where w is the number of roots of unity in K, h is the class number of K, R is the regulator of K, and where the upper bound is [Lou01, Theorem 1].

We have $\zeta_K(0)\ll q^{1/2}(\log q)^{d-1}$ by (40) and (43) (and indeed $\zeta_K(0)=0$ if K is not imaginary quadratic), and we apply Theorem 4 with

$$\delta=1,\ A=\frac{d}{2},\ R_\phi(X)=X(\log q)^{d-1}+O\big(q^{1/2}(\log q)^{d-1}\big).$$

We have $\zeta_K(s)\le\zeta(s)^d=\sum_n d_d(n)n^{-s}$ coefficientwise, and

$$\sum_{n<Z}d_d(n)\ll_d Z(\log Z)^{d-1}, \tag{44}$$

so that we may take

$$B_\psi(Z)=Zq^{1/2}(\log(Zq))^{d-1}$$

to conclude that

$$\#\{\mathfrak{a}\ :\ N(\mathfrak{a})<X\}-X\operatorname{Res}_{s=1}\zeta_K(s)-O\big(q^{1/2}(\log q)^{d-1}\big)\ll \\ X^{1-\frac{1}{d}-\eta}(\log q)^{d-1}+q^{1/2}X^{\frac12-\frac{1}{2d}}\cdot(X^{d\eta})^{\frac12-\frac{1}{2d}}(\log qX^{d\eta})^{d-1}.$$

We choose $X^\eta=X^{\frac{d-1}{d(d+1)}}q^{-\frac{1}{d+1}}$; formally, this is equivalent to applying Theorem 2 with $\delta_1\ll(\log q)^{d-1}$ and $\widehat{\delta_1}\ll q^{1/2}(\log qX)^{d-1}$. (We may not literally apply Theorem 2 as stated because this $\widehat{\delta_1}$ depends on X.) We also note that $\log(qX^{d\eta})\ll_d\log(X)$ whenever $q\le X$ (and if $q>X$, our conclusion does not beat the trivial bound (44)).

Putting everything together, we have

$$\#\{\mathfrak{a}\ :\ N(\mathfrak{a})<X\}=\frac{2^{r_1}(2\pi)^{r_2}hR}{w|\operatorname{Disc}(K)|^{1/2}}X+O\Big(|\operatorname{Disc}(K)|^{\frac{1}{d+1}}X^{\frac{d-1}{d+1}}(\log X)^{d-1}\Big).$$

5 Counting lattice points: Proof of Theorem 1

5.1 Background on Epstein zeta functions

We assemble some background material on Epstein zeta functions which will be needed in the proof. Epstein's original paper is [Eps03]; our formulation of his results can be found (for example) in [BBS14], but to our knowledge the only reference for the proofs is Epstein's original work. We also refer to [Cas97] for a good reference on lattices and the geometry of numbers.

If $\Lambda\subseteq\mathbb{R}^d$ is a rank d lattice, then we choose a matrix $L\in\mathrm{GL}_d(\mathbb{R})$ for which $\Lambda=\{Lx\ :\ x\in\mathbb{Z}^d\}$, and define $\det\Lambda=|\det L|$. (L is not uniquely defined, but $\det\Lambda$, Λ^*, and $\zeta(s,\Lambda)$ will be.)

We define the *dual lattice* Λ^* to be the set of all vectors $u\in\mathbb{R}^d$ such that $u^Tv\in\mathbb{Z}$ for every $v\in\Lambda$. It is easy to show that Λ^* is actually a lattice of rank d, and in fact it is given by

$$\Lambda^*=\{(L^T)^{-1}x : x\in\mathbb{Z}^d\}.$$

Thus Λ is also the dual lattice of Λ^*, and $\det \Lambda \det \Lambda^* = 1$.

The function $v \mapsto |v|^2$ is a positive definite quadratic form on Λ: if $v = Lx$ where $x \in \mathbb{Z}^d$, then $|v|^2 = Lx \cdot Lx = x^T(L^T L)x$. Writing $Q = L^T L$ for the matrix associated to this quadratic form, we have $|v|^2 = Q[x] := x^T Q x$ and $\det Q = \det(L^T L) = (\det \Lambda)^2$.

Then the *Epstein zeta function* associated to Λ (or to Q) is defined by the Dirichlet series

$$\zeta(s, \Lambda) := \zeta(s, Q) := \sum_{v \in \Lambda - \{0\}} |v|^{-2s} = \sum_{x \in \mathbb{Z}^d - \{0\}} Q[x]^{-s}. \tag{45}$$

It converges absolutely for $\mathrm{Re}(s) > \frac{d}{2}$, has analytic continuation to $\mathbb{C}$ apart from a simple pole at $s = \frac{d}{2}$ with residue

$$\mathrm{Res}_{s=\frac{d}{2}} \zeta(s, \Lambda) = \frac{1}{\sqrt{|\det Q|}} \frac{\pi^{d/2}}{\Gamma(d/2)} = \frac{1}{|\det \Lambda|} \frac{\pi^{d/2}}{\Gamma(d/2)},$$

and satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta(s, \Lambda) = (\det \Lambda)^{-1}\pi^{s-\frac{d}{2}}\Gamma(\tfrac{d}{2} - s)\zeta(\tfrac{d}{2} - s, \Lambda^*). \tag{46}$$

5.2 Conclusion of the proof

In the introduction, we noted that we can prove Theorem 1 by rescaling both $\zeta(s, \Lambda)$ and the output of Theorem 2. But by using Theorem 4, it is possible to avoid any scaling.

We apply Theorem 4 with $X = R^2$, $\phi(s) = \zeta(s, \Lambda)$, $\psi(s) = (\det \Lambda)^{-1}\pi^{\frac{d}{2}-2s}\zeta(s, \Lambda^*)$, $\delta = \frac{d}{2}$, $A = 1$. We obtain

$$N(\Lambda, R) = \frac{\pi^{d/2}}{\Gamma(d/2+1)} \frac{R^d}{|\det(\Lambda)|} + O_d\bigg(\frac{1}{|\det \Lambda|} R^{d-1-2\eta} + \frac{1}{|\det \Lambda|} C'_{\zeta(\cdot,\Lambda^*)} R^{(1+2\eta)\left(\frac{d}{2}-\frac{1}{2}\right)}\bigg), \tag{47}$$

where $C'_{\zeta(\cdot,\Lambda^*)}$ is a positive constant guaranteeing the bound

$$\sum_{\substack{v \in \Lambda^* \\ 0 < |v|^2 \leq Z}} 1 \leq C'_{\zeta(\cdot,\Lambda^*)} Z^{d/2}, \tag{48}$$

and upon choosing $R^{2\eta} = R^{\frac{d-1}{d+1}}(C'_{\zeta(\cdot,\Lambda^*)})^{-\frac{2}{d+1}}$ we obtain

$$N(\Lambda, R) = \frac{\pi^{d/2}}{\Gamma(d/2+1)} \frac{R^d}{|\det(\Lambda)|} + O_d\bigg(\frac{1}{|\det \Lambda|}(C'_{\zeta(\cdot,\Lambda^*)})^{\frac{2}{d+1}}\ R^{d \cdot \frac{d-1}{d+1}}\bigg), \tag{49}$$

We will see that the condition $\eta \geq -\frac{1}{2A}$ holds whenever $R > r_{\mathrm{bas}}$; it remains to bound $C'_{\zeta(\cdot,\Lambda^*)}$, which we do in the next lemma.

Lemma 10. *For any complete lattice $\Lambda \subseteq \mathbb{R}^d$, let $\lambda_1(\Lambda)$ denote the length of the shortest nontrivial vector in Λ. The number of lattice points in Λ satisfying $|v| \leq X$ is bounded by*

$$\sum_{\substack{v \in \Lambda \\ 0 < |v| \leq X}} 1 \ll_d \frac{X^d}{\lambda_1(\Lambda)^d}. \tag{50}$$

Therefore, in the notation above, $C'_{\zeta(\cdot,\Lambda)}$ can be taken as

$$C'_{\zeta(\cdot,\Lambda)} = \frac{c_d}{\lambda_1(\Lambda)^d}$$

for some absolute constant c_d depending only on the dimension d.

Proof. Assume $X \leq \lambda_1(\Lambda)$ (otherwise, the bound is trivial), and define $R_j := \{x \in \mathbb{R}^d : j\lambda_1(\Lambda) \leq |x| < (j+1)\lambda_1(\Lambda)\}$ to be a set of d-dimensional annuli, so that

$$\sum_{\substack{v\in\Lambda\\|v|\leq X}} 1 = \sum_{j\geq 1} \sum_{\substack{v\in\Lambda\cap R_j\\|v|\leq X}} 1 \leq \sum_{j\leq \lfloor X/\lambda_1(\Lambda)\rfloor} \#\{v \in \Lambda \cap R_j\}.$$

To bound $\#\{v \in \Lambda \cap R_j\}$, consider n-spheres of radius $\frac{\lambda_1(\Lambda)}{2}$ around each v being counted: their interiors are disjoint and lie within the annulus $\{\frac{|x|}{\lambda_1(\Lambda)} \in [j-\frac{1}{2}, j+\frac{3}{2}]\}$, so that

$$\#\{v \in \Lambda \cap R_j\} \ll_d \left(j+\frac{3}{2}\right)^d - \left(j-\frac{1}{2}\right)^d \ll_d j^{d-1},$$

yielding the bound

$$\sum_{\substack{v\in\Lambda\\|v|\leq X}} 1 \ll_d \sum_{j\leq \lfloor X/\lambda_1(\Lambda)\rfloor} j^{d-1} \ll_d \frac{X^d}{\lambda_1(\Lambda)^d}, \tag{51}$$

where the implicit constants depend only on the dimension d, and not on Λ. □

Remark 11. *Observe that, owing to the shape of the functional equation of the Epstein zeta function, the proof of Theorem 1 requires as input a simpler but similar statement.*

One can improve the bound in Theorem 1, at the expense of complicating its statement, by incorporating a stronger bound than Lemma 10. For example, by Widmer's bound [Wid10, Theorem 5.4], we have

$$\sum_{\substack{v\in\Lambda\\|v|\leq X}} 1 \ll_d 1 + \max_{1\leq k\leq d} \frac{X^k}{\lambda_1(\Lambda)\cdots\lambda_k(\Lambda)}. \tag{52}$$

Lemma 12. *Suppose Λ is any rank d lattice in $\mathbb{R}^d$, and let Λ^* denote its dual lattice. Let $r_{\text{bas}}(\Lambda^*)$ denote the infimum of all $r \in \mathbb{R}^+$ such that the ball $B(r)$ contains a basis for Λ^*. Then*

$$\lambda_1(\Lambda) \cdot r_{\text{bas}}(\Lambda^*) \geq 1.$$

Proof. Recall the definition of the dual lattice, $\Lambda^* := \{w \in \mathbb{R}^d : \forall v \in \Lambda, \langle v, w\rangle \in \mathbb{Z}\}$. Let $v \in \Lambda$ be of minimal length, so that $\|v\| = \lambda_1(\Lambda)$. Suppose $w_1, \ldots, w_d$ is a set of n linearly independent elements in Λ^* fitting within $\overline{B}(V_{\Lambda^*})$. Then there exists i such that $\langle w_i, v\rangle \neq 0$. Then by the definition of Λ^* above, we have $\langle w_i, v\rangle \in \mathbb{Z}$, and thus $|w_i||v| \geq 1$. □

By Lemmas 10 and 12, we can take

$$C'_{\zeta(\cdot,\Lambda^*)} = c_d/\lambda_1(\Lambda^*)^d \leq c_d\, r_{\text{bas}}(\Lambda)^d. \tag{53}$$

The condition $\eta \geq -\frac{1}{2A}$ is equivalent to $R^{\frac{d-1}{d+1}}\left(C'_{\zeta(\cdot,\Lambda^*)}\right)^{-\frac{2}{d+1}} \geq R^{-1}$, which by (53) is true if $r_{\text{bas}}(\Lambda) \ll_d R$. We may then allow $r_{\text{bas}} < R$ by multiplying R^η by a factor that is $O_d(1)$, which multiplies the error term in (49) by another (harmless) factor of $O_d(1)$.

Our result therefore follows by inserting (53) into (49).

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