RELATIONS AMONG DIRICHLET SERIES WHOSE COEFFICIENTS ARE CLASS NUMBERS OF BINARY CUBIC FORMS II

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Abstract. As a continuation of the authors and Wakatsuki's previous paper [5], we study relations among Dirichlet series whose coefficients are class numbers of binary cubic forms. We show that for any integral models of the space of binary cubic forms, the associated Dirichlet series satisfies a simple explicit relation to that of the dual other than the usual functional equation. As an application, we write the functional equations of these Dirichlet series in self dual forms.

1. Introduction

The theory of zeta functions for the space of binary cubic forms was initiated by Shintani [7] as a fine example of zeta functions of prehomogeneous vector spaces [6]. He introduced 4 Dirichlet series $\xi_{1,1}(s)$, $\xi_{1,2}(s)$, $\xi^*_{1,1}(s)$, $\xi^*_{1,2}(s)$ whose coefficients are class numbers of integral binary cubic forms, and established their remarkable beautiful analytic properties. These 4 zeta functions he introduced are for the "standard" integral models, and our purpose is to study the zeta functions for *all* integral models.

Let us recall the definition of the zeta function. Let $V_\mathbb{Q}$ be the space of binary cubic forms over the rational number field $\mathbb{Q}$;

$$V_\mathbb{Q} := \{x(u,v) = au^3 + bu^2v + cuv^2 + dv^3 \mid a,b,c,d \in \mathbb{Q}\}.$$

We express elements of $V_\mathbb{Q}$ as $x = x(u,v) = au^3 + bu^2v + cuv^2 + dv^3$. We identify $V_\mathbb{Q}$ with $\mathbb{Q}^4$ via $V_\mathbb{Q} \ni x \mapsto (a,b,c,d) \in \mathbb{Q}^4$ and write as $x = (a,b,c,d)$ also. Let $P(x)$ denote the discriminant of $x \in V_\mathbb{Q}$:

$$P(x) := \mathrm{Disc}(x(u,v)) = b^2c^2 + 18abcd - 4ac^3 - 4b^3d - 27a^2d^2.$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $V_\mathbb{Q}$ by the linear change of variables, and $P(x)$ is invariant under the action.

We recall the classification of $\mathrm{SL}_2(\mathbb{Z})$-invariant lattices in $V_\mathbb{Q}$. We put

$$\begin{aligned}
L_1 &:= \{x \in V_\mathbb{Q} \mid a,b,c,d \in \mathbb{Z}\} = \mathbb{Z}^4,\\
L_2 &:= \{(a,b,c,d) \in L_1 \mid a+b+d, a+c+d \in 2\mathbb{Z}\},\\
L_3 &:= \{(a,b,c,d) \in L_1 \mid a+b+c, b+c+d \in 2\mathbb{Z}\},\\
L_4 &:= \{(a,b,c,d) \in L_1 \mid b+c \in 2\mathbb{Z}\},\\
L_5 &:= \{(a,b,c,d) \in L_1 \mid a,d,b+c \in 2\mathbb{Z}\},
\end{aligned}$$

and

$$\begin{aligned}
&L_1^* := \{x \in V_\mathbb{Q} \mid a,d \in \mathbb{Z}, b,c \in 3\mathbb{Z}\},\\
&L_2^* := L_1^* \cap L_3, \quad L_3^* := L_1^* \cap L_2, \quad L_4^* := L_1^* \cap L_5, \quad L_5^* := L_1^* \cap L_4.
\end{aligned}$$

We have $L_4 \supset L_2$, $L_4 \supset L_5$, $L_3 \supset L_5$ and similar relations for L_i^*'s.

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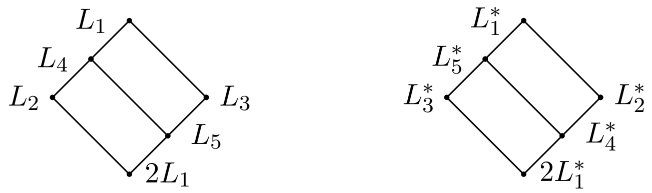

In the previous paper [5] the authors and Wakatsuki showed that up to $\mathbb{Q}^\times$-multiplication, this is a complete list of $\mathrm{SL}_2(\mathbb{Z})$-invariant lattices in $V_\mathbb{Q}$. Hence there are 10 different integral models of $V_\mathbb{Q}$. The notation L_i^* is because it is isomorphic to the contragradient representation $\mathrm{Hom}(L_i, \mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$.

The zeta functions are defined as follows.

Definition 1.1. For $1 \le i \le 5$ and $1 \le j \le 2$, we define

$$\xi_{i,j}(s) := \sum_{\substack{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash L_i \\ (-1)^{j-1} P(x) > 0}} \frac{|\mathrm{Stab}(x)|^{-1}}{|P(x)|^s}, \qquad \xi^*_{i,j}(s) := \sum_{\substack{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash L_i^* \\ (-1)^{j-1} P(x) > 0}} \frac{|\mathrm{Stab}(x)|^{-1}}{|P(x)/27|^s},$$

and call them the *zeta functions* associated with L_i or L_i^*. Here $|\mathrm{Stab}(x)|$ denote the number of stabilizers of x in $\mathrm{SL}_2(\mathbb{Z})$.

Note that $|\mathrm{Stab}(x)|$ is either 1 or 3 and that for $x \in L_i^*$, $P(x)$ is a multiple of 27.

Shintani [7] showed that for $i = 1$, the standard integral models L_1 and L_1^*, these zeta functions have holomorphic continuations to the whole complex plane except for simple poles at $s = 1, 5/6$, and satisfy a functional equation. He also computed the residues explicitly. In [5], we proved similar analytic properties for $2 \le i \le 5$.

However, despite of Shintani's extensive study of $\xi_{1,j}(s)$ and $\xi^*_{1,j}(s)$, one other significant property was remain unrevealed until 1990's. The following identity was conjectured by the first author [4] and proved by Nakagawa [3].

Theorem 1.2 (Conjectured in [4], proved in [3]). *We have*

$$\xi^*_{1,1}(s) = \xi_{1,2}(s), \qquad \xi^*_{1,2}(s) = 3\xi_{1,1}(s). \tag{1}$$

Although (1) is quite simple, no elementary proof of this theorem is known to the present. In fact, Nakagawa proved them as a consequence of the sophisticated use of class field theory. As we will describe in Theorem 1.5, this theorem has an important application to the functional equation. Hence it is natural to ask whether there exist similar relations of the zeta functions for other integral models. We will give the affirmative answer to this problem.

To state our results, we find it convenient to put

$$\xi_i(s) := \begin{pmatrix} \xi_{i,1}(s) \\ \xi_{i,2}(s) \end{pmatrix}, \quad \xi^*_i(s) := \begin{pmatrix} \xi^*_{i,1}(s) \\ \xi^*_{i,2}(s) \end{pmatrix}, \quad A := \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}.$$

Then (1) is written as $\xi_1^*(s) = A \cdot \xi_1(s)$. For $i = 2, 3$, the authors and Wakatsuki proved the following in the previous paper [5].

Theorem 1.3 ([5]). *We have*

$$\begin{aligned} \xi_2^*(s) &= A \cdot \xi_2(s), \\ \xi_3^*(s) &= A \cdot \xi_3(s). \end{aligned} \tag{2}$$

On the other hand, for $i = 4, 5$, $\xi_i^*(s)$ and $A \cdot \xi_i(s)$ do not coincide. These discrepancies themselves are not surprising since the indices $[L_1 : L_i]$ and $[L_1^* : L_i^*]$ do not coincide.

However, in view of (1), (2) for $i = 1, 2, 3$, one may believe that some corresponding formulas should exist for $i = 4, 5$.

Indeed, we find such formulas in certain *linear combinations* of the zeta functions. The following is a main result of this paper.

Theorem 1.4 (Main Theorem)**.** *We put*

$$\begin{aligned}
\theta(s) &:= \xi_1(s) - 2\xi_3(s) - \xi_4(s) + 4\xi_5(s),\\
\eta(s) &:= 2^{2s}\left(\xi_4(s) - \xi_2(s) - \xi_5(s) + 2^{1-4s}\xi_1(s)\right),\\
\theta^*(s) &:= 2^{2s}\left(\xi_5^*(s) - \xi_3^*(s) - \xi_4^*(s) + 2^{1-4s}\xi_1^*(s)\right),\\
\eta^*(s) &:= \xi_1^*(s) - 2\xi_2^*(s) - \xi_5^*(s) + 4\xi_4^*(s).
\end{aligned}$$

Then

$$\begin{aligned}
\theta^*(s) &= A \cdot \theta(s),\\
\eta^*(s) &= A \cdot \eta(s).
\end{aligned} \tag{3}$$

We now give an application of these identities to the functional equations. We put

$$\begin{aligned}
\Delta_+(s) &:= \left(\frac{2^4 3^3}{\pi^4}\right)^{s/2} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}-\frac{1}{12}\right)\Gamma\left(\frac{s}{2}+\frac{1}{12}\right),\\
\Delta_-(s) &:= \left(\frac{2^4 3^3}{\pi^4}\right)^{s/2} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{5}{12}\right)\Gamma\left(\frac{s}{2}+\frac{7}{12}\right).
\end{aligned}$$

Then by plugging (1), (2) into the functional equations, the followings were obtained:

Theorem 1.5 ([4], [3], [5])**.** *Let $i = 1, 2, 3$. For each sign, we put*

$$\xi_{i,\pm}(s) := \sqrt{3}\xi_{i,1}(s) \pm \xi_{i,2}(s).$$

Let $a_1 = 0, a_2 = a_3 = 2$. Then they satisfy the functional equations

$$2^{a_i s}\Delta_\pm(s)\xi_{i,\pm}(s) = 2^{a_i(1-s)}\Delta_\pm(1-s)\xi_{i,\pm}(1-s). \tag{4}$$

Similarly, as a consequence of (3), we have the following.

Theorem 1.6 (Corollary to Theorem 1.4)**.** *We put*

$$\theta_\pm(s) := \sqrt{3}\theta_1(s) \pm \theta_2(s) \quad \textit{and} \quad \eta_\pm(s) := \sqrt{3}\eta_1(s) \pm \eta_2(s),$$

where $\theta(s) = (\theta_1(s), \theta_2(s))$ and $\eta(s) = (\eta_1(s), \eta_2(s))$. Then

$$\begin{aligned}
2^s\Delta_\pm(s)\theta_\pm(s) &= 2^{1-s}\Delta_\pm(1-s)\theta_\pm(1-s),\\
2^s\Delta_\pm(s)\eta_\pm(s) &= 2^{1-s}\Delta_\pm(1-s)\eta_\pm(1-s).
\end{aligned} \tag{5}$$

Hence the functional equations of the zeta functions are expressed in self dual forms for all integral models. In view of (4) and (5), we may say that the "conductor" of the Dirichlet series $\xi_{1,\pm}(s)$, $\xi_{2,\pm}(s)$, $\xi_{3,\pm}(s)$, $\theta_\pm(s)$ and $\eta_\pm(s)$ are $2^4 3^3$, $2^8 3^3$, $2^8 3^3$, $2^6 3^3$ and $2^6 3^3$, respectively[1]. We can describe the poles and residues of these Dirichlet series.

[1] Either of these 10 Dirichlet series are of the form $\sum_{n\geq 1} a_n/n^s$, and we can confirm that the greatest common divisor of $\{n \mid a_n \neq 0\}$ is 1 by using the table of the coefficients in [5].

Theorem 1.7. *The Dirichlet series $\xi_{1,+}(s)$, $\xi_{2,+}(s)$, $\xi_{3,+}(s)$, $\theta_+(s)$ and $\eta_+(s)$ are holomprhic except for simple poles at $s=1$ and $s=5/6$, while $\xi_{1,-}(s)$, $\xi_{2,-}(s)$, $\xi_{3,-}(s)$, $\theta_-(s)$ and $\eta_-(s)$ are holomorphic except for a simple pole at $s=1$. The residues are given as follows:*

	$\xi_{1,+}(s)$	$\xi_{2,+}(s)$	$\xi_{3,+}(s)$	$\theta_+(s)$	$\eta_+(s)$
Residue at $s=1$	$\frac{2\sqrt{3}+3}{18}\pi^2$	$\frac{2\sqrt{3}+3}{72}\pi^2$	$\frac{2\sqrt{3}+3}{72}\pi^2$	$\frac{7\sqrt{3}+9}{72}\pi^2$	$\frac{5\sqrt{3}+9}{72}\pi^2$
Residue at $s=5/6$	$\frac{\Gamma(1/3)^3\zeta(1/3)}{3\pi}$	$\frac{\Gamma(1/3)^3\zeta(1/3)}{12\pi}$	$\frac{\Gamma(1/3)^3\zeta(1/3)}{12\pi}$	$\frac{\Gamma(1/3)^3\zeta(1/3)}{3\sqrt[3]{2}\pi}$	$\frac{\Gamma(1/3)^3\zeta(1/3)}{3\sqrt[3]{2}\pi}$
Conductor	2^43^3	2^83^3	2^83^3	2^63^3	2^63^3

	$\xi_{1,-}(s)$	$\xi_{2,-}(s)$	$\xi_{3,-}(s)$	$\theta_-(s)$	$\eta_-(s)$
Residue at $s=1$	$\frac{2\sqrt{3}-3}{18}\pi^2$	$\frac{2\sqrt{3}-3}{72}\pi^2$	$\frac{2\sqrt{3}-3}{72}\pi^2$	$\frac{7\sqrt{3}-9}{72}\pi^2$	$\frac{5\sqrt{3}-9}{72}\pi^2$
Conductor	2^43^3	2^83^3	2^83^3	2^63^3	2^63^3

It is an interesting phenomenon that the latter 5 Dirichlet series are holomorphic at $s=5/6$.

Our basic approach to prove Theorem 1.4 is to reduce to Theorem 1.2, as we did in the previous paper [5] to prove Theorem 1.3. However we need to argue more carefully since the relations between $\xi_4(s), \xi_5(s)$ and $\xi_1(s)$ are not as direct as those of $\xi_2(s), \xi_3(s)$ and $\xi_1(s)$. We look closely certain subsets of L_1 which are no longer $\mathrm{SL}_2(\mathbb{Z})$-invariant but invariant under certain congruence subgroups such as $\Gamma_0(2)$ or $\Gamma(2)$, and study them in terms of the *induction* in the category of G-sets. The zeta functions behaves quite well with respect to this induction, and these enables us to bring $\xi_4(s), \xi_5(s)$ and $\xi_1(s)$ into connection.

We note that for $i=1$, curious algebraic interpretations of the set of integer orbits of L_1 and L_1^* were known. Precisely, $\mathrm{GL}_2(\mathbb{Z})\backslash L_1$ has a canonical bijection to the set of cubic rings, while $\mathrm{SL}_2(\mathbb{Z})\backslash L_1^*$ essentially corresponds to the set of 3-torsions in ideal class groups of quadratic rings. Indeed, these interpretations were key ingredient for Nakagawa's proof of Theorem 1.2 in terms of class field theory. Such algebraic interpretations of integer orbits for many other prehomogeneous vector spaces were discovered rather systematically in Bhargava's surprising work of *higher composition laws* [1]. In consideration of these results, we expect that there might exist interesting interpretations for integer orbits of L_i, L_i^* for $2 \le i \le 5$ also. We hope the theory of integer orbits will be pursued further in the future.

This paper is organized as follows. In Section 2, we introduce the notion of *induction*. After that we study the set

$$\{x \in L_1 \mid P(x) \equiv l \mod 32\}, \quad \{x \in L_1^* \mid P(x)/27 \equiv -l \mod 32\}, \quad l = 4, 20$$

in some detail. We prove in Proposition 2.4 that actions of $\mathrm{SL}_2(\mathbb{Z})$ to these sets are induced from actions of $\Gamma_0(2)$ to their certain subsets. The proof of Theorem 1.4 is given in Section 3. In Theorem 3.8 we express partial zeta functions associated with the sets above in terms of linear combinations of $\xi_i(s)$ or $\xi_i^*(s)$. This enables us to reduce Theorem 1.4 to Theorem 1.2. In Section 4, we prove Theorems 1.6 and 1.7.

Notations. The notations introduced above are used throughout this paper. For a finite set X, we denote its cardinality by $|X|$. If a group G acts on a set X, then for $x \in X$ we put $G_x = \{g \in G \mid gx = x\}$. In this paper we often consider congruence relations in $\mathbb{Z}$. If $a - a' \in N\mathbb{Z}$ then we write $a \equiv a'(N)$ as well as $a \equiv a' \mod N$. For $a, a', a'', \dots \in \mathbb{Z}$, "$a \mod N \equiv a' \equiv a'' \equiv \cdots$" means $a - a', a' - a'', \dots \in N\mathbb{Z}$.

The congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are denoted by

$$\Gamma(N) = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} p & q \\ r & s \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\},$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, r \equiv 0 \ (N) \right\}, \quad \Gamma^0(N) = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, q \equiv 0 \ (N) \right\}.$$

Hence $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. Finally, we put $\mathcal{E} := 2\mathbb{Z}$ and $\mathcal{O} := 2\mathbb{Z}+1$, the set of even integers and odd integers, respectively.

2. Expressions in induced forms

To prove the main theorem, we use the notion of "induction" in the category of G-sets. For the convenience of the reader, we summarize its definition and basic properties. The situation is quite similar to the induction of representations of finite groups. We omit the elementary proofs of the basic facts.

Let G be a group. Assume that its subgroup H acts on a set Y. Then up to equivalence, there exists a unique pair $(\iota, \widetilde{Y})$ where $\widetilde{Y}$ is a G-set and $\iota\colon Y \hookrightarrow \widetilde{Y}$ is an injective H-homomorphism which satisfy the following conditions;

1. the map $\tilde{\iota}\colon H\backslash Y \to G\backslash \widetilde{Y}$ induced from ι is bijective, and
2. for all $y \in Y$, $H_y = G_{\iota(y)}$.

The pair $(\iota, \widetilde{Y})$ is constructed as follows: Consider an equivalence relation $\sim$ on $G \times Y$ so that $(g, y) \sim (g', y')$ if and only if there exists $h \in H$ such that $g' = gh^{-1}$ and $y' = hy$. Let $\widetilde{Y}$ be the set of equivalence classes. The equivalence class of (g, y) is again denoted by (g, y). The well-defined map $G \times \widetilde{Y} \ni (g', (g, x)) \mapsto (g'g, x) \in \widetilde{Y}$ defines an action of G on $\widetilde{Y}$. Let $\iota\colon Y \ni y \mapsto (e, y) \in \widetilde{Y}$, where $e \in G$ is the identity. Then the pair $(\iota, \widetilde{Y})$ satisfies the desired properties. We denote this $\widetilde{Y}$ by $G \times_H Y$.

Let G be a group acting on a set X, and a subset $Y \subset X$ is invariant under the action of a subgroup $H \subset G$. Then we can consider a natural map $G \times_H Y \ni (g, y) \mapsto gy \in X$ of G-sets. When this map is bijective, we write $X = G \times_H Y$ and say that (G, X) *is induced from* (H, Y). We have the followings.

Lemma 2.1. *With the notations above, $X = G \times_H Y$ if and only if*

1. *the map $G \times Y \ni (g, y) \to gy \in X$ is surjective, and*
2. *for $y \in Y$ and $g \in G$, $gy \in Y$ if and only if $g \in H$.*

Lemma 2.2. *If X' is a G-invariant subset of $G \times_H Y$, then $X' = G \times_H (Y \cap X')$.*

Remark 2.3. In the category theoretic terminology, the correspondence $\{H\text{-set}\} \ni Y \mapsto G \times_H Y \in \{G\text{-set}\}$ is the left adjoint functor of the restriction functor $\{G\text{-set}\} \ni X \mapsto X \in \{H\text{-set}\}$, i.e., $\mathrm{Hom}_H(Y, X) \cong \mathrm{Hom}_G(G \times_H Y, X)$.

We now consider the space of binary cubic forms. For $l, N \in \mathbb{Z}$, we put

$$L_1^{\equiv l(N)} := \{x \in L_1 \mid P(x) \equiv l \ (N)\}, \qquad (L_1^*)^{\cong l(N)} := \{x \in L_1^* \mid P(x)/27 \equiv l \ (N)\}.$$

The purpose of this section is to prove the following.

Proposition 2.4. *We put*

$$X_1 = \{(a, b, c, d) \in \mathbb{Z}^4 \mid b \in \mathcal{O}, c \in 2\mathcal{O}, d \in 4\mathcal{E}\},$$
$$X_2 = \{(a, b, c, d) \in \mathbb{Z}^4 \mid b \in \mathcal{O}, c \in 2\mathcal{O}, d \in 4\mathcal{O}\},$$
$$X_3 = \{(a, b, c, d) \in \mathbb{Z}^4 \mid a \in \mathcal{O}, b, c \in \mathcal{E}, d \in 2\mathcal{O}\}$$

and $X_i^* = X_i \cap L_1^*$ *for* $i = 1, 2, 3$. *Then*

$$L_1^{\equiv 4(32)} = \Gamma(1) \times_{\Gamma_0(2)} X_1, \qquad L_1^{\equiv 20(32)} = \Gamma(1) \times_{\Gamma_0(2)} (X_2 \sqcup X_3),$$
$$(L_1^*)^{\cong -20(32)} = \Gamma(1) \times_{\Gamma_0(2)} X_1^*, \qquad (L_1^*)^{\cong -4(32)} = \Gamma(1) \times_{\Gamma_0(2)} (X_2^* \sqcup X_3^*).$$

We start with a lemma.

Lemma 2.5. *For* $x = (a, b, c, d) \in \mathbb{Z}^4$, $P(x) \equiv 4\ (16)$ *if and only if one of the followings holds;*

(1) $a, b, c, d \in \mathcal{O}, a + b + c + d \in 2\mathcal{O}$,
(2) $b, c \in \mathcal{E}, ad \in 2\mathcal{O}$,
(3) $a \in 2\mathcal{E}, b \in 2\mathcal{O}, c \in \mathcal{O}$,
(4) $d \in 2\mathcal{E}, c \in 2\mathcal{O}, b \in \mathcal{O}$,
(5) $b + c \in \mathcal{O}, a + c, b + d \in 2\mathcal{E}$.

Proof. We write $P = P(x)$. We have $P = (bc + ad)^2 + 4R + 16(abcd - 2a^2d^2)$ where $R = a^2d^2 - ac^3 - b^3d$. Hence $P \mod 16 \equiv (bc + ad)^2 + 4R$ and if $P \equiv 4\ (16)$ then $bc + ad \in \mathcal{E}$. We note that in this case $(bc + ad)^2 \equiv 0\ (16)$ or $\equiv 4\ (16)$ according as $bc + ad \in 2\mathcal{E}$ or $bc + ad \in 2\mathcal{O}$. Also if $n \in \mathcal{O}$ then $n^2 \equiv 1\ (8)$ in general.

Assume $bc, ad \in \mathcal{O}$. Then $a, b, c, d \in \mathcal{O}$ and so $R \mod 4 \equiv 1 - ac - bd \equiv 1, 3$. Hence $P \equiv 4\ (16)$ if and only if $bc + ad \in 2\mathcal{E}$ and $1 - ac - bd \in 4\mathbb{Z} + 1$. Under the condition $a, b, c, d \in \mathcal{O}$, this is equivalent to $a + b + c + d \in 2\mathcal{O}$. This is the case (1).

For the rest we consider the case $bc, ad \in \mathcal{E}$. In this case $R \mod 4 \equiv -ac^3 - b^3d$. First assume $b, c \in \mathcal{E}$. Then $R \equiv 0\ (4)$ and hence $P \equiv 4\ (16)$ if and only if $bc + ad \in 2\mathcal{O}$. Hence $ad \in 2\mathcal{O}$ and we get the condition (2). Next assume $b \in \mathcal{E}$ and $c \in \mathcal{O}$. Since $R \equiv -ac\ (4)$, $P \equiv 4\ (16)$ if and only if either (i) $ad + bc \in 2\mathcal{O}, ac \in 4\mathbb{Z}$ or (ii) $ad + bc \in 2\mathcal{E}, ac + 1 \in 4\mathbb{Z}$. In the case (i), since $c \in \mathcal{O}$, we have $a \in 2\mathcal{E}$ and so $ad + bc \in 2\mathcal{O}$ if and only if $b \in 2\mathcal{O}$. This is the case (3). In the case (ii), since $a \in \mathcal{O}$, we have $d \in \mathcal{E}$. Under the condition $a, c \in \mathcal{O}, b, d \in \mathcal{E}$, (ii) hold if and only if $a + c, b + d \in 2\mathcal{E}$. Hence we get the condition (A) : $b \in \mathcal{E}, c \in \mathcal{O}, a + c, b + d \in 2\mathcal{E}$. Finally we assume $b \in \mathcal{O}$ and $c \in \mathcal{E}$. By the same argument, $P \equiv 4\ (16)$ if and only if either (4) or (B) : $b \in \mathcal{O}, c \in \mathcal{E}, a + c, b + d \in 2\mathcal{E}$ is satisfied. Since (A) or (B) is equivalent to the condition (5), we have the lemma. □

Lemma 2.6. *We put*

$$X_1' = \{(a, b, c, d) \in \mathbb{Z}^4 \mid c \in \mathcal{O}, b \in 2\mathcal{O}, a \in 4\mathcal{E}\},$$
$$X_1'' = \{(a, b, c, d) \in \mathbb{Z}^4 \mid b + c \in \mathcal{O}, a + c \in 2\mathcal{E}, a + b + c + d \in 4\mathcal{E}\},$$
$$X_2' = \{(a, b, c, d) \in \mathbb{Z}^4 \mid c \in \mathcal{O}, b \in 2\mathcal{O}, a \in 4\mathcal{O}\},$$
$$X_2'' = \{(a, b, c, d) \in \mathbb{Z}^4 \mid b + c \in \mathcal{O}, a + c \in 2\mathcal{E}, a + b + c + d \in 4\mathcal{O}\},$$
$$X_3' = \{(a, b, c, d) \in \mathbb{Z}^4 \mid d \in \mathcal{O}, b, c \in \mathcal{E}, a \in 2\mathcal{O}\},$$
$$X_3'' = \{(a, b, c, d) \in \mathbb{Z}^4 \mid a, b, c, d \in \mathcal{O}, a + b + c + d \in 2\mathcal{O}\}.$$

Then

$$L_1^{\equiv 4(32)} = X_1 \sqcup X_1' \sqcup X_1'', \quad L_1^{\equiv 20(32)} = X_2 \sqcup X_2' \sqcup X_2'' \sqcup X_3 \sqcup X_3' \sqcup X_3''.$$

Proof. If $P(x) \equiv 4\ (16)$ then $P(x) \equiv 4\ (32)$ or $P(x) \equiv 20\ (32)$. Hence we can prove this lemma by examining each of five cases listed in Lemma 2.5. Since the argument is elemental and simple, we briefly sketch the outline of the proof.

In case (1), Since $R \mod 8 \equiv a^2d^2 - ac^3 - b^3d \equiv 1 - (ac + bd)$, we have $P \mod 32 \equiv (ad + bc)^2 - 4(ac + bd) + 20$. We note that $ad + bc, ac + bd \in 2\mathcal{E}$ (see the proof of the previous lemma). Moreover, $ad+bc+ac+bd=(a+b)(c+d)\in 8\mathbb{Z}$ since $a+b, c+d\in\mathcal{E}$ and $a+b+c+d\in 2\mathcal{O}$. Hence $P\equiv 20$ (32). So this case corresponds to X_3''.

In case (2), since $ad+bc\in 2\mathcal{O}$, $(ad+bc)^2\equiv 4$ (32). Also $R\equiv 4$ (8). Hence $P\equiv 20$ (32). So this case corresponds to X_3 and X_3'. In case (3), since $ad+bc\in 2\mathcal{O}$ also, $(ad+bc)^2\equiv 4$ (32). Moreover, $R \mod 8\equiv -ac^3\equiv -a$. Hence $P\equiv 4-4a$ (32). The case $a\in 4\mathcal{E}$ corresponds to X_1' and the case $a\in 4\mathcal{O}$ corresponds to X_2'. In case (4), by the same argument we obtain X_1 and X_2.

In case (5), first assume that $b\in\mathcal{E}, c\in\mathcal{O}$ Then $a\in\mathcal{O}, d\in\mathcal{E}$. We put $a+c=4m$ and $b+d=4n$ where $n,m\in\mathbb{Z}$. Then since

$$\begin{aligned}ad+bc \mod 8 &\equiv ad+(4m-a)(4n-d)\equiv 2ad-4an\equiv 2d-4n,\\ R \mod 8 &\equiv d^2-ac\equiv 2d-a(4m-a)\equiv 2d+a^2-4am\equiv 2d-4m+1,\end{aligned}$$

we have

$$P \mod 32\equiv 4(2d-4n)+4(2d-4m+1)\equiv 16(m+n)+4\equiv 4(a+b+c+d)+4.$$

If we assume $b\in\mathcal{O}, c\in\mathcal{E}$, by the same argument we have $P \mod 32\equiv 4(a+b+c+d)+4$. The case $a+b+c+d\in 4\mathcal{E}$ corresponds to X_1'' and the case $a+b+c+d\in 4\mathcal{O}$ corresponds to X_2''. This completes the proof. □

We now give the proof of Proposition 2.4.

Proof of Proposition 2.4. Let $\tau=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ and $\sigma=\begin{pmatrix}1&1\\-1&0\end{pmatrix}$. Then $\{e,\tau,\sigma\}$ is a complete representative of $\Gamma(1)/\Gamma_0(2)$. Since

$$\tau^{-1}(a,b,c,d)=(-d,c,-b,a),\quad \sigma^{-1}(a,b,c,d)=(-d,c+3d,-b-2c-3d,a+b+c+d)$$

by a simple computation we have $X_i'=\tau X_i$ and $X_i''=\sigma X_i$ for $i=1,2,3$. Hence by Lemma 2.1 it is enough to show that for $\gamma\in\Gamma(1)$ and $x\in X_i$, $x'=\gamma x\in X_i$ if and only if $\gamma\in\Gamma_0(2)$. Let $x=(a,b,c,d)$, $x'=(a',b',c',d')$ and $\gamma=\begin{pmatrix}p&q\\r&s\end{pmatrix}\in\Gamma(1)$. Then

$$\begin{pmatrix}a'\\b'\\c'\\d'\end{pmatrix}=\begin{pmatrix}p^3&p^2q&pq^2&q^3\\3p^2r&p^2s+2pqr&q^2r+2pqs&3q^2s\\3pr^2&qr^2+2prs&ps^2+2qrs&3qs^2\\r^3&r^2s&rs^2&s^3\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}.$$

We consider the case $i=1$. Let $x\in X_1$. It is easy to see that if $\gamma\in\Gamma_0(2)$ then $x'\in X_1$. Conversely, assume $x'\in X_1$. Then $b' \mod 2\equiv pra+psb$. So $b'\in\mathcal{O}$ implies $p\in\mathcal{O}$. Hence $b' \mod 2\equiv ra+sb$, $d' \mod 2\equiv ra+rsb$. So $b'-d'\in\mathcal{O}$ implies $r\in\mathcal{E}$, namely $\gamma\in\Gamma_0(2)$. The cases $i=2,3$ are similarly proved. Hence by Lemma 2.6, we obtain the first two formulas of the proposition. Since $L_1^{\equiv l(32)}\cap L_1^*=(L_1^*)^{\cong 3l(32)}$, the rest two follow from $L_1^{\equiv 4(32)}\cap L_1^*=(L_1^*)^{\cong 12(32)}=(L_1^*)^{\cong -20(32)}$, $L_1^{\equiv 20(32)}\cap L_1^*=(L_1^*)^{\cong 60(32)}=(L_1^*)^{\cong -4(32)}$ and Lemma 2.2. □

3. Proof of the main theorem

In this section we prove Theorem 1.4. We start with a definition.

Definition 3.1. For a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ and a Γ-invariant subset X of a lattice, we define

$$\xi_j(X,\Gamma,s) := \sum_{\substack{x\in\Gamma\backslash X\\ (-1)^{j-1}P(x)>0}} \frac{|\Gamma_x|^{-1}}{|P(x)|^s}, \qquad \xi(X,\Gamma,s) := \begin{pmatrix}\xi_1(X,\Gamma,s)\\ \xi_2(X,\Gamma,s)\end{pmatrix}$$

and call them *partial zeta functions* for the pair (X,Γ).

In this section the complex variable s is always fixed and so we mostly drop s and write as $\xi(X,\Gamma)$. If the expressions of X,Γ contain parentheses we may also write as $\xi[X,\Gamma]$. By definition, $\xi_i = \xi[L_i,\Gamma(1)]$ and $\xi_i^* = 3^{3s}\xi[L_i^*,\Gamma(1)]$. We define as follows.

Definition 3.2. We put

$$\xi_1^{\equiv l(N)} := \xi[L_1^{\equiv l(N)},\Gamma(1)], \qquad (\xi_1^*)^{\cong l(N)} := \xi[(L_1^*)^{\cong l(N)},\Gamma(1)].$$

The crucial step of our proof of the main theorem is to express $\xi_1^{\equiv l(32)}$ (resp. $(\xi_1^*)^{\cong -l(32)}$) for $l = 4, 20$ in terms of linear combinations of ξ_i's (resp. ξ_i^*'s). After we prepare necessary tools, we will do this in Theorem 3.8 by using Proposition 2.4.

To study the zeta functions, It will be convenient to consider the following twisted action of $G_{\mathbb{Q}} := \mathrm{GL}_2(\mathbb{Q})$ on $V_{\mathbb{Q}}$ which is compatible with the action of $\mathrm{SL}_2(\mathbb{Z})$:

$$(g\cdot x)(u,v) = \frac{1}{\det g}\cdot x(pu+rv, qu+sv), \qquad x\in V_{\mathbb{Q}}, \quad g = \begin{pmatrix} p & q\\ r & s\end{pmatrix} \in G_{\mathbb{Q}}.$$

Then $P(gx) = (\det g)^2 P(x)$. The followings are basic properties of the partial zeta functions.

Proposition 3.3. *The followings hold.*

1. *If X, X' are Γ-invariant and $X\cap X' = \emptyset$, $\xi(X\sqcup X',\Gamma) = \xi(X,\Gamma)+\xi(X',\Gamma)$.*
2. *If X is Γ-invariant and $g\in G_{\mathbb{Q}}$, $\xi(X,\Gamma) = (\det g)^{2s}\xi(gX, g\Gamma g^{-1})$.*
3. *If $\Gamma'\subset\Gamma$ and X is Γ'-invariant, $\xi(X,\Gamma') = \xi(\Gamma\times_{\Gamma'} X,\Gamma)$.*
4. *If $\Gamma'\subset\Gamma$ and X is Γ-invariant, $\xi(X,\Gamma') = [\Gamma:\Gamma']\xi(X,\Gamma)$.*

Proof. Since (1), (2) and (3) immediately follow from the definition, we consider (4). Let $X^\pm = \{x\in X \mid \pm P(x) > 0\}$. We have

$$\sum_{y\in\Gamma'\backslash X^\pm}\frac{|\Gamma'_y|^{-1}}{|P(y)|^s} = \sum_{x\in\Gamma\backslash X^\pm}\sum_{y\in\Gamma'\backslash\Gamma x}\frac{|\Gamma'_y|^{-1}}{|P(y)|^s} = \sum_{x\in\Gamma\backslash X^\pm}\frac{|\Gamma_x|^{-1}}{|P(x)|^s}\sum_{y\in\Gamma'\backslash\Gamma x}\frac{|\Gamma_x|}{|\Gamma'_y|}.$$

Hence (4) follows from the following result in elementary group theory. □

Lemma 3.4. *Assume a group G acts on a set X and H be an index finite subgroup of G. Then for $x\in X$ with $|G_x|<\infty$,*

$$\sum_{y\in H\backslash Gx}\frac{|G_x|}{|H_y|} = [G:H],$$

where in the summation of the left hand side, y runs through all the representatives of H-orbits in Gx.

Proof. Consider the canonical bijections $H\backslash Gx \simeq H\backslash(G/G_x) \simeq (H\backslash G)/G_x$. If $y = gx, g\in G$, then we have $|H_y| = |(g^{-1}Hg\cap G_x)|$ because $H_y = H\cap gG_xg^{-1} = g(g^{-1}Hg\cap G_x)g^{-1}$. Since $g^{-1}Hg\cap G_x$ is the group of stabilizers of $Hg\in H\backslash G$ in G_x, this implies that $|G_x|/|H_y|$ is equal to the cardinality of the G_x-orbit of Hg in $H\backslash G$. Hence to sum up all the representative in the left hand side is nothing but counting all the elements of the quotient set $H\backslash G$ exactly one time for each. □

Remark 3.5. The formula in Proposition 3.3 (3) indicates an advantage of using the induction. The formula in (4) says that $\xi(X,\Gamma)$ is essentially determined by X. In this sense we also say $\xi(X,\Gamma)$ as a partial zeta function for X, without referring to Γ.

We consider partial zeta functions for the quotient classes of L_1 by $2L_1$. Each class $(p,q,r,s)+2L_1$ is invariant under the action of $\Gamma(2)$.

Definition 3.6. For $p,q,r,s\in\{0,1\}$, we put $\xi_{pqrs} := \xi[(p,q,r,s)+2L_1,\Gamma(2)]$.

If necessary, we also regard p,q,r,s as elements of $\mathbb{Z}/2\mathbb{Z}$. It is easy to see that the number of $\Gamma(1)$-orbits of $L_1/2L_1$ is six. By Proposition 3.3 (2), this implies that there are six different partial zeta functions ξ_{pqrs}. More precisely, we can take six partial zeta functions $\xi_{0000},\xi_{0001},\xi_{0010},\xi_{0110},\xi_{0111},\xi_{1011}$ as representatives, and others are given by

$$\xi_{0001}=\xi_{1000}=\xi_{1111},\qquad \xi_{0010}=\xi_{0100}=\xi_{0011}=\xi_{1100}=\xi_{0101}=\xi_{1010},$$
$$\xi_{0111}=\xi_{1110}=\xi_{1001},\qquad \xi_{1011}=\xi_{1101}.$$

The relations between ξ_i's and ξ_{pqrs}'s are given as follows.

Proposition 3.7. (1) *We have*

$$6\xi_1=\xi_{0000}+3\xi_{0001}+6\xi_{0010}+\xi_{0110}+3\xi_{0111}+2\xi_{1011},$$
$$6\xi_2=\xi_{0000}+3\xi_{0111},\quad 6\xi_3=\xi_{0000}+\xi_{0110}+2\xi_{1011},$$
$$6\xi_4=\xi_{0000}+3\xi_{0001}+\xi_{0110}+3\xi_{0111},\quad 6\xi_5=\xi_{0000}+\xi_{0110},\quad 6\cdot 2^{-4s}\xi_1=\xi_{0000}.$$

(2) *We have*

$$\xi_{0000}=6\cdot 2^{-4s}\xi_1,\quad \xi_{0001}=2(\xi_4-\xi_2-\xi_5+2^{-4s}\xi_1),\quad \xi_{0010}=\xi_1+\xi_5-\xi_3-\xi_4,$$
$$\xi_{0110}=6(\xi_5-2^{-4s}\xi_1),\quad \xi_{0111}=2(\xi_2-2^{-4s}\xi_1),\quad \xi_{1011}=3(\xi_3-\xi_5).$$

Proof. Since $[\Gamma(1),\Gamma(2)]=6$, $6\xi_i=[\Gamma(1),\Gamma(2)]\cdot\xi[L_i,\Gamma(1)]=\xi[L_i,\Gamma(2)]$. Hence by dividing L_i into the disjoint union of quotient classes modulo $2L_1$, we have (1). For example,

$$6\xi_3=\sum_{\substack{p,q,r,s\in\mathbb{Z}/2\mathbb{Z}\\ p+q+r=q+r+s=0}}\xi_{pqrs}=\xi_{0000}+\xi_{0110}+\xi_{1011}+\xi_{1101}=\xi_{0000}+\xi_{0110}+2\xi_{1011}.$$

The formulas in (2) are easily obtained from (1). □

Now we will prove the following formulas. The authors call the following relations as kaleidoscopic relations in a joke but for meaning their sophisticated symmetry.

Theorem 3.8.

$$\begin{aligned}
\xi_1^{\equiv 4(32)} &= 3\cdot 2^{-2s}(\xi_5-2^{-4s}\xi_1),\\
(\xi_1^*)^{\cong -20(32)} &= 3\cdot 2^{-2s}(\xi_4^*-2^{-4s}\xi_1^*),\\
\xi_1^{\equiv 20(32)} &= (\xi_4-\xi_2-\xi_5+2^{1-4s}\xi_1)+2^{-4s}(\xi_1-\xi_4-2\xi_3+4\xi_5)\\
&\quad -2^{-2s}(\xi_1-\xi_3-2\xi_2+5\cdot 2^{-4s}\xi_1),\\
(\xi_1^*)^{\cong -4(32)} &= (\xi_5^*-\xi_3^*-\xi_4^*+2^{1-4s}\xi_1^*)+2^{-4s}(\xi_1^*-\xi_5^*-2\xi_2^*+4\xi_4^*)\\
&\quad -2^{-2s}(\xi_1^*-\xi_2^*-2\xi_3^*+5\cdot 2^{-4s}\xi_1^*).
\end{aligned}$$

Proof. For subsets A, B, C, D of $\mathbb{Z}$, we write $(A, B, C, D) = \{(a, b, c, d) \in \mathbb{Z}^4 \mid a \in A, b \in B, c \in C, d \in D\}$. Let $g = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$. Then by Proposition 3.3,

$$\begin{aligned} \xi[(A, B, 2C, 4D), \Gamma_0(2)] &= (\det g)^{2s} \xi[g(A, B, 2C, 4D), g\Gamma_0(2)g^{-1}] \\ &= 2^{-2s}\xi[(2A, B, C, D), \Gamma^0(2)]. \end{aligned}$$

Hence by Propositions 2.4, 3.3 and 3.7,

$$\begin{aligned} \xi_1^{\equiv 4(32)} &= \xi[L_1^{\equiv 4(32)}, \Gamma] = \xi[\Gamma \times_{\Gamma_0(2)} X_1, \Gamma] = \xi[X_1, \Gamma_0(2)] \\ &= \xi[(\mathbb{Z}, 2\mathbb{Z}+1, 4\mathbb{Z}+2, 8\mathbb{Z}), \Gamma_0(2)] = 2^{-2s}\xi[(2\mathbb{Z}, 2\mathbb{Z}+1, 2\mathbb{Z}+1, 2\mathbb{Z}), \Gamma^0(2)] \\ &= 2^{-2s}[\Gamma^0(2), \Gamma(2)]^{-1} \cdot \xi_{0110} = 3 \cdot 2^{-2s}(\xi_5 - 2^{-4s}\xi_1) \end{aligned}$$

and we have the first formula. Similarly, the third formula follows from

$$\xi_1^{\equiv 20(32)} = \xi[\Gamma \times_{\Gamma_0(2)} (X_2 \sqcup X_3), \Gamma] = \xi[X_2, \Gamma_0(2)] + \xi[X_3, \Gamma_0(2)]$$

and

$$\begin{aligned} \xi[X_2, \Gamma_0(2)] &= \xi[(\mathbb{Z}, 2\mathbb{Z}+1, 4\mathbb{Z}+2, 8\mathbb{Z}+4), \Gamma_0(2)] \\ &= 2^{-2s}\xi[(2\mathbb{Z}, 2\mathbb{Z}+1, 2\mathbb{Z}+1, 2\mathbb{Z}+1), \Gamma^0(2)] \\ &= 2^{-2s}[\Gamma^0(2) : \Gamma(2)]^{-1} \cdot \xi_{0111} = 2^{-2s}(\xi_2 - 2^{-4s}\xi_1), \\ \xi[X_3, \Gamma_0(2)] &= \xi[(2\mathbb{Z}+1, 2\mathbb{Z}, 2\mathbb{Z}, 4\mathbb{Z}+2), \Gamma_0(2)] \\ &= \xi[(2\mathbb{Z}+1, 2\mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z}), \Gamma_0(2)] - \xi[(2\mathbb{Z}+1, 2\mathbb{Z}, 2\mathbb{Z}, 4\mathbb{Z}), \Gamma_0(2)] \\ &= 2^{-1}\xi_{1000} - \xi[(\mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z}, 4\mathbb{Z}), \Gamma_0(2)] + \xi[(2\mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z}, 4\mathbb{Z}), \Gamma_0(2)] \\ &= 2^{-1}\xi_{1000} - 2^{-2s}\xi[(2\mathbb{Z}, 2\mathbb{Z}, \mathbb{Z}, \mathbb{Z}), \Gamma^0(2)] + 2^{-4s}\xi[(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, 2\mathbb{Z}), \Gamma_0(2)] \\ &= 2^{-1}\xi_{1000} - 2^{-1-2s}\textstyle\sum_{r,s=0,1} \xi_{00rs} + 2^{-1-4s}\sum_{p,q,r=0,1} \xi_{pqr0} \\ &= 2^{-1}\xi_{0001} - 2^{-1-2s}(\xi_{0000} + \xi_{0001} + 2\xi_{0010}) \\ &\qquad + 2^{-1-4s}(\xi_{0000} + \xi_{0001} + 4\xi_{0010} + \xi_{0110} + \xi_{0111}) \\ &= (\xi_4 - \xi_2 - \xi_5 + 2^{-4s}\xi_1) - 2^{-2s}(\xi_1 - \xi_2 - \xi_3 + 2^{2-4s}\xi_1) \\ &\qquad + 2^{-4s}(2\xi_1 - 2\xi_3 - \xi_4 + 4\xi_5). \end{aligned}$$

By considering the intersection with L_1^* of each subset in L_1, the rest two formulas are proved similarly. □

Remark 3.9. In the previous paper [5], we proved $\xi_1^{\equiv 5(8)} = \xi_2 - 2^{-4s}\xi_1$ and $\xi_1^{\equiv 1(8)} = \xi_3 - 2^{-4s}\xi_1$. Hence with Theorem 3.8, we have

$$\begin{aligned} \xi_2 &= \xi_1^{\equiv 5(8)} + 2^{-4s}\xi_1, \\ \xi_3 &= \xi_1^{\equiv 1(8)} + 2^{-4s}\xi_1, \\ (1 - 2^{-4s})\xi_4 &= \xi_1^{\equiv 20(32)} + 2^{-2s}(1 - 2^{-2s})(1 + 2^{1-4s})\xi_1 \\ &\qquad + (1 - 2^{1-2s})\left(\xi_1^{\equiv 5(8)} - 2^{-2s}\xi_1^{\equiv 1(8)} + 3^{-1}(1 + 2^{1-2s})2^{2s}\xi_1^{\equiv 4(32)}\right), \\ \xi_5 &= 3^{-1}2^{2s}\xi_1^{\equiv 4(32)} + 2^{-4s}\xi_1. \end{aligned}$$

Thus the coefficients of Dirichlet series $\xi_2, \xi_3, \xi_4, \xi_5$ are expressed in terms of those of ξ_1. This is also valid for zeta functions for dual lattices.

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. We first note that Nakagawa's formula $\xi_1^* = \xi_1 A$ of Theorem 1.2 implies $(\xi_1^*)^{\cong -l(N)} = A\xi_1^{\equiv l(N)}$. The reason for the switch from $l \mod N$ to $-l \mod N$ is because A replaces forms of positive discriminants by forms of negative discriminants and vice-versa. Recall that we have put

$$\theta := \xi_1 - \xi_4 - 2\xi_3 + 4\xi_5, \qquad \eta := 2^{2s}\left(\xi_4 - \xi_2 - \xi_5 + 2^{1-4s}\xi_1\right),$$
$$\theta^* := 2^{2s}\left(\xi_5^* - \xi_3^* - \xi_4^* + 2^{1-4s}\xi_1^*\right), \qquad \eta^* := \xi_1^* - \xi_5^* - 2\xi_2^* + 4\xi_4^*,$$

and our goal is to get $\theta^* = \theta A$ and $\eta^* = A\eta$. By Theorems 1.2, 1.3 and 3.8,

$$\begin{aligned} 2^{2s}A\xi_1^{\equiv 20(32)} &= A\eta + 2^{-2s}A\theta - A(\xi_1 - \xi_3 - 2\xi_2 + 5\cdot 2^{-4s}\xi_1) \\ &= A\eta + 2^{-2s}A\theta - (\xi_1^* - \xi_3^* - 2\xi_2^* + 5\cdot 2^{-4s}\xi_1^*). \end{aligned}$$

Therefore since $A\xi_1^{\equiv 20(32)} = (\xi_1^*)^{\cong -20(32)}$, we have

$$A\eta + 2^{-2s}A\theta = \xi_1^* - 2\xi_2^* - \xi_3^* + 3\xi_4^* + 2^{1-4s}\xi_1^* = \eta^* + 2^{-2s}\theta^*.$$

Similarly, from $A\xi_1^{\equiv 4(32)} = (\xi_1^*)^{\cong -4(32)}$, we have

$$\theta^* + 2^{-2s}\eta^* = A(\xi_1 - \xi_2 - 2\xi_3 + 3\xi_5 + 2^{1-4x}\xi_1) = A\theta + 2^{-2s}A\eta.$$

These two equalities are equivalent to $\theta^* = A\theta$ and $\eta^* = A\eta$. □

4. Analytic properties of the zeta functions

Now we will prove Theorems 1.6 and 1.7.

Proof of Theorems 1.6 and 1.7. By [5, Theorem 4.2], we have the functional equation

$$\xi_i(1-s) = 2^{2a_i s}[L_1 : L_i]^{-1}M(s)\xi_i^*(s)$$

where

$$M(s) := \frac{3^{3s-2}}{2\pi^{4s}}\Gamma(s)^2\Gamma(s-\frac{1}{6})\Gamma(s+\frac{1}{6})\begin{pmatrix}\sin 2\pi s & \sin \pi s\\ 3\sin \pi s & \sin 2\pi s\end{pmatrix}$$

and $a_1 = 0, a_2 = a_3 = a_4 = a_5 = 2$. Hence

$$\begin{aligned} \theta(1-s) &= \xi_1(1-s) - 2\xi_3(1-s) - \xi_4(1-s) + 4\xi_5(1-s) \\ &= 2^{4s-1}M(s)\left(2^{1-4s}\xi_1^*(s) - \xi_3^*(s) - \xi_4^*(s) + \xi_5^*(s)\right) \\ &= 2^{2s-1}M(s)\theta^*(s) \\ &= 2^{2s-1}M(s)A\theta(s). \end{aligned}$$

Note that the last equality follows from Theorem 1.4. Similarly, we have

$$\eta(1-s) = 2^{2s-1}M(s)A\eta(s).$$

We put

$$\Delta(s) = \begin{pmatrix}\Delta_+(s) & 0\\ 0 & \Delta_-(s)\end{pmatrix}, \quad T = \begin{pmatrix}\sqrt{3} & 1\\ \sqrt{3} & -1\end{pmatrix}.$$

Then, since $\Delta(1-s)TM(s)A = \Delta(s)T$ (this symmetrization of $M(s)$ is due to Datskovsky and Wright [2]), we have

$$2^{1-s}\Delta(1-s)T\theta(1-s) = 2^s\Delta(s)T\theta(s),$$
$$2^{1-s}\Delta(1-s)T\eta(1-s) = 2^s\Delta(s)T\eta(s),$$

and Theorem 1.6 is proved. Finally, Theorem 1.7 immidiately follows from the residue formulas of $\xi_{i,j}(s)$ given in [5, Theorem 4.2]. We note that

$$\frac{\sqrt[3]{2}\pi\Gamma(1/3)\zeta(2/3)}{3\Gamma(2/3)} = \frac{\Gamma(1/3)^3\zeta(1/3)}{2\pi}.$$

Interestingly, the residues at $s = 5/6$ of $\xi_{1,-}(s)$, $\theta_-(s)$, $\xi_{2,-}(s)$, $\xi_{3,-}(s)$ and $\eta_-(s)$ vanish. This finishes the proof. □

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