ISOTHERMICITY FOR DISCRETE SURFACES IN THE EUCLIDEAN AND MINKOWSKI 3-SPACES

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Abstract. In this report we explain why a certain notion of isothermicity for discrete surfaces in Euclidean 3-space is natural. We also consider isothermicity of discrete surfaces in Minkowski 3-space.

Introduction

In this report on the contents of a talk given in December of 2008 at Osaka City University, we give an already known description of isothermicity for discrete surfaces in Euclidean 3-space, and we apply this description for the case of discrete surfaces in Minkowski 3-space as well. We derive a motivation for this description from the case of smooth surfaces in Euclidean 3-space.

1. Smooth isothermic surfaces in Euclidean 3-space, and their mean curvature.

Let

$$x = x_1(u,v)i + x_2(u,v)j + x_3(u,v)k$$

be a surface in Euclidean 3-space $\mathrm{Im}(\mathfrak{Q}) \approx R^3$ with metric $4(dx_1^2 + dx_2^2 + dx_3^2)$, where $\mathfrak{Q}$ denotes the quaternions and $\mathrm{Im}(\mathfrak{Q})$ denotes the imaginary quaternions. Assume (u,v) is a conformal curvature-line coordinate system. Every constant mean curvature (CMC) surface can be parametrized this way, away from umbilic points. We call such coordinates *isothermic* coordinates.

The choice of metric $4(dx_1^2 + dx_2^2 + dx_3^2)$ instead of the more common $dx_1^2 + dx_2^2 + dx_3^2$ comes from a unification of notation with cases where the ambient space is spherical 3-space S^3 or hyperbolic 3-space H^3. Although we do not consider those other two space forms here, we keep the unifying notation, for the benefit of a reader who would like to look at those other cases (see [4] or [7]).

Although the phrase "isothermic coordinates" means simply conformal curvature-line coordinates, we also use the phrase "isothermic surface" to mean any surface for which isothermic coordinates exist, even if those isothermic coordinates have not been determined yet.

A formula for the mean curvature is given in the next lemma:

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Lemma 1. *The mean curvature H of x, with $\triangle x = \partial_u \partial_u x + \partial_v \partial_v x$, is*

$$H = \tfrac{1}{2}(k_1 + k_2) = \tfrac{-1}{2}|x_u|^{-2} \operatorname{Re}\{\triangle x \cdot n\} ,$$

where the $k_j \in R$ are the principal curvatures, i.e. $\partial_u n = -k_1 \partial_u x$ and $\partial_v n = -k_2 \partial_v x$. Here, n is the unit normal vector to x.

Remark 1. Letting x_{1u} denote $\frac{d}{du}(x_1)$, and similarly taking other notations, the unit normal vector n to the surface takes the explicit form

$$n = \frac{1}{2} \cdot \frac{(x_{2u}x_{3v} - x_{3u}x_{2v})i + (x_{3u}x_{1v} - x_{1u}x_{3v})j + (x_{1u}x_{2v} - x_{2u}x_{1v})k}{\sqrt{(x_{2u}x_{3v} - x_{3u}x_{2v})^2 + (x_{3u}x_{1v} - x_{1u}x_{3v})^2 + (x_{1u}x_{2v} - x_{2u}x_{1v})^2}} .$$

Note the presence of the factor $\frac{1}{2}$ in front of this expression, which is due to our unusual choice of metric for R^3.

Proof. The first equality of the equation in the lemma is of course just the definition of the mean curvature. We now prove the second equality. The first fundamental form (g_{ij}) satisfies $\langle x_u, x_v \rangle = 0 = g_{12} = g_{21}$, and

$$g_{11} = \langle x_u, x_u \rangle = 4|x_u|^2 = 4|x_v|^2 = \langle x_v, x_v \rangle = g_{22} .$$

Then the second fundamental form (b_{ij}) satisfies

$$b_{11} = \langle x_{uu}, n \rangle = -4 \operatorname{Re}\{x_{uu} \cdot n\} ,$$

$$b_{12} = b_{21} = \langle x_{uv}, n \rangle = 0 ,$$

$$b_{22} = \langle x_{vv}, n \rangle = -4 \operatorname{Re}\{x_{vv} \cdot n\} .$$

The result follows. $\square$

2. Christoffel transforms for smooth surfaces

We now define the Christoffel transform x^*, which for a CMC surface in R^3 gives the parallel CMC surface. Let x be a surface in R^3 with mean curvature H and unit normal n. The Christoffel transform x^* satisfies that

- x^* is defined on the same domain as x,
- x^* has the same conformal structure as x,
- x and x^* have opposite orientations, and
- x and x^* have parallel tangent planes at corresponding points.

One can check that it automatically follows that the curvature directions at corresponding points of x and x^* will themselves also be parallel.

This definition above turns out to be equivalent to the following definition, and the existence of the integrating factor ρ below is equivalent to the existence of isothermic coordinates. Then, once we have x^*, we will see that we can take x^* so that $dx^* = x_u^{-1} du - x_v^{-1} dv$.

Definition 1. A Christoffel transform x^* of an umbilic-free surface x in R^3 is a surface that satisfies $dx^* = \rho(dn + H dx)$ for some nonzero real-valued function ρ on the surface x (here x^* is determined only up to translations and homotheties).

Remark 2. The Christoffel transform is also sometimes called the “dual surface”, and taking the Christoffel transform can be called “dualizing”.

Remark 3. We did not allow umbilic points on x in the above definition, because they can be troublesome. In particular, the case that x is a round sphere (i.e. is completely umbilic) is a very special one in this discussion.

Proposition 1. *Away from umbilics of x, the Christoffel transform x^* exists if and only if x is isothermic.*

Proof. First we prove one direction, by assuming x is isothermic and then showing x^* exists.

Take x to be isothermic, and take isothermic coordinates u, v for x, so $x_{uv} = Ax_u + Bx_v$ for some A, B. Then

$$d(x_u^{-1}du - x_v^{-1}dv) = 16 g_{11}^{-2}(x_u x_{uv} x_u + x_v x_{uv} x_v) du \wedge dv = 0 \ .$$

This implies that there exists an x^* such that

$$dx^* = x_u^{-1}du - x_v^{-1}dv \ .$$

Also,

$$dn + H dx = \tfrac{1}{8}(b_{11} - b_{22})(x_u^{-1}du - x_v^{-1}dv) \ ,$$

implying that x^* is a Christoffel transform, since $b_{11} - b_{22} \neq 0$ at non-umbilic points.

Now we prove the other direction, by assuming x^* exists and then showing that x has isothermic coordinates.

For any choice of coordinates u, v for $x = x(u, v)$, the Codazzi equations are

$$(b_{11})_v - (b_{12})_u = \Gamma^1_{12} b_{11} + (\Gamma^2_{12} - \Gamma^1_{11}) b_{12} - \Gamma^2_{11} b_{22} \ ,$$
$$(b_{12})_v - (b_{22})_u = \Gamma^1_{22} b_{11} + (\Gamma^2_{22} - \Gamma^1_{21}) b_{12} - \Gamma^2_{21} b_{22} \ .$$

(See, for example, page 97 of [6].) Here the Christoffel symbols are

$$\Gamma^h_{ij} = \frac{1}{2}\sum_{k=1}^{2} g^{hk}(\partial_{u_j} g_{ik} + \partial_{u_i} g_{jk} - \partial_{u_k} g_{ij}) \ ,$$

where $u_1 = u$ and $u_2 = v$. Because we are avoiding any umbilic points of x, we may assume that u and v are curvature line coordinates for x (see, for example, Appendix B-5 of [8]), and so $g_{12} = b_{12} = 0$. It follows that

$$\Gamma^1_{11} = \frac{\partial_u g_{11}}{2g_{11}} \ , \ \ \Gamma^2_{22} = \frac{\partial_v g_{22}}{2g_{22}} \ , \ \ \Gamma^2_{11} = -\frac{\partial_v g_{11}}{2g_{22}} \ ,$$
$$\Gamma^1_{22} = -\frac{\partial_u g_{22}}{2g_{11}} \ , \ \ \Gamma^1_{12} = \Gamma^1_{21} = \frac{\partial_v g_{11}}{2g_{11}} \ , \ \ \Gamma^2_{12} = \Gamma^2_{21} = \frac{\partial_u g_{22}}{2g_{22}} \ .$$

Denoting the principal curvatures by k_j, the Codazzi equations simplify to

$$2(k_1)_v = \frac{\partial_v g_{11}}{g_{11}} \cdot (k_2 - k_1) \ , \quad 2(k_2)_u = \frac{\partial_u g_{22}}{g_{22}} \cdot (k_1 - k_2) \ . \tag{1}$$

Then existence of x^* gives

$$d(\rho dn + \rho H dx) = 0 \ ,$$

from which it follows that

$$\begin{pmatrix} 0 & \frac{b_{11}}{g_{11}} - \frac{b_{22}}{g_{22}} \\ \frac{b_{22}}{g_{22}} - \frac{b_{11}}{g_{11}} & 0 \end{pmatrix} \begin{pmatrix} \rho_u \\ \rho_v \end{pmatrix} = \rho \cdot \begin{pmatrix} \left(\frac{b_{11}}{g_{11}} + \frac{b_{22}}{g_{22}}\right)_v \\ \left(\frac{b_{11}}{g_{11}} + \frac{b_{22}}{g_{22}}\right)_u \end{pmatrix} \ .$$

Then because $\rho_{uv} = \rho_{vu}$ (i.e. it does not matter which order we take mixed derivatives in), we have

$$\left(\frac{(k_2+k_1)_v}{k_1-k_2}\right)_u = \left(\frac{(k_1+k_2)_u}{k_2-k_1}\right)_v ,$$

which implies

$$\frac{2(((k_1)_v)_u + ((k_2)_u)_v)}{k_1-k_2} + 2(k_2-k_1)^{-2}\left((k_1)_v(k_2-k_1)_u + (k_2)_u(k_2-k_1)_v\right) = 0 .$$

Substituting the Codazzi equations (1) into this, we have

$$\left(\log\frac{g_{11}}{g_{22}}\right)_{uv} = 0 .$$

In particular, there exist positive functions $f_1(u)$ and $f_2(v)$ depending only on u and v, respectively, so that

$$(f_1(u))^2 g_{11} = (f_2(v))^2 g_{22} .$$

Writing $u = u(\hat{u})$ and $v = v(\hat{v})$ for new curvature line coordinates $\hat{u}$ and $\hat{v}$, we have $\hat{g}_{12} = \hat{b}_{12} = 0$ and $\hat{g}_{11} = (u_{\hat{u}})^2 g_{11}$ and $\hat{g}_{22} = (v_{\hat{v}})^2 g_{22}$, for the fundamental form entries $\hat{g}_{ij}$ and $\hat{b}_{ij}$ in terms of $\hat{u}$ and $\hat{v}$. We can choose $\hat{u}$ and $\hat{v}$ so that $u_{\hat{u}} = f_1(u(\hat{u}))$ and $v_{\hat{v}} = f_2(v(\hat{v}))$ hold. Then $\hat{g}_{11} = \hat{g}_{22}$ and so $\hat{u}, \hat{v}$ are isothermic coordinates. □

Corollary 1. *Away from umbilic points, one Christoffel transform x^* of an isothermic $x = x(u,v)$ can be taken as a solution of $dx^* = x_u^{-1}du - x_v^{-1}dv$.*

Because of $dx^* = \rho(dn + Hdx)$, we have

$$0 = d^2x^* = d\rho \wedge (dn + Hdx) + \rho \cdot dH \wedge dx ,$$

which gives, with respect to isothermic coordinates (u,v), that

$$\rho_u = -\frac{g_{11}\partial_u H}{g_{11}H - b_{22}} \cdot \rho , \quad \rho_v = -\frac{g_{11}\partial_v H}{g_{11}H - b_{11}} \cdot \rho . \tag{2}$$

The existence of x^* then automatically implies the compatibility condition $(\rho_u)_v = (\rho_v)_u$, with ρ_u and ρ_v as just above. This pair of equations tells us that ρ is uniquely determined once its value is chosen at a single point, and thus the solution ρ is unique up to scalar multiplication by a constant factor. Thus the Christoffel transform in Corollary 1 is essentially the unique choice, up to homothety and translation in R^3. As a result of this, without loss of generality, we can now simply take the definition of x^* as follows:

Definition 2. The Christoffel transform of a surface x with isothermic coordinates (u,v) is any x^* (defined in R^3 up to translation) such that $dx^* = x_u^{-1}du - x_v^{-1}dv$.

Remark 4. The function ρ in Definition 1 is generally a constant scalar multiple of the multiplicative inverse of the mean curvature of x^*, seen as follows: The Christoffel transform of the Christoffel transform $(x^*)^*$, with respect to Definition 2, satisfies that

$$d((x^*)^*) = (x_u^*)^{-1}du - (x_v^*)^{-1}dv = (x_u^{-1})^{-1}du - (-x_v^{-1})^{-1}dv = x_u du + x_v dv = dx ,$$

so $(x^*)^*$ should be the original surface x, up to translation and homothety, with respect to Definition 1. So by scaling and translating appropriately, we may assume $(x^*)^* = x$. Also, if the normal of x is n, then the normal of x^* is $-n$. We have

$$dx = d((x^*)^*) = \rho^*(dn^* + H^*dx^*) = \rho^*(-dn + H^*\rho(dn + Hdx)) ,$$

and so

$$(1-\rho\rho^* HH^*)dx = (H^*\rho\rho^* - \rho^*)dn \ .$$

Since dx and dn are linearly independent away from umbilic points, it follows that

$$\rho H^* = \rho^* H = 1 \ .$$

Remark 5. When H is constant and we have isothermic coordinates, the equations in (2) tell us that ρ is constant. Thus if $x^{||} = x + H^{-1}n$ is the parallel CMC surface, then x^* and $x^{||}$ differ by only a homothety and translation of R^3. Thus the Christoffel transform is essentially the same as the parallel CMC surface to x, as expected.

Remark 6. The round cylinder gives one simple example of a Christoffel transform's orientation reversing property. For the cylinder $x(u,v) = (\cos u)i + (\sin u)j + vk$ in R^3, the normal vector is $n = (-\cos u)i + (-\sin u)j$, and the Christoffel transform is $x^*(u,v) = (-\cos u)i + (-\sin u)j + vk$ with its normal vector $n^* = (\cos u)i + (\sin u)j$. Thus $n^* = -n$.

3. Discrete isothermic surfaces

Here we will give a definition of discrete isothermic surfaces in discrete differential geometry, which is now well known.

The phrase "discrete differential geometry" is sometimes abbreviated as "DDG", and many researchers now work in this and related fields. Here we list some of those researchers, but we first note that this list includes only people whose work is in some way related to the viewpoint presented in these notes – and even with this restriction is by no means a complete list: Sergey Agafonov, Andreas Asperl, Alexander Bobenko, Christoph Bohle, Folkmar Bornemann, Ulrike Buecking, Fran Burstall, Adam Doliwa, Charles Gunn, Udo Hertrich-Jeromin, Michael Hofer, Tim Hoffmann, Ivan Izmestiev, Michael Joswig, Axel Kilian, Yang Liu, Vladimir Matveev, Christian Mercat, Franz Pedit, Paul Peters, Ulrich Pinkall, Konrad Polthier, Helmut Pottmann, Jurgen Richter-Gebert, Wolfgang Schief, Jean-Marc Schlenkev, Nicholas Schmitt, Oded Schramm, Peter Schroeder, Boris Springborn, John Sullivan, Yuri Suris, Johannes Wallner, Wenping Wang, Max Wardetzky.

Consider a discrete surface $\mathfrak{f}_p \in \mathrm{Im}(\mathfrak{Q})$, which we can consider to be a discrete surface in Euclidean 3-space, since $\mathrm{Im}(\mathfrak{Q}) \approx R^3$. Here p is any point in a discrete lattice domain (locally always a subdomain of Z^2). Consider any quadrilateral in the lattice with vertices p, q, r, s (i.e. the points (m,n), $(m+1,n)$, $(m+1,n+1)$, $(m,n+1)$ for some $m,n \in Z$) ordered counterclockwise about the quadrilateral.

We change the notation "x" for surfaces in the previous section to "$\mathfrak{f}$" here. This is for distinguishing between smooth surfaces, always denoted by "x", and discrete surfaces, always denoted by "$\mathfrak{f}$".

It would be natural to assume that the points $\mathfrak{f}_p$, $\mathfrak{f}_q$, $\mathfrak{f}_r$ and $\mathfrak{f}_s$ are coplanar, so that they are the vertices of a planar quadrilateral in R^3, and thus the surface is comprized of planar quadrilaterals connecting continuously along edges. It is even better if the points $\mathfrak{f}_p$, $\mathfrak{f}_q$, $\mathfrak{f}_r$ and $\mathfrak{f}_s$ are concircular (i.e. all lie in one circle), because then we could extend the notion of a surface comprized of planar quadrilaterals to the cases of other ambient spaces, such as S^3 or H^3. In fact, once the vertices are concircular, there is no need to think about "planar faces", as all the needed information is encoded in the circle itself.

We will soon restrict to the concircular case, but for the moment we make no assumptions about the positioning of $\mathfrak{f}_p$, $\mathfrak{f}_q$, $\mathfrak{f}_r$ and $\mathfrak{f}_s$.

We define the cross ratio of this quadrilateral as

$$q_{pqrs} = (\mathfrak{f}_q - \mathfrak{f}_p)(\mathfrak{f}_r - \mathfrak{f}_q)^{-1}(\mathfrak{f}_s - \mathfrak{f}_r)(\mathfrak{f}_p - \mathfrak{f}_s)^{-1} \ .$$

This cross ratio is not invariant with respect to conformal transformations of R^3, but such an invariance *almost* holds, in the sense that we can produce a conformally invariant version of the cross ratio by changing it into a complex-valued object, defined up to conjugation, as follows:

$$\hat{q}_{pqrs} = \mathrm{Re}(q_{pqrs}) \pm i||\mathrm{Im}(q_{pqrs})|| \ .$$

Lemma 2. *$\hat{q}_{pqrs}$ is a Möbius invariant.*

Proof. Applying the following maps to the space $\mathrm{Im}(\mathfrak{Q})$:

$$ai + bj + ck \to rai + rbj + rck \ ,$$

$$ai + bj + ck \to ai + bj + ck + (a_0 i + b_0 j + c_0 k) \ ,$$

$$ai + bj + ck \to -ai + bj + ck \ ,$$

$$ai + bj + ck \to (\cos(\theta)a - \sin(\theta)b)i + (\sin(\theta)a + \cos(\theta)b)j + ck \ ,$$

$$ai + bj + ck \to (\cos(\theta)a - \sin(\theta)c)i + bj + (\sin(\theta)a + \cos(\theta)c)k \ ,$$

$$ai + bj + ck \to ai + (\cos(\theta)b - \sin(\theta)c)j + (\sin(\theta)b + \cos(\theta)c)k \ ,$$

$$ai + bj + ck \to (ai + bj + ck)/(a^2 + b^2 + c^2) \ ,$$

where θ, r, a_0, b_0, c_0 are any real constants, and a, b, c represent coordinates of $\mathrm{Im}(\mathfrak{Q}) \approx R^3$, we find that both $\mathrm{Re}(q)$ and $||\mathrm{Im}(q)||^2$ are preserved in all seven cases. These seven maps are a dilation, a translation, a reflection, three rotations, and an inversion, respectively, that generate the full Möbius group. It follows that $\hat{q}$ is a Möbius invariant. □

A direct computation gives the following general formula for the cross ratio:

Lemma 3. *For $p_1, p_2, p_3, p_4 \in Im(\mathfrak{Q})$, we define $s_{ij} = (p_i - p_j)^2$ and then we have*

$$\hat{q}_{p_1 p_2 p_3 p_4} = \frac{s_{12}s_{34} - s_{13}s_{24} + s_{14}s_{23} \pm \sqrt{\mathcal{E}}}{2 s_{14} s_{23}} \ ,$$

where $\mathcal{E} = s_{12}^2 s_{34}^2 + s_{13}^2 s_{24}^2 + s_{14}^2 s_{23}^2 - 2s_{13}s_{14}s_{23}s_{24} - 2s_{12}s_{14}s_{23}s_{34} - 2s_{12}s_{13}s_{24}s_{34}$.

Because

$$\mathcal{E} = \tfrac{1}{2}(s_{12}s_{34} - s_{14}s_{23})^2 + \tfrac{1}{2}(s_{12}s_{34} - s_{13}s_{24})^2 + \tfrac{1}{2}(s_{13}s_{24} - s_{14}s_{23})^2 - s_{12}s_{23}s_{34}s_{14} - s_{12}s_{24}s_{13}s_{34} - s_{13}s_{14}s_{23}s_{24} \ ,$$

it is not clear from straightforward algebraic considerations that $\mathcal{E} \leq 0$. However, this does indeed hold, for geometric reasons (see [4] or [7] for a proof of this):

Lemma 4. *$\mathcal{E} \leq 0$.*

Now let us assume that for every quadrilateral with vertices p, q, r, s, the image points $\mathfrak{f}_p, \mathfrak{f}_q, \mathfrak{f}_r, \mathfrak{f}_s$ are concircular, lying in a circle $\mathcal{C}$. This makes the cross ratios all real-valued. In fact, once the cross ratio is real, then the value $\hat{q}$ of the cross ratio, along with the values of $\mathfrak{f}_p$ and $\mathfrak{f}_q$ and $\mathfrak{f}_s$, determine the point $\mathfrak{f}_r \in \mathcal{C}$ uniquely. In this way, the cross ratio gives a parametrization of the circle containing $\mathfrak{f}_p$, $\mathfrak{f}_q$ and $\mathfrak{f}_s$.

We consider the following additional condition:

Definition 3. When, for every quadrilateral, we can write the cross ratio as

$$q_{pqrs} = a_{pq}/a_{ps} \in R$$

so that the cross ratio factorizing function a_{pq} defined on the edges of $\mathfrak{f}$ satisfies

$$a_{pq} = a_{sr} \in R \text{ and } a_{ps} = a_{qr} \in R \; , \tag{3}$$

then we say that $\mathfrak{f}$ is *discrete isothermic*.

Note that the a_{pq} are symmetric, i.e. $a_{pq} = a_{qp}$ for any adjacent p and q.

4. Justification of the notion of discrete isothermic surfaces

One viewpoint on what a "discrete isothermic surface" is, as in Definition 3, is as follows: Take a smooth surface x with unit normal n. Give it curvature line coordinates $x = x(u, v)$, so $x_u \perp x_v$. (Curvature line coordinates always exist away from umbilics.) Then the fundamental forms are

$$I = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} \; , \quad II = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix} \; .$$

One can always stretch the coordinates, so that $x = x(u, v) = x(\tilde{u}(u), \tilde{v}(v))$ for any monotonic functions $\tilde{u}$ depending only on u, and $\tilde{v}$ depending only on v. Note that $\langle x_{\tilde{u}}, x_{\tilde{v}} \rangle = 0$, and $x_{\tilde{u}\tilde{v}} = x_{uv} \frac{du}{d\tilde{u}} \frac{dv}{d\tilde{v}}$ implies $\langle x_{\tilde{u}\tilde{v}}, n \rangle = 0$, so $(\tilde{u}, \tilde{v})$ are also curvature line coordinates. The surface is then isothermic if and only if there exist $\tilde{u}$, $\tilde{v}$ such that the metric becomes conformal, i.e. $\langle x_{\tilde{u}}, x_{\tilde{u}} \rangle = -4x_{\tilde{u}}^2 = -4x_{\tilde{v}}^2 = \langle x_{\tilde{v}}, x_{\tilde{v}} \rangle$, and this is equivalent to

$$\frac{g_{11}}{g_{22}} = \frac{a(u)}{b(v)} \; ,$$

where the function a depends only on u, and b depends only on v.

Now consider the cross ratio q_ϵ of the four points $x(u, v)$, $x(u + \epsilon, v)$, $x(u + \epsilon, v + \epsilon)$ and $x(u, v+\epsilon)$. Using that $x_u \perp x_v$ implies $x_u x_v + x_v x_u = 0$, i.e. that $x_u x_v^{-1} = -x_v^{-1} x_u$, we see that

$$\lim_{\epsilon \to 0} q_\epsilon = -\frac{g_{11}}{g_{22}} \; . \tag{4}$$

So x is isothermic if and only if

$$\lim_{\epsilon \to 0} q_\epsilon = -\frac{a(u)}{b(v)} \; ,$$

where again a is some function that depends only on u, and b depends only on v. This description of isothermicity does not involve any stretching of $\tilde{u}$ or $\tilde{v}$, which we would not be able to do in the discrete case anyways, and now Definition 3 is a natural discretization of this description in the smooth case: The corresponding statement for discrete surfaces, where stretching of coordinates is no longer possible, is that the surface is discrete isothermic if and only if the cross ratio factorizing function can be chosen so that $a_{pq} = a_{rs}$ and $a_{ps} = a_{qr}$ for vertices p, q, r, s (in order) about a given quadrilateral.

There is another perspective on isothermicity, coming from a theorem proven by Bobenko and Pinkall [2]:

Theorem 1. *Let $x(u, v)$ be a smooth surface in R^3, and define the diagonal cross ratio*

$$q_\epsilon^d = (x(u+\epsilon, v-\epsilon) - x(u-\epsilon, v-\epsilon))(x(u+\epsilon, v+\epsilon) - x(u+\epsilon, v-\epsilon))^{-1} \times$$

$$(x(u-\epsilon, v+\epsilon) - x(u+\epsilon, v+\epsilon))(x(u-\epsilon, v-\epsilon) - x(u-\epsilon, v+\epsilon))^{-1} .$$

Then

$$q_\epsilon^d = -1 + \mathcal{O}(\epsilon)$$

if and only if (u, v) are conformal coordinates for x, and

$$q_\epsilon^d = -1 + \mathcal{O}(\epsilon^2)$$

if and only if (u, v) are isothermic coordinates for x.

The superscript "d" in q_ϵ^d stands for "diagonal", because we are using diagonal elements to define this cross ratio, unlike with the previous q_ϵ.

Proof. Without loss of generality, we may assume $x(u, v) = 0$, and then for $\rho_u, \rho_v \in \{\pm 1\}$, we have

$$x(u + \rho_u\epsilon, v + \rho_v\epsilon) = \epsilon\rho_u x_u + \epsilon\rho_v x_v + \tfrac{1}{2}\epsilon^2(x_{uu} + x_{vv} + 2\rho_u\rho_v x_{uv}) + \mathcal{O}(\epsilon^3) ,$$

so

$$q_\epsilon^d = x_u x_v^{-1} x_u x_v^{-1} +$$

$$\epsilon(x_u x_v^{-1} x_{uv} x_v^{-1} + x_u x_v^{-1} x_u x_v^{-1} x_{uv} x_v^{-1} - x_{uv} x_v^{-1} x_u x_v^{-1} - x_u x_v^{-1} x_{uv} x_v^{-1} x_u x_v^{-1}) + \mathcal{O}(\epsilon^2) .$$

If the coordinates are conformal, then $x_u x_v^{-1} x_u x_v^{-1} = -1$, and we have

$$q_\epsilon^d = -1 + \epsilon x_u^{-4}(x_u x_v x_{uv}(x_u + x_v) + x_u^2 x_{uv}(x_u - x_v)) + \mathcal{O}(\epsilon^2) .$$

Now, if the coordinates are isothermic, then $b_{12} = 0$, and so there exist scalar functions α and β so that

$$x_{uv} = \alpha x_u + \beta x_v .$$

From this it follows that $q_\epsilon^d = -1 + \epsilon \cdot 0 + \mathcal{O}(\epsilon^2)$. □

This theorem leads to the following definition for discrete isothermic surfaces in the narrow sense: $\mathfrak{f}$ is discrete isothermic if

$$q_{pqrs} = -1$$

for all quadrilaterals, with vertices $\mathfrak{f}_p, \mathfrak{f}_q, \mathfrak{f}_r, \mathfrak{f}_s$.

However, with this definition, transformations, such as the Calapso transform for discrete surfaces, of isothermic surfaces will not remain isothermic. Hence the broader definition given in Definition 3 has been found to be more suitable.

5. Christoffel transforms of discrete surfaces

When $\mathfrak{f}$ is an isothermic surface in $R^3 \approx \mathrm{Im}(\mathfrak{Q})$, we can define the Christoffel transform $\mathfrak{f}^*$ (also in R^3) of $\mathfrak{f}$ as follows:

Definition 4. Let $\mathfrak{f}$ be a discrete isothermic surface in R^3, then the *Christoffel transform* $\mathfrak{f}^*$ of $\mathfrak{f}$ satisfies

$$d\mathfrak{f}^*_{pq} d\mathfrak{f}_{pq} = a_{pq} \ . \tag{5}$$

Here, for any object $\mathfrak{F}$ defined on vertices, $d\mathfrak{F}_{pq}$ denotes the difference

$$d\mathfrak{F}_{pq} := \mathfrak{F}_q - \mathfrak{F}_p$$

of the values of $\mathfrak{F}$ at the vertices q and p.

To see that this definition is natural, we consider the Christoffel transform x^* of a smooth surface x in R^3 with isothermic coordinates (u, v). In the smooth case, we may assume x and x^* satisfy

$$dx = x_u du + x_v dv \ , \quad dx^* = x_u^{-1} du - x_v^{-1} dv \ .$$

So

$$dx^*(\partial_u) dx(\partial_u) = 1 \quad \text{and} \quad dx^*(\partial_v) dx(\partial_v) = -1 \ .$$

We also have

$$\lim_{\epsilon \to 0} q_\epsilon = -1 = \frac{dx^*(\partial_u) dx(\partial_u)}{dx^*(\partial_v) dx(\partial_v)} \ ,$$

by Equation (4). In the discrete case, we loosened the -1 in the right-hand side of Equation (4) to the a_{pq}/a_{ps} in the right-hand side of $q_{pqrs} = a_{pq}/a_{ps}$, as in Definition 3. In this way, it is natural to consider that

$$\frac{a_{pq}}{a_{ps}} = \frac{d\mathfrak{f}^*_{pq} d\mathfrak{f}_{pq}}{d\mathfrak{f}^*_{ps} d\mathfrak{f}_{ps}} \ ,$$

where $d\mathfrak{f}_{pq}$, $d\mathfrak{f}^*_{pq}$, $d\mathfrak{f}_{ps}$, $d\mathfrak{f}^*_{ps}$ now represent discrete analogs of $dx(\partial_u)$, $dx^*(\partial_u)$, $dx(\partial_v)$, $dx^*(\partial_v)$, and so Definition 4 becomes natural.

We can then prove the following:

Theorem 2. [1] *If $\mathfrak{f}$ is a discrete isothermic surface, then there exists a Christoffel transform $\mathfrak{f}^*$.*

Proof. First of all, $\mathfrak{f}^*$ exists if and only if the compatibility condition

$$d\mathfrak{f}^*_{pq} + d\mathfrak{f}^*_{qr} = d\mathfrak{f}^*_{ps} + d\mathfrak{f}^*_{sr} \tag{6}$$

holds, that is to say, we can apply "discrete integration" of $d\mathfrak{f}^*$ to obtain $\mathfrak{f}^*$.

We now assume $\mathfrak{f}$ is discrete isothermic and prove that $\mathfrak{f}^*$ exists, i.e. that Equation (6) holds with $d\mathfrak{f}^*$ defined as in Equation (5). By Equation (5), Equation (6) is equivalent to having

$$a_{pq} d\mathfrak{f}_{pq}^{-1} + a_{qr} d\mathfrak{f}_{qr}^{-1} = a_{ps} d\mathfrak{f}_{ps}^{-1} + a_{sr} d\mathfrak{f}_{sr}^{-1}$$

hold. Because $a_{pq} = a_{sr}$ and $a_{ps} = a_{qr}$ (using isothermicity), the above equation is equivalent to

$$\frac{a_{pq}}{a_{ps}} (d\mathfrak{f}_{pq}^{-1} - d\mathfrak{f}_{sr}^{-1}) = d\mathfrak{f}_{ps}^{-1} - d\mathfrak{f}_{qr}^{-1} \ .$$

The cross ratio is $a_{pq}a_{ps}^{-1} = d\mathfrak{f}_{pq}d\mathfrak{f}_{qr}^{-1}d\mathfrak{f}_{rs}d\mathfrak{f}_{sp}^{-1} = d\mathfrak{f}_{qr}^{-1}d\mathfrak{f}_{rs}d\mathfrak{f}_{sp}^{-1}d\mathfrak{f}_{pq} = d\mathfrak{f}_{qr}^{-1}d\mathfrak{f}_{pq}d\mathfrak{f}_{sp}^{-1}d\mathfrak{f}_{rs}$, and so the equation becomes

$$d\mathfrak{f}_{qr}^{-1}d\mathfrak{f}_{rs}d\mathfrak{f}_{sp}^{-1} + d\mathfrak{f}_{qr}^{-1}d\mathfrak{f}_{pq}d\mathfrak{f}_{sp}^{-1} = d\mathfrak{f}_{ps}^{-1} - d\mathfrak{f}_{qr}^{-1} \ ,$$

that is to say,

$$d\mathfrak{f}_{rs} + d\mathfrak{f}_{pq} + d\mathfrak{f}_{qr} + d\mathfrak{f}_{sp} = 0 \ .$$

But this follows from the fact that $\mathfrak{f}$ exists and so $d\mathfrak{f}$ is closed. □

Lemma 5. *Let $\mathfrak{f}$ be a discrete isothermic surface. Then the Christoffel transform $\mathfrak{f}^*$ of $\mathfrak{f}$ is isothermic with the same cross ratios as $\mathfrak{f}$.*

Proof. Let q, q^* be the cross ratios of $\mathfrak{f}$, $\mathfrak{f}^*$ respectively. Then

$$q^* = d\mathfrak{f}^*_{pq}(d\mathfrak{f}^*_{qr})^{-1}d\mathfrak{f}^*_{rs}(d\mathfrak{f}^*_{sp})^{-1} = a_{pq}d\mathfrak{f}_{pq}^{-1}(a_{qr}d\mathfrak{f}_{qr}^{-1})^{-1}a_{rs}d\mathfrak{f}_{rs}^{-1}(a_{sp}d\mathfrak{f}_{sp}^{-1})^{-1} =$$

$$(a_{pq}/a_{qr})(a_{rs}/a_{sp})d\mathfrak{f}_{pq}^{-1}(d\mathfrak{f}_{qr}^{-1})^{-1}d\mathfrak{f}_{rs}^{-1}(d\mathfrak{f}_{sp}^{-1})^{-1} = q^2(d\mathfrak{f}_{sp}^{-1}d\mathfrak{f}_{rs}d\mathfrak{f}_{qr}^{-1}d\mathfrak{f}_{pq})^{-1} \ .$$

We then have

$$q^* = q^2(d\mathfrak{f}_{pq}d\mathfrak{f}_{qr}^{-1}d\mathfrak{f}_{rs}d\mathfrak{f}_{sp}^{-1})^{-1} = q^2 \cdot q^{-1} = q \ .$$

□

6. Smooth CMC surfaces in R^3 and $R^{2,1}$, without quaternions

Consider a smooth surface

$$x(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$$

in R^3 or $R^{2,1}$, with unit normal n. Suppose the surface is spacelike, in the case of $R^{2,1}$. Also, suppose that the surface is isothermic, with isothermic coordinates u, v. Then conformality implies

$$I = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$$

with $E = x_u \circ x_u$, where $\circ$ denotes the inner product associated with R^3 or $R^{2,1}$. Then the second fundamental form is

$$II = \begin{pmatrix} n \circ x_{uu} & n \circ x_{vu} \\ n \circ x_{uv} & n \circ x_{vv} \end{pmatrix} \ ,$$

and having isothermic coordinates implies $n_u = -k_1 x_u$ and $n_v = -k_2 x_v$, and so

$$II = \begin{pmatrix} k_1 E & 0 \\ 0 & k_2 E \end{pmatrix} \ .$$

The Hopf differential is, with $z = u + iv$,

$$\mathcal{Q} = n \circ x_{zz} = \frac{1}{4} n \circ (x_{uu} - x_{vv} - 2i x_{uv}) = \frac{1}{4} n \circ (x_{uu} - x_{vv})$$

$$= \frac{1}{4}(b_{11} - b_{22}) = \frac{E}{4}(k_1 - k_2) \ .$$

If the mean curvature H is constant, then it is well known that $\mathcal{Q}_{\bar{z}} = 0$, so $\mathcal{Q} = (E/4)(k_1 - k_2) \in R$ is constant. As a remark to motivate the next lemma and proposition, if the constant H is nonzero, then the parallel CMC surface is $x^{||} = x + H^{-1}n$, so the equation $dx^* = h \cdot d(Hx + n)$, for $h = 2(E(k_1 - k_2))^{-1}$ constant, is solvable for x^*.

Lemma 6. *If x is isothermic in R^3 or $R^{2,1}$ with isothermic coordinates u, v, then x^* exists, solving $dx^* = -\frac{x_u}{E}du + \frac{x_v}{E}dv$.*

Proof. We want to show "$d^2x^* = 0$", i.e.

$$d(-\frac{x_u}{E}du + \frac{x_v}{E}dv) = 0,$$

i.e. $2x_{uv}E - x_uE_v - x_vE_u = 0$. We can get this by noting that $b_{12} = 0$ implies $x_{uv} = ax_u + bx_v$ for some reals a and b, and that $x_u \circ x_v = 0$. □

Proposition 2. *Let x be an isothermic immersion in R^3 or $R^{2,1}$, with x^* as in the previous lemma. Then x is CMC H if and only if $dx^* = h(Hdx + dn)$ for some constant h.*

Proof.

$$-\frac{x_u}{E}du + \frac{x_v}{E}dv = h(Hdx + dn) \ , \quad h \text{ constant}$$

is equivalent to

$$(k_1 + k_2) = 2H \ , \ \text{ and } \ h = 2E^{-1}(k_1 - k_2)^{-1} \text{ is constant} \ .$$

The first of these is clearly true, and h is constant if and only if the Hopf differential $\mathcal{Q}$ is constant, which is true if and only if x is CMC. □

Corollary 2. *An isothermic immersion x in R^3 or $R^{2,1}$ is CMC if and only if*

$$-\frac{x_u}{E}du + \frac{x_v}{E}dv = h(Hdx + dn)$$

for some real constants h and H.

7. Discrete isothermic CMC surfaces in R^3, without quaternions

The notion of constant mean curvature for discrete isothermic surfaces in R^3 was given in [1]. For the case that the ambient space is a general simply-connected complete 3-dimensional Riemannian manifold of constant sectional curvature, this notion of "discrete CMC" was extended in [4]. To make this extension, [4] used discrete versions of linear conserved quantities, analogous to the way smooth CMC surfaces possess smooth linear conserved quantities. We do not explain the linear conserved quantities here, but rather use an equivalent property to define "discrete CMC", a property that was proven to be equivalent in [4] (see also [7]) and can be stated without any use of conserved quantities. As we state the following definition only in the case that the ambient space is R^3, it is also equivalent to the definitions found in [1].

Starting with the equation

$$d\mathfrak{f}^*_{pq} = a_{pq}\frac{-d\mathfrak{f}_{pq}}{|d\mathfrak{f}_{pq}|^2}$$

we have the following:

Definition 5. A discrete isothermic surface $\mathfrak{f}$ in R^3 is CMC if and only if there exist constants $h, H \in R$ and n_p with $|n_p|^2 = 1$ and $d\mathfrak{f}_{pq}n_q + n_pd\mathfrak{f}_{pq} = 0$ so that

$$h(dn_{pq} + Hd\mathfrak{f}_{pq}) = \frac{-a_{pq}d\mathfrak{f}_{pq}}{|d\mathfrak{f}_{pq}|^2} \ .$$

However, $d\mathfrak{f}_{pq}n_q + n_p d\mathfrak{f}_{pq} = 0$ is still a quaternionic equation. But this equation is equivalent to $d\mathfrak{f}_{pq} \wedge n_q + n_p \wedge d\mathfrak{f}_{pq} - d\mathfrak{f}_{pq} \circ (n_p + n_q) = 0$, where $d\mathfrak{f}_{pq} \wedge n_q + n_p \wedge d\mathfrak{f}_{pq} \in \mathrm{Im}(\mathfrak{Q})$ and $d\mathfrak{f}_{pq} \circ (n_p + n_q) \in R$. So, instead, the equations we want are $d\mathfrak{f}_{pq} \wedge n_q + n_p \wedge d\mathfrak{f}_{pq} = 0$ and $d\mathfrak{f}_{pq} \circ (n_p + n_q) = 0$. Then we can restate the previous definition, without any use of quaternions, as:

Theorem 3. *A discrete isothermic surface $\mathfrak{f}$ in R^3 is CMC if and only if there exist constants $h, H \in R$ and vectors n_p so that*

- $|n_p|^2 = 1$,
- $d\mathfrak{f}_{pq} \wedge n_q + n_p \wedge d\mathfrak{f}_{pq} = 0$,
- $d\mathfrak{f}_{pq} \circ (n_p + n_q) = 0$, *and*
- $h(dn_{pq} + H d\mathfrak{f}_{pq}) = \frac{-a_{pq} d\mathfrak{f}_{pq}}{|d\mathfrak{f}_{pq}|^2}$.

The second item in the above theorem actually follows from the fourth item, because the second item is just saying that $d\mathfrak{f}_{pq}$ is parallel to dn_{pq}.

8. Discrete CMC surfaces in $R^{2,1}$

We now propose possible definitions for discrete isothermic surfaces and discrete spacelike CMC surfaces in $R^{2,1}$.

To define the a_{pq} in the case of $R^{2,1}$, we need to define some analogue of the cross ratio, call it $q = q_{pqrs}$. Then we can define the a_{pq} in the usual way.

We now consider how to define the cross ratio on quadrilaterals. We could consider quadrilaterals in spacelike planes, without rotating those planes to horizontal. However, we choose in the argument below to rotate the planes to horizontal, so that the metric will be exactly the Euclidean metric that is so familiar to us.

We assume that the points p, q, r, s lie in a "circle" in $R^{2,1}$ lying in a spacelike plane. In general, such a circle is

$$\left\{ \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\gamma & 0 & \sinh\gamma \\ 0 & 1 & 0 \\ \sinh\gamma & 0 & \cosh\gamma \end{pmatrix} \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ 0 \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \middle| \, \theta \in [0, 2\pi) \right\} ,$$

where $x_0, y_0, z_0, \rho, \gamma, \beta$ are all real constants. By a rigid motion, we can move the "circle" to the horizontal circle

$$\{(\rho\cos\theta, \rho\sin\theta, 0) \,|\, \theta \in [0, 2\pi)\} .$$

Then the vertices p, q, r, s go to points $(\rho\cos\theta_j, \rho\sin\theta_j, 0)$ for $j = 1, 2, 3, 4$, respectively.

After doing this, we can compute the cross ratio in the usual way for the space R^3 (that is, we can replace the metric for $R^{2,1}$ with the metric for R^3 and then compute the cross ratio, which is allowed because the circle is now horizontal and therefore "Euclidean"):

$$q_{pqrs} = \sin(\frac{\theta_1 - \theta_2}{2}) \csc(\frac{\theta_2 - \theta_3}{2}) \sin(\frac{\theta_3 - \theta_4}{2}) \csc(\frac{\theta_4 - \theta_1}{2}) .$$

Remark 7. This q_{pqrs} is invariant under isometries of $R^{2,1}$, but is not Moebius invariant (unlike the case of R^3).

Once the q_{qprs} are defined, then the a_{pq} can be defined by

$$q_{pqrs} = a_{pq}/a_{ps} ,$$

and then we could use the same equations as for the R^3 case, that is, the equations in (3), to determine when the surface is discrete isothermic, with spacelike quadrilaterals.

Then, after restricting to discrete isothermic surfaces, we could define discrete spacelike CMC surfaces in $R^{2,1}$ by imitating the equations from the case of discrete CMC surfaces in R^3, as found in the previous section. This is justified by looking at smooth CMC surfaces in R^3 and $R^{2,1}$, which have exactly the same equations – only the metric changes, see Corollary 2.

So the equations we want for defining a discrete spacelike CMC surface in $R^{2,1}$ are as follows: there exist $h, H \in R$ and "normals" n_p so that

(1) $n_p \circ n_p = -1$,
(2) $d\mathfrak{f}_{pq} \wedge n_q + n_p \wedge d\mathfrak{f}_{pq} = 0$,
(3) $d\mathfrak{f}_{pq} \circ (n_p + n_q) = 0$, and
(4) $h(dn_{pq} + H d\mathfrak{f}_{pq}) = \frac{-a_{pq} d\mathfrak{f}_{pq}}{|d\mathfrak{f}_{pq}|^2}$,

where here $\circ$ represents the $R^{2,1}$ inner product, and $\wedge$ is the $R^{2,1}$ cross product, and $|\cdot|$ is the $R^{2,1}$ norm.

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