Correction in Lemma 3.2's proof

In the paper entitled "Discrete Constant Mean Curvature Surfaces and their Index" written by Konrad Polthier and myself (Wayne Rossman) and published in J. Reine Angew. Math 549 (2002), 47-77, Lemma 3.2 is the following:

> **Lemma** Given two points (a, b) and (d, e) in $\mathbb{R}^2$ with $b \neq e$, and an angle θ with $|\theta| < \pi$, there exists an r so that these two points lie on some vertical translate of the modified cosh curve
> $$\gamma(t) = \left(r \cosh \left[\frac{t}{e-b} \operatorname{arccosh} \left(1 + \frac{1}{r^2} \frac{(e-b)^2}{1+\cos\theta} \right) \right], t \right) \quad ,$$
> $t \in \mathbb{R}$, if and only if $2ad > \frac{(e-b)^2}{1+\cos\theta}$.

This lemma is correct as stated, but inside the second half of its proof in the paper, containing a list of four and then six properties, there are some erroneous statements. To correct this, here we outline an alternate approach to the proof, which again involves only entirely elementary arguments.

The first half of the proof in the paper gives the equations

$$\alpha_1(r) := \frac{1 + \frac{\hat{\delta}^2}{r^2} + \frac{\hat{\delta}}{r}\sqrt{2 + \frac{\hat{\delta}^2}{r^2}}}{\left(\frac{d}{r} + \sqrt{\frac{d^2}{r^2} - 1} \right) \left(\frac{a}{r} + \sqrt{\frac{a^2}{r^2} - 1} \right)} = 1$$

and

$$\alpha_2(r) := \frac{\left(1 + \frac{\hat{\delta}^2}{r^2} + \frac{\hat{\delta}}{r}\sqrt{2 + \frac{\hat{\delta}^2}{r^2}} \right) \left(\frac{a}{r} + \sqrt{\frac{a^2}{r^2} - 1} \right)}{\frac{d}{r} + \sqrt{\frac{d^2}{r^2} - 1}} = 1 \ ,$$

either of which being satisfied for some $r \in (0, a]$ is equivalent to the existence of a vertical translate of $\gamma(t)$ containing both of the points (a, b) and (d, e), for that r. Recall that we have assumed that $0 < a \leq d$ and $e > 0$ and $b \in [-e, e)$.

The function $\alpha_1(r)$ for $r \in (0, a]$ is increasing with respect to r and has limiting value $\hat{\delta}^2/(2ad)$ as $r \to 0$. The function $\alpha_2(r)$ for $r \in (0, a]$

$$\texttt{Do}\Big[\Big\{\texttt{a = 1; d = 2; }\delta\texttt{ = Sqrt[(0.4 j + 2) a d]; Print["a=", a, " d=", d, " }\delta\texttt{=", }\delta\texttt{];}$$

$$\texttt{Plot}\Big[\Big\{\frac{\left(1+\left(\frac{\delta}{r}\right)^2+\sqrt{\left(1+\left(\frac{\delta}{r}\right)^2\right)^2-1}\right)\left(\frac{a}{r}+\sqrt{\left(\frac{a}{r}\right)^2-1}\right)}{\frac{d}{r}+\sqrt{\left(\frac{d}{r}\right)^2-1}},$$

$$\frac{1+\left(\frac{\delta}{r}\right)^2+\sqrt{\left(1+\left(\frac{\delta}{r}\right)^2\right)^2-1}}{\left(\frac{d}{r}+\sqrt{\left(\frac{d}{r}\right)^2-1}\right)\left(\frac{a}{r}+\sqrt{\left(\frac{a}{r}\right)^2-1}\right)}\Big\},$$

$$\texttt{\{r, 0, a\}, PlotRange} \rightarrow \texttt{\{0.5, 6\}}\Big]\Big\},\ \texttt{\{j, -1, 1\}}\Big]$$

FIGURE 1. The Mathematica program that produces the graphics in the next figure.

is decreasing with respect to r and has limiting value $+\infty$ as $r \to 0$. Furthermore, $\alpha_1(a) = \alpha_2(a)$ is positive and finite. It follows that one of the two functions $\alpha_1(r)$ or $\alpha_2(r)$ will attain the value 1 for some $r \in (0, a]$ if and only if $\hat{\delta}^2 < 2ad$, proving the lemma.

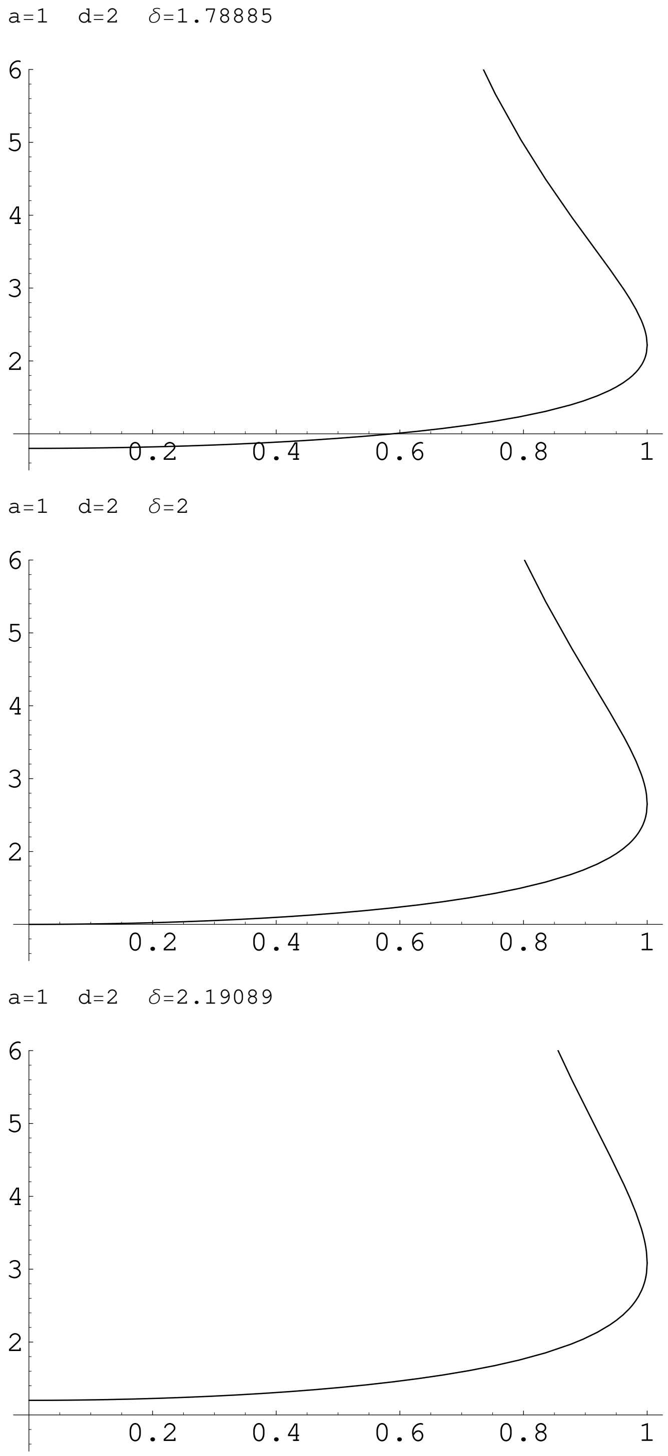

FIGURE 2. Both graphs $(r, \alpha_1(r))$ and $(r, \alpha_2(r))$ in the $\mathbb{R}^2$ plane for $r \in (0, a]$ with $a = 1$ and $d = 2$ and $\hat{\delta}$ equal to either $\sqrt{8ad/5}$ or $\sqrt{2ad}$ or $\sqrt{12ad/5}$.