CONSTANT MEAN CURVATURE (CMC) SURFACES IN EUCLIDEAN 3-SPACE AND HYPERBOLIC 3-SPACE

WAYNE ROSSMAN

ABSTRACT. We outline some of the techniques used to study the surfaces in the title, and we note some of the directions this kind of research has taken recently. We do not try to give complete information when it can be found in other sources – instead we try to be thorough in our referencing. In two particular cases where the author is unaware of a good reference (the maximum principle in our context, and a Bernstein type result in $\mathbb{H}^3$), a more complete explanation is given.

1. DEFINITION OF CMC SURFACES IN $\mathbb{R}^3$

A surface M immersed in Euclidean 3-space $\mathbb{R}^3$ has an induced metric (the first fundamental form) $\langle\cdot,\cdot\rangle : T_p(M) \times T_p(M) \to \mathbb{R}$, which is a bilinear map. If (u,v) is a local coordinate chart of M, and if the basis $\{M_u = (\frac{\partial M}{\partial u})_p, M_v = (\frac{\partial M}{\partial v})_p\}$ is chosen for $T_p(M)$, then the metric is represented by the matrix

$$(I) = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle M_u, M_u\rangle & \langle M_u, M_v\rangle \\ \langle M_v, M_u\rangle & \langle M_v, M_v\rangle \end{pmatrix} .$$

The second fundamental form of M is a symmetric bilinear map from $T_p(M) \times T_p(M)$ to the normal space $N_p(M) \approx \mathbb{R}$ represented by the matrix

$$(II) = \begin{pmatrix} l & m \\ m & n \end{pmatrix} = -\begin{pmatrix} \langle N_u, M_u\rangle & \langle N_u, M_v\rangle \\ \langle N_v, M_u\rangle & \langle N_v, M_v\rangle \end{pmatrix} = \begin{pmatrix} \langle N, M_{uu}\rangle & \langle N, M_{vu}\rangle \\ \langle N, M_{uv}\rangle & \langle N, M_{vv}\rangle \end{pmatrix} ,$$

where N is a local unit normal vector to M. The eigenvalues k_1, k_2 and the corresponding eigenvectors in $T_p(M)$ of the matrix $(I)^{-1}(II)$ are the principal curvatures and the corresponding principal curvature directions of the surface M at the point p.

The shape operator $v \to -D_vN$ is a linear map from $T_p(M)$ to $T_p(M)$ represented by the matrix $(I)^{-1}(II)$. Thus k_1, k_2 are the eigenvalues of the shape operator.

Definition 1.1. *The determinant and the trace of the shape operator are the* Gaussian curvature *K and the* mean curvature *H, respectively. Thus $K = k_1k_2$ and $H = \frac{k_1+k_2}{2}$. A surface has* constant mean curvature *if H is constant, and a surface is* minimal *if H is identically zero.*

Hence the Gaussian and mean curvatures satisfy the equations

$$K = \frac{ln - m^2}{EG - F^2} , \quad H = \frac{Gl - 2Fm + En}{2(EG - F^2)} .$$

2. Variational formulations of CMC surfaces

There are equivalent (more geometric) definitions for minimal and CMC surfaces. The mean curvature H at a point can be defined as the average of the normal curvatures in all tangent directions at that point, thus a minimal surface has average normal curvature zero at every point. This suggests a physical interpretation of a minimal surface, for which we quote [HoMe2]:

> "Loosely speaking, one imagines the surface as made up of very many rubber bands, stretched out in all directions; on a minimal surface the forces due to the rubber bands balance out, and the surface does not need to move to reduce tension."

This is actually true only locally. One could define a minimal surface to be a surface M for which at every point there exists a neighborhood $U \subseteq M$ of this point such that U is a least-area surface for its boundary ∂U. This property explains why the theory of minimal surfaces mathematically describes physical soap films that do not enclose bounded pockets of air. Since nonminimal CMC surfaces have nonzero constant average normal curvature (which, for instance, could be the result of constant air pressure on a physical soap film), nonminimal CMC surfaces are mathematical models for physical soap films that do enclose bounded pockets of air.

Another way to define minimal surfaces is that M is minimal if for every compact subdomain $\mathcal{U} \subseteq M, \mathcal{U}$ is critical for area among all variations through surfaces with fixed boundary equal to $\partial\mathcal{U}$. Similary, nonminimal CMC surfaces are surfaces such that any compact subdomain is critical for area among all volume preserving variations through surfaces fixing the boundary of the subdomain [BC2].

In the final section we give the first and second variational formulas, which will justify the above variational formulations.

3. Nonminimal CMC surfaces in $\mathbb{R}^3$

The sphere and the cylinder are the simplest examples of complete nonminimal CMC surfaces in $\mathbb{R}^3$. Another famous set of examples are those found by Delaunay (see Figure 1). Delaunay surfaces are nonminimal CMC periodic surfaces of revolution with two ends. The profile curve for any embedded Delaunay surface is the trace of one of the foci of some ellipse that is rolled along a line in the plane [E]. It is known that any embedded nonminimal CMC surface with two ends and finite topology must be a Delaunay surface [KKS]. Furthermore, any CMC complete properly immersed annular end must be asymptotic to a Delaunay surfaces; in particular, it is cylindrically bounded [KKS], [M].

Kapouleas has recently found a wide variety of examples of nonminimal CMC surfaces with both finite and infinite topology [Kap].

Wente has found compact immersed CMC curvature surfaces without boundary and with positive genus in $\mathbb{R}^3$ [W] (see Figure 2).

If a CMC surface is represented by a local conformal coordinate chart, then the Gauss map from the surface to the unit sphere is a harmonic map. Kenmotsu has found a

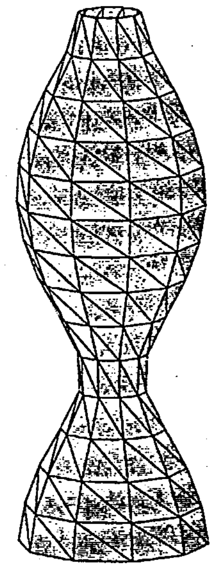

Figure 1. An embedded Delaunay surface

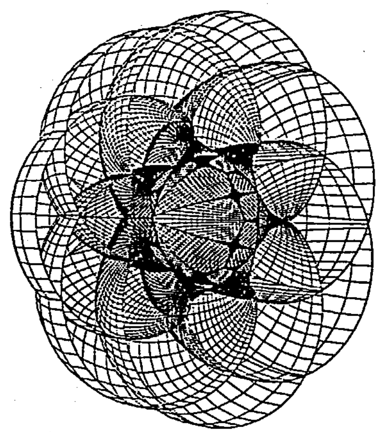

Figure 2. An example of a Wente surface. In this picture half of the surface is cut away to show the inner parts of the surface. The entire surface consists of the piece shown and its reflection through the plane of the cut.

Weierstrass-type representation for nonminimal CMC surfaces in terms of the harmonic Gauss map [Ke].

4. Minimal surfaces in $\mathbb{R}^3$

The plane, catenoid, and helicoid (Figure 3) are the most well known examples of complete minimal surfaces. Other well known examples are Enneper's surface (Figure 3) and Scherk's doubly periodic surface (Figure 5). Other examples are: the Costa surface (a figure can be found in [HoMe3]), Scherk's singly periodic surface (Figure 5), the genus 1 helicoid [HKW] (Figure 8), the Jorge-Meeks genus 0 n-noids [JoMe] (Figure 6), the genus 1 n-noids [BR] (Figure 6), the higher genus Enneper surfaces ([ChGa], [Sa]), Riemann's staircase (see, for example, [HoMe]) (Figure 4), and there are many others.

Bernstein's theorem tells us that the only complete minimal surface that can be a graph is the plane (see, for example, [O]).

The strong half-space theorem has been proven by Hoffman and Meeks [HoMe2]: Any complete properly immersed nonplanar minimal surface in $\mathbb{R}^3$ cannot lie in a half-space of $\mathbb{R}^3$.

Schoen has shown that any complete connected finite total curvature minimal surface with two embedded ends must be a catenoid [Scn]. He has also shown that any complete embedded minimal end with finite total curvature must be asymptotic to either a catenoid or a plane [Scn].

Ros, Perez, and Lopez have shown that any embedded finite total curvature complete minimal surface of genus zero must be the catenoid or the plane [PR].

Jorge and Xavier have shown existence of complete minimal surfaces lying between two planes [BCol]. This is related to the Calabi conjecture that there does not exist a complete bounded minimal surface. Recently, there has been an argument presented that a counterexample to Calabi's conjecture exists [Nad], but the question appears to remain open.

There is a still open conjecture called the Meeks conjecture: Any compact minimal surface with boundary consisting of two convex curves in parallel planes must be an annulus.

The Costa surface was shown to be embedded by Hoffman and Meeks [HoMe3]. This provides a counterexample to an old longstanding conjecture that the plane, catenoid, and helicoid are the only complete embedded minimal surfaces with finite topology.

5. The Weierstrass Representation

Minimal surfaces can be described by a pair of meromorphic functions on a Riemann surface, as in the lemma below. This lemma is proven in [O]. Other references are [L], [HoMe].

Lemma 5.1. (Weierstrass representation) *A regular minimal surface in* $\mathbb{R}^3$ *can be represented in the form*

$$\Psi(w) = Re \int_{w_0}^{w} \begin{pmatrix} (1-g^2)fdz \\ i(1+g^2)fdz \\ 2fgdz \end{pmatrix},$$

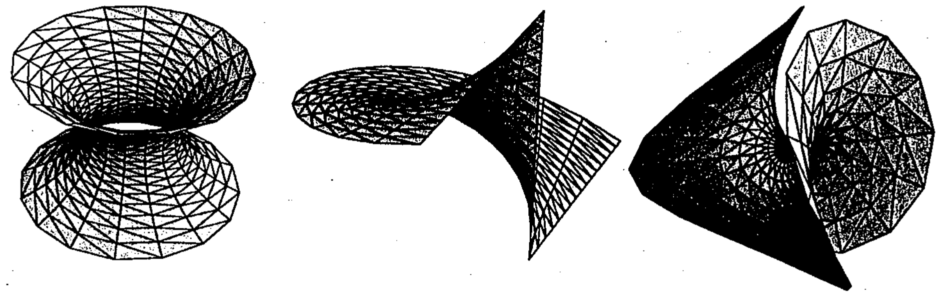

FIGURE 3. The catenoid, a portion of a helicoid bounded by a vertical line segment and two horizontal rays, and Enneper's surface

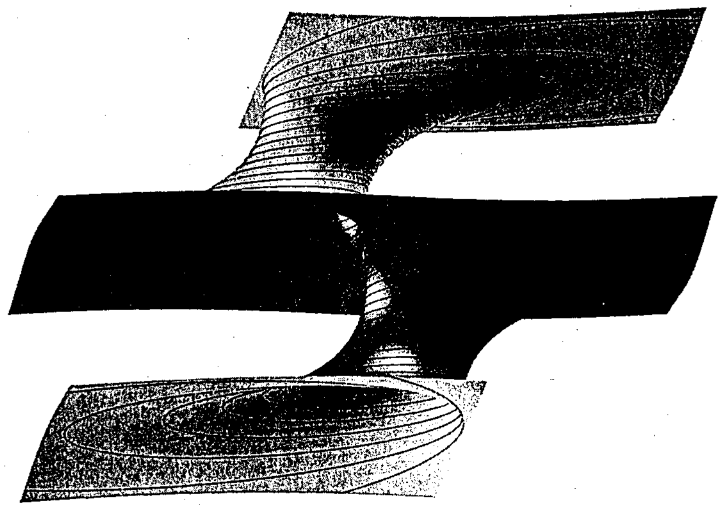

FIGURE 4. A Riemann example, also called "Riemann's staircase"

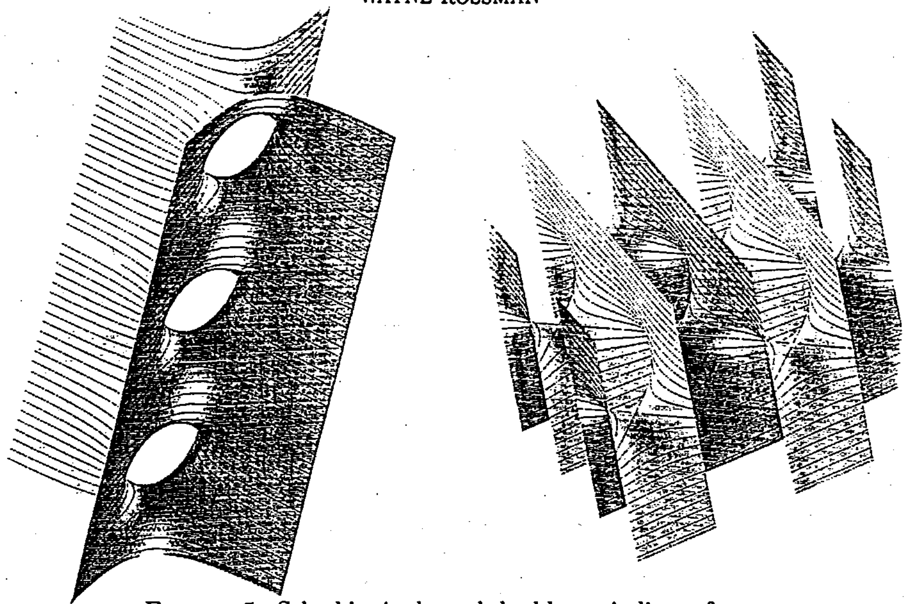

FIGURE 5. Scherk's singly and doubly periodic surfaces

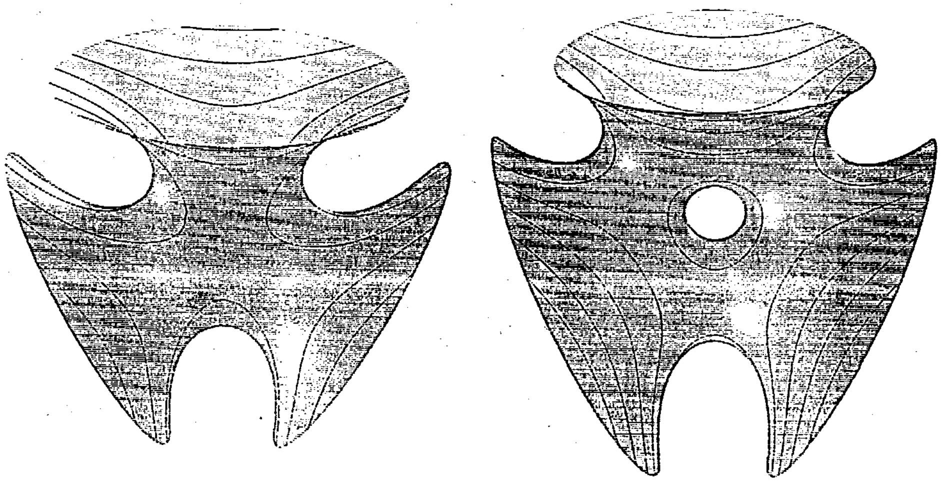

FIGURE 6. The genus 0 and genus 1 trinoids

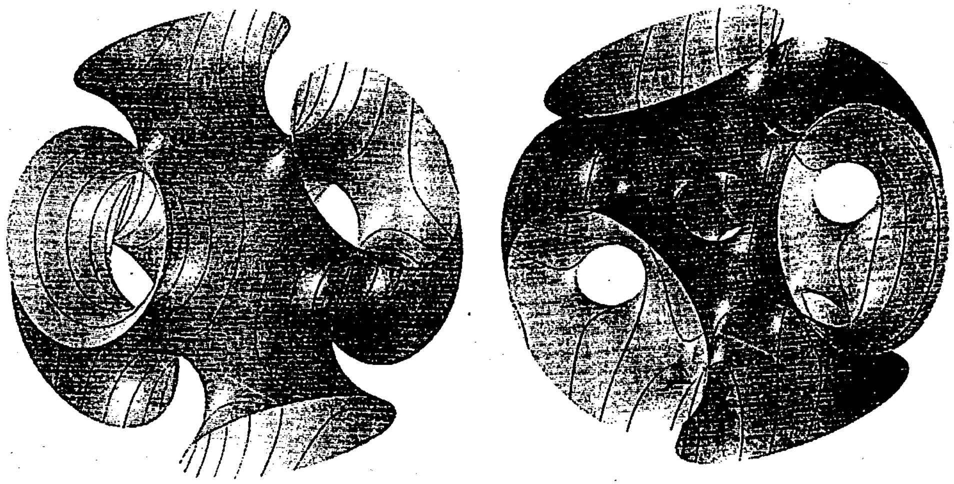

FIGURE 7. The genus 0 and genus 7 octoids

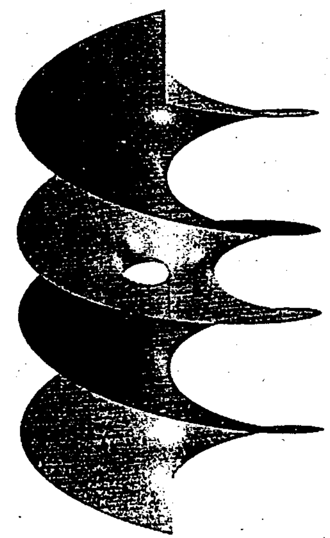

FIGURE 8. The genus 1 helicoid

the domain Σ being a Riemann surface with points and/or disks removed, and the integral being taken along a path from $w_0 \in \Sigma$ to $w \in \Sigma$. The function g is meromorphic on Σ, and the function f is holomorphic on Σ. Furthermore, f vanishes only at the poles of g, and the order of its zero at such a point is exactly twice the order of the pole of g.

Applying this integral to closed curves in Σ, one has a period vector associated to each loop, and for nontrivial loops this vector might be nonzero. Thus $\Psi(w)$ may only be well defined on a covering space of Σ. Periodic minimal surfaces can be constructed by having one or more nonvanishing periods.

The function g has a nice geometric interpretation; it is the composition of the Gauss map with stereographic projection to the complex plane $\mathbb{C}$. This implies, in particular, that the Gauss map is conformal for minimal surfaces. There is further geometric information encoded in the functions f and g, since the metric and the Gaussian curvature on the surface can be described in terms of them [O]:

$$ds^2 = |f|^2(1+|g|^2)^2|dz|^2 ,$$

$$K = -\left(\frac{4|\frac{dg}{dz}|}{|f|(1+|g|^2)^2}\right)^2 .$$

The Hopf differential $Q = dg \cdot f dz$ is a meromorphic 2-form, and the complexification of the second fundamental form is written in terms of Q:

$$II = -Q - \overline{Q} .$$

Changing f to $e^{i\theta} \cdot f$ gives us a 1-parameter family of minimal surfaces with parameter θ. This family is called the *conjugate family*. When $\theta = \frac{\pi}{2}$, we have the *conjugate surface*. Since ds^2 is unaffected by θ, the surfaces in this family are all isometric. If the minimal surface with Weierstrass data g, f contains a straight line (resp. planar geodesic), this curve becomes a planar geodesic (resp. straight line) on the conjugate surface [Ka3]. This follows from the following fact, proved in [Ka3]: Let $\alpha(t)$ be a curve in $\Psi(w)$, and let $\alpha'(t)$ be its tangent vector field, then

$$\alpha(t) \text{ is an asymptotic curve } \Leftrightarrow Q(\alpha'(t), \alpha'(t)) \in i\mathbb{R} ,$$

$$\alpha(t) \text{ is a principal curve } \Leftrightarrow Q(\alpha'(t), \alpha'(t)) \in \mathbb{R} .$$

This property is useful for proving existence of minimal surfaces containing a relatively large number of planar geodesics. One can prove existence by proving existence of simply-connected pieces of the conjugate surface bounded by the straight lines that the conjugate surface must contain. This is called the *conjugate surface construction* [BR], [Ka1], [Ka2], [Ka3], [Ka4], [R].

An essential tool in the conjugate surface construction is the Schwarz reflection principle. It says that a minimal surface can be extended smoothly across its planar geodesic (resp. straight line) boundary by reflection in the plane containing the boundary (resp. rotation by π radians about the boundary line).

Now we list Weierstrass data for some examples:

plane: $\Sigma = \mathbb{C}, g = 0, f = 1$.
Enneper surface: $\Sigma = \mathbb{C}, g = z, f = 1$.
higher winding order Enneper surface: $\Sigma = \mathbb{C}, g = z^n, f = 1$.

catenoid: $\Sigma = \mathbb{C} \setminus \{0\}, g = z, f = \frac{1}{z^2}$.
helicoid: $\Sigma = \mathbb{C} \setminus \{0\}, g = z, f = \frac{i}{z^2}$.
trinoid: $\Sigma = (\mathbb{C} \cup \{\infty\}) \setminus \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}, g = z^2, f = \frac{1}{(z^3-1)^2}$.
Scherk's singly periodic example: $\Sigma = (\mathbb{C} \cup \{\infty\}) \setminus \{\pm 1, \pm i\}, g = z, f = \frac{i}{z^4-1}$.
Scherk's doubly periodic example: $\Sigma = (\mathbb{C} \cup \{\infty\}) \setminus \{\pm 1, \pm i\}, g = z, f = \frac{1}{z^4-1}$.
genus 1 Enneper surface: $\bar{\Sigma}_\lambda = \{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = \frac{z}{(z+\lambda)(z-\lambda)}\}$, $\lambda \in \mathbb{R}$, $\Sigma = \bar{\Sigma}_\lambda \setminus \{(\infty, 0)\}$, $g = w$, $f = \frac{1}{w}$.
Costa surface: $\bar{\Sigma} = \{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = \frac{z}{(z^2-1)}\}$, $\Sigma = \bar{\Sigma} \setminus \{(z,w) \mid z = \pm 1, \infty\}$, $g = \frac{B}{w}$, $f = \frac{z}{w}$, $B \in \mathbb{R}$.
Riemann example: $\bar{\Sigma}_\lambda = \{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = z(z-\lambda)(z+\frac{1}{\lambda})\}$, $\lambda > 0$, $\Sigma = \bar{\Sigma}_\lambda \setminus \{(0,0), (\infty,\infty)\}$, $g = z$, $f = \frac{i}{zw}$.

The catenoid and Enneper's surface are the only examples with Gauss map that is one to one. Thus they are the only examples with total absolute curvature $\int_M |K| dA = 4\pi$.

The Cohn-Vossen inequality says that $\int_M K dA \leq 2\pi\chi(M)$. This was strengthened, when M is a complete minimal surface, by Osserman to the Osserman inequality: $\int_M K dA \leq 2\pi(\chi(M) - k)$, where k is the number of ends. And finally in the case that M is a minimal surface with finite total curvature and with ends that are asymptotic to either a plane or a catenoid, an equality was given by Jorge and Meeks [JoMe]:

$$\int_M K dA = 2\pi(\chi(M) - \sum_{p_j} b_j) ,$$

where b_j is the multiplicity (winding order) at each end p_j. b_j is defined in [JoMe] and is a positive integer, and is 1 if and only if the end is embedded.

6. Weierstrass Data for the Genus-1 Trinoid

We give this example to show how one can attack the period problems numerically on a computer.

The genus 1 trinoid can be represented by the following Weierstrass data: $\bar{\Sigma} = \{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = \frac{(z-1)(z-\lambda_1)}{z}\}$, $\Sigma = \bar{\Sigma} \setminus \{(z,w) \in \bar{\Sigma} \mid z = \lambda_2, \infty\}$. Since there are two points in $\bar{\Sigma}$ such that $z = \lambda_2$ and only one point in $\bar{\Sigma}$ such that $z = \infty$, we have removed three points from $\bar{\Sigma}$ to produce Σ. We can then choose $g = \frac{k_1}{w}(z - \lambda_3)$ and $f = \frac{k_2 w}{(z-\lambda_2)^2}$. We are assuming that $\lambda_i \in \mathbb{R}$ and $0 > \lambda_1 > \lambda_2 > \lambda_3$. The value $k_1 = \frac{-1}{\sqrt{3}} \frac{w(\lambda_2)}{(\lambda_2 - \lambda_3)}$ is determined by the fact that the angles between the normals of the ends are all $\frac{2\pi}{3}$.

To solve the period problems, we need to have the planar geodesics α_1 and α_3 in the same plane; we also need to have α_2, α_4, and α_5 all within a single plane (see Figure 9). This can be accomplished by the proper choice of λ_1, λ_2, and λ_3. Using the MESH graphics program [MESH] and the Simplex algorithm, one can find values for λ_i so that these period problems are solved, producing a symmetric genus-1 trinoid. Surprisingly, one can also find one other set of values for λ_i which solves the period problems. This surface is not as symmetric, and suggests the existence of a larger family of less symmetric n-noids of genus 1 (see Figure 10).

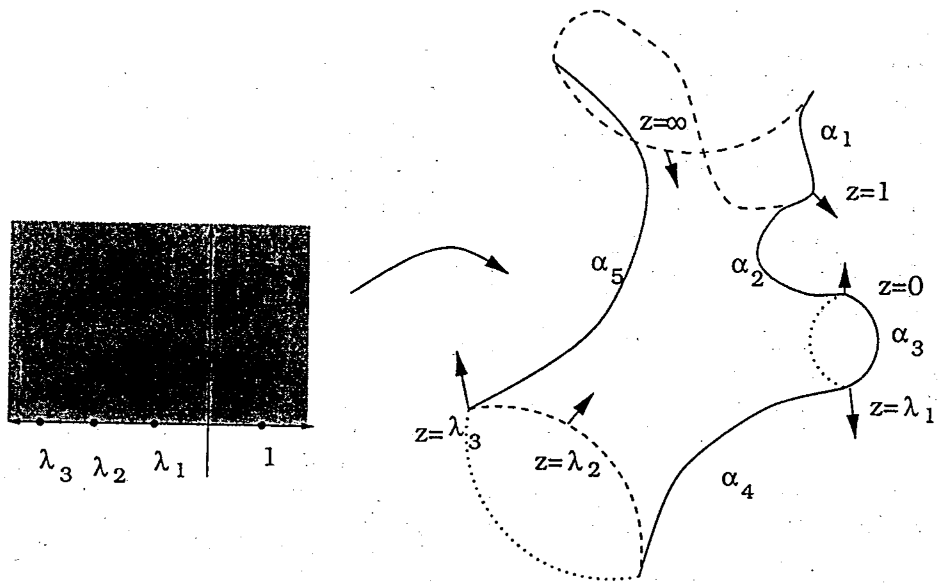

FIGURE 9. The image of the Weierstrass Data for the genus 1 trinoid

By adjusting the value of k_1 one can still solve the period problem with the Simplex method and produce genus-1 trinoids where the normals at the ends do not form angles of $\frac{2\pi}{3}$ (see Figure 10).

Using the conjugate surface construction, one can make a 1-dimensional period problem for the most symmetric case and solve it to prove existence of a genus 1 n-noid with dihedral symmetry group of size $4n$. The argument for this is given in [BR].

7. FLUX

Here we define a vector associated to boundary curves of CMC surfaces in $\mathbb{R}^3$.

Definition 7.1. *Let $C_1, ..., C_k$ be the boundary curves of an embedded surface M in $\mathbb{R}^3$ with constant mean curvature H. Let $Q_1, ..., Q_k$ be disks such that $\partial Q_i = C_i$ and $M \cup Q_1 \cup ... \cup Q_k$ is the boundary of a bounded region in $\mathbb{R}^3$. Let $\vec{\eta}$ be the outward pointing conormal of M along C_i, and let $\vec{v}$ be the outward pointing normal on Q_i relative to the bounded region. Then the* flux *$w(C_i)$ (also called* weight*) of the boundary curve C_i is*

$$w(C_i) = \int_{C_i} \vec{\eta} ds - H \int_{Q_i} \vec{v} dA .$$

We now state two lemmas that tell us that flux is well-defined and is an invariant of homology, and one can find proofs in [KKS]. Another reference is [HoMe2]. The proof of the second lemma is an application of Stoke's theorem.

Lemma 7.1. *Flux is well defined, that is, $w(C_i)$ is independent of the choice of Q_i.*

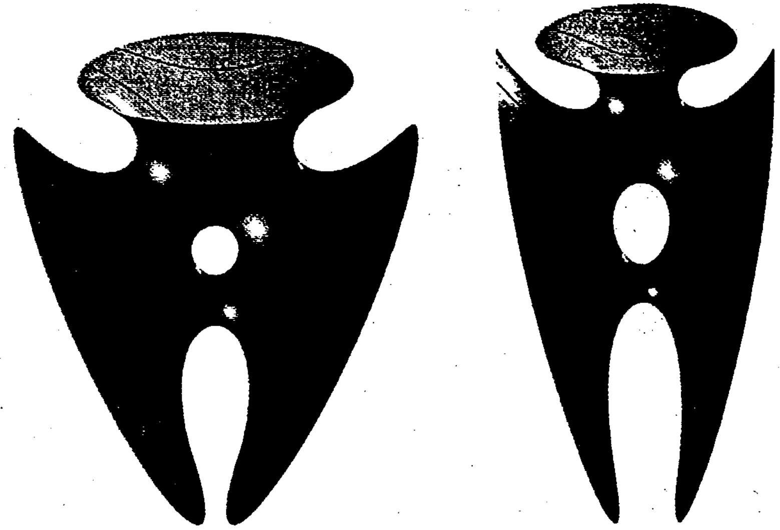

FIGURE 10. Less symmetric trinoids of genus 1: (left) with angles of $\frac{2\pi}{3}$ between the normals of the ends, and (right) where those angles are $\frac{3\pi}{5}$, $\frac{3\pi}{5}$, and $\frac{4\pi}{5}$

Thus for a closed curve C in a complete CMC surface M, the flux is defined by choice of the conormal $\vec{\eta}$. The lemma below implies that there is a well defined flux associated to each annular end of a CMC surface (see Figure 11), and that they "balance".

Lemma 7.2. *Flux is a homology invariant, that is, $w(C)$ is independent of the choice of curve within the homology class of C with oriented conormal. Moreover, with the curves C_i defined as above,*

$$\sum_{i=1}^{k} w(C_i) = 0 .$$

The flux is an useful tool that has been used in many situations. We have already mentioned the results of [KKS] and [Kap], in which flux was a central ingredient. The flux is also used in many other papers, especially in the works of Brito, Earp, Hoffman, Shin Kato, Karcher, Korevaar, Kusner, Meeks, Polthier, Rosenberg, Umehara, and Yamada.

8. HYPERBOLIC n-SPACE

The ambient spaces in which we consider CMC surfaces are $\mathbb{R}^3$ and hyperbolic 3-space $\mathbb{H}^3$. As $\mathbb{H}^n$ is less universally studied than $\mathbb{R}^n$, we give a brief description of $\mathbb{H}^n$ here. We then describe some useful tools for the study of CMC surfaces in $\mathbb{H}^3$, such as the maximum principle and the Bryant representation, in later sections.

The group of isometries of $\mathbb{R}^n$ which fix the origin is the orthogonal group $O(n) = \{A \in M_{n\times n} \mid A^T A = I\} = \{A \in M_{n\times n} \mid \langle \vec{x}, \vec{x} \rangle = \langle A\vec{x}, A\vec{x} \rangle \ \forall \vec{x} \in \mathbb{R}^n\}$. Thus the group of

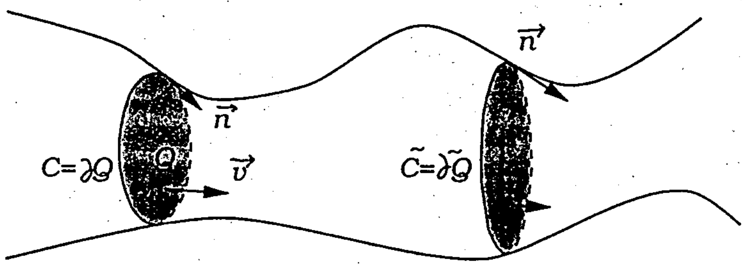

FIGURE 11. The Homological Invariance of Flux; C Homologous to $\tilde{C} \Rightarrow w(C) = w(\tilde{C})$

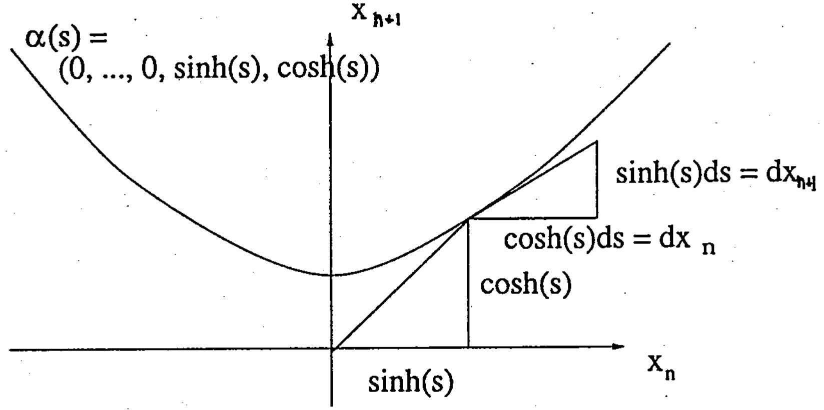

FIGURE 12. The Hyperbolic Sine and Cosine Functions

isometries can be described by actions of matrices. There is a similar description for the isometries of $\mathbb{H}^n$ if one uses the Minkowski model. Let $\mathbb{R}^{n,1}$ be the space $\mathbb{R}^{n+1}$ with the Minkowski metric

$$\langle (x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}) \rangle = x_1y_1 + ... + x_ny_n - x_{n+1}y_{n+1} ,$$

and consider the hypersurface $\{\vec{x} \in \mathbb{R}^{n,1} \mid \langle \vec{x}, \vec{x} \rangle = -1, x_{n+1} > 0\}$ which is one of two sheets of a hyperboloid. This hypersurface $\mathcal{M}^n$ with the induced metric is called the Minkowski model for hyperbolic space. Let $O(n,1) = \{A \in M_{(n+1)\times(n+1)} \mid \langle \vec{x}, \vec{x} \rangle = \langle A\vec{x}, A\vec{x} \rangle \; \forall \vec{x} \in \mathbb{R}^{n,1}\} = \{A \in M_{(n+1)\times(n+1)} \mid A^{\mathcal{T}}A = I\}$ be the orthogonal group of $\mathbb{R}^{n,1}$. By $\mathcal{T}$ we mean the Lorentz transpose: for $A = (a_{ij})$, $\mathcal{T}$ is the transformation $a_{ij} \to a_{ji}$ if

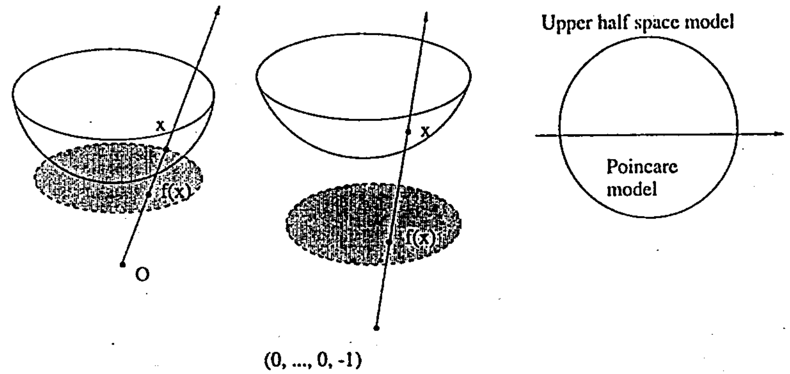

FIGURE 13. The Klein Model, Poincare Model, and Upper-Halfspace Model for $\mathbb{H}^n$

$i \neq n+1$ and $j \neq n+1$ or if $i = j = n+1$, and $a_{ij} \to -a_{ji}$ if $i = n+1$ or $j = n+1$ but not $i = j = n+1$. For any point $p \in \mathcal{M}^n$, there exists a matrix $A \in SO(n)$ such that the matrix

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in O(n,1)$$

preserves $\mathcal{M}^n$ and maps p to a point of the form $(0, ..., 0, \sinh(s), \cosh(s))$, $s \in \mathbb{R}$. Then if B is the 2×2 matrix

$$B = \begin{pmatrix} \cosh(-s) & \sinh(-s) \\ \sinh(-s) & \cosh(-s) \end{pmatrix} ,$$

the matrix

$$\begin{pmatrix} I_{n-1} & 0 \\ 0 & B \end{pmatrix} \in O(n,1)$$

preserves $\mathcal{M}^n$ and maps the point $(0, ..., 0, \sinh(s), \cosh(s))$ to the point $(0, ..., 0, 1)$. Thus one can move an arbitrary point of $\mathcal{M}^n$ to the point $(0, ..., 0, 1)$ by an isometry, and one can conclude that this model has constant sectional curvature simply by checking that the model has constant sectional curvature at $(0, ..., 0, 1)$. If π_1, π_2 are any two plane sections of $T_{(0,...,0,1)}(\mathcal{M}^n)$, there exists an isometry $\phi \in O(n.1)$ of $\mathcal{M}^n$ fixing $(0, ..., 0, 1)$ such that $\phi_*(\pi_1) = \pi_2$, therefore this model has constant sectional curvature at $(0, ..., 0, 1)$. The 2-dimensional submanifold of $\mathcal{M}^n$ consisting of geodesic arcs through $(0, ..., 0, 1)$ and tangent at $(0, ..., 0, 1)$ to π_1 is isometric to $\mathcal{M}^2$. Thus to see that the constant sectional curvature is -1, one need only check that this is the value of the Gaussian curvature of $\mathcal{M}^2$ at $(0, 0, 1)$.

Intersecting $\mathcal{M}^n$ with the plane $\{x_1 = ... = x_{n-1} = 0\}$, we obtain a curve that can be parametrized by unit speed by $\alpha(s) = (0, ..., 0, \sinh(s), \cosh(s))$, $\alpha(0) = (0, ..., 0, 1)$, and $\frac{d\alpha}{ds}(0) = (0, ..., 0, 1, 0)$. (In fact, the functions hyperbolic sine and hyperbolic cosine can be defined by these conditions. The curve $\alpha(s)$ satisfies $x_n^2 - x_{n+1}^2 = -1$, so $\cosh^2(s) -$

$\sinh^2(s) = 1$, and by implicit differentiation

$$\frac{dx_{n+1}}{dx_n} = \frac{x_n}{x_{n+1}} \Longrightarrow \frac{d(\cosh(s))}{d(\sinh(s))} = \frac{\sinh(s)}{\cosh(s)} .$$

Since $|\alpha'(s)|^2 = (\frac{dx_n}{ds})^2 - (\frac{dx_{n+1}}{ds})^2 = 1$, it follows that

$$\frac{d}{ds}(\sinh(s)) = \cosh(s) ,$$

$$\frac{d}{ds}(\cosh(s)) = \sinh(s) .$$

(An analogous analysis can be carried out for the functions sine and cosine on the unit circle in the Euclidean plane.) Once we know how to differentiate $\cosh(s)$ and $\sinh(s)$, we know the power series expansions of these functions about $s = 0$. Comparing these series with the power series expansions for e^s and e^{-s} about $s = 0$, we conclude

$$\cosh(s) = \frac{e^s + e^{-s}}{2} , \tag{8.1}$$

$$\sinh(s) = \frac{e^s - e^{-s}}{2} , \tag{8.2}$$

which are the standard definitions of $\cosh(s)$ and $\sinh(s)$. See Figure 12.) Since any geodesic segment in $\mathcal{M}^n$ can be moved by an isometry to $\alpha(s)$, $0 \leq s \leq a$ for some value of a, equations 8.2 and 8.3 imply that any geodesic segment can be extended to an infinite length geodesic. Therefore $\mathcal{M}^n$ is complete. Thus $\mathcal{M}^n$ is a complete simply connected constant sectional curvature -1 manifold, and so $\mathcal{M}^n = \mathbb{H}^n(-1)$ is hyperbolic n-space with constant sectional curvature -1.

The geodesics of $\mathcal{M}^n$ are formed by the intersections of $\mathcal{M}^n$ with 2-planes in $\mathbb{R}^{n,1}$ which pass through the origin of $\mathbb{R}^{n,1}$.

Let $\mathcal{K}$ be the n-dimensional ball in $\mathbb{R}^{n,1}$ lying in the hyperplane $\{x_{n+1} = 1\}$ with radius 1 and center at $(0, ..., 0, 1)$. By Euclidean stereographic projection from $\vec{0} \in \mathbb{R}^{n,1}$ of $\mathcal{M}^n$ to $\mathcal{K}$, one has the Klein model $\mathcal{K}$ for $\mathbb{H}^n$. $\mathcal{K}$ is given the metric that makes this stereographic projection an isomety. This model has the advantage that its geodesics are the same as the geodesics of the same n-ball with the Euclidean metric and hence they are "straight" lines. However, it has the disadvantage that it is not conformal to the Euclidean metric. (See Figure 13.)

Let $\mathcal{P}$ be the n-dimensional ball in $\mathbb{R}^{n,1}$ lying in the hyperplane $\{x_{n+1} = 0\}$ with radius 1 and center at $\vec{0}$. By Euclidean stereographic projection from $(0, ..., 0, -1) \in \mathbb{R}^{n,1}$ of $\mathcal{M}^n$ to $\mathcal{P}$, one has the Poincare model $\mathcal{P}$ for $\mathbb{H}^n$. $\mathcal{P}$ is given the metric that makes this stereographic projection an isomety. This model is the Euclidean unit ball B^n in $\mathbb{R}^n$ with the metric $d\hat{s}^2 = \frac{4}{(1-r^2)^2}ds^2$, where r is Euclidean distance to the origin and $ds^2 = \sum_{i=1}^n dx_i^2$ is the Euclidean metric. Thus the metric of this model is conformal to the Euclidean metric on the unit ball and therefore angles are preserved between these two metrics. The geodesics in the Poincare model are segments of Euclidean lines and circles that intersect the "boundary at infinity" ∂B^n at right angles. (See Figure 13.)

One can obtain the upper-halfspace model for $\mathbb{H}^n$ from the Poincare model by the Moebius transformation of $\mathbb{R}^n$ which maps the unit ball B^n (with the Poincare metric) centered at the origin to the upper half of $\mathbb{R}^n$ $\{x_n > 0\}$ and maps $\vec{0}$ to $(0, ..., 0, 1)$ and fixes $\partial B^n \cap \{x_n = 0\}$. The metric induced on the upper-half space by this transformation is $d\hat{s}^2 = \frac{1}{x_n^2}ds^2$, where ds^2 is the Euclidean metric on the upper-halfspace. Thus the upper-half space model is conformal to the Euclidean upper-halfspace (hence angles appear "correct" from a Euclidean point of view), and again the geodesics are Euclidean lines and circles that intersect the hyperplane "boundary at infinity" $\{x_n = 0\}$ at right angles. The isometries of the upper-halfspace model are generated by horizontal Euclidean translations, Euclidean rotations about vertical axes, Euclidean dilations about points in the hyperplane $\{x_n = 0\}$, and Euclidean inversions through hyperspheres (and hyperplanes) intersecting the hyperplane $\{x_n = 0\}$ orthogonally. (See Figure 13.)

In the case $n = 3$ we consider another representation, the Hermitean model of $\mathbb{H}^3$, which will be useful for stating Bryant's theorem below. We first recall the following definitions: The Lie group $SL(2, \mathbb{C})$ is all 2×2 matrices with complex entries and determinant 1. $s\ell(2, \mathbb{C})$ is the associated Lie algebra, thus is the tangent space of $SL(2, \mathbb{C})$ at the identity matrix. $s\ell(2, \mathbb{C})$ consists of all 2×2 complex matrices with trace 0. A 2×2 matrix μ is in $SU(2)$ if $\mu\mu^*$ is the identity matrix, where $\mu^* = \bar{\mu}^t$. Equivalently, $\mu \in SU(2)$ if

$$\mu = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix} ,$$

for some $p, q \in \mathbb{C}$ such that $|p|^2 + |q|^2 = 1$.

$R^{3,1}$ can be mapped to the space of 2×2 Hermitian symmetric matrices by

$$\phi : (x_1, x_2, x_3, x_4) \longrightarrow \begin{pmatrix} x_4 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_4 - x_3 \end{pmatrix} .$$

For $\vec{x} \in \mathbb{R}^{3,1}$, $\langle \vec{x}, \vec{x} \rangle = -\det(\phi(\vec{x}))$. Thus ϕ maps the Minkowsky model $\mathcal{M}^3$ to the set of Hermitean symmetric matrices with determinant 1. Therefore we have the Hermitean model

$$\mathcal{H} = \{A\bar{A}^t \mid A \in SL(2, \mathbb{C})\} ,$$

and $\mathcal{H}$ is given the metric so that ϕ is an isometry from $\mathcal{M}^3$ to $\mathcal{H}$.

The group $SL(2, \mathbb{C})$ represents the isometry group of $\mathbb{H}^3$ in the model $\mathcal{H}$. A matrix $h \in SL(2, \mathbb{C})$ acts isometrically on $x \in \mathcal{H} = \mathbb{H}^3$ by

$$h \cdot x = h x h^* ,$$

where $h^* = \bar{h}^t$. The kernel of this action is $\pm I$, hence $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm I\}$ is the isometry group of $\mathbb{H}^3$ and is the identity component of $SO(3, 1)$.

Above we have described $\mathbb{H}^n(-1)$, the hyperbolic space with constant sectional curvature -1. One can extend this description to the hyperbolic space $\mathbb{H}^n(-c^2)$ with constant sectional curvature $-c^2$. We may define $\mathbb{H}^n(-c^2) = \{\vec{x} \in \mathbb{R}^{n,1} \mid \langle \vec{x}, \vec{x} \rangle = \frac{-1}{c^2}, x_{n+1} > 0\}$ with the induced metric from $R^{n,1}$.

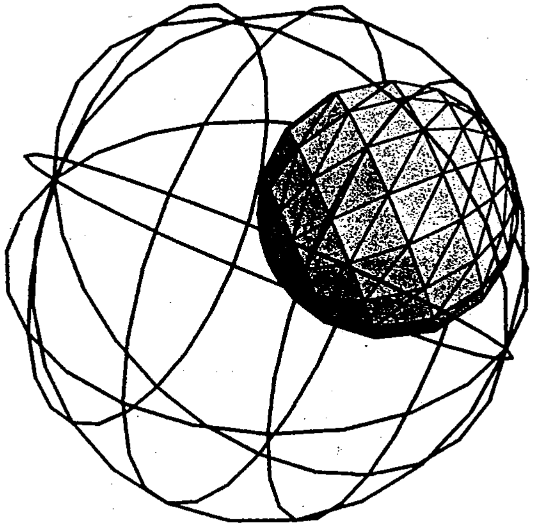

FIGURE 14. The horosphere in the Poincare model

9. CMC SURFACES IN $\mathbb{H}^3$

To define CMC surfaces in $\mathbb{H}^3$, we can proceed in the same way as we did for the Euclidean case in the first section. We only need to replace the "true" Euclidean vectors M_u and M_v by the linear differentials $(\frac{\partial}{\partial u})_p$ and $(\frac{\partial}{\partial v})_p$, and replace N_u and N_v by $\nabla_{\frac{\partial}{\partial u}} N$ and $\nabla_{\frac{\partial}{\partial v}} N$, where ∇ is the Riemannian connection associated with the hyperbolic metric. Using ∇, the shape operator becomes $v \to -\nabla_v N$.

We consider $\mathbb{H}^n$ with the Poincare model in the Euclidean unit ball B^n. The totally geodesic hypersurfaces (hyperplanes) are the intersections of B^n with $(n-1)$-dimensional Euclidean spheres and planes in $\mathbb{R}^n$ which meet ∂B^n orthogonally. The *hyperspheres* are the intersections of B^n with arbitrary $(n-1)$-dimensional Euclidean spheres and planes in $\mathbb{R}^n$ having nonempty nontangent intersection with ∂B^n, and they are CMC H hypersurfaces with $|H| < 1$. The spheres in $\mathbb{H}^n$ are the $(n-1)$-dimensional Euclidean spheres in the interior of B^n, and they are CMC H hypersurfaces with $|H| > 1$. The horospheres are the $(n-1)$-dimensional Euclidean spheres that are tangent to ∂B^n at one point, and they are CMC 1 hypersurfaces (see Figure 14).

As in the Euclidean case, there exist complete two-ended periodic CMC surfaces of revolution in hyperbolic space $\mathbb{H}^3$, and they are called hyperbolic Delaunay surfaces (see Figure 15). If a Delaunay surface has constant mean curvature H, then $H > 1$. Just like the results in [KKS], corresponding results are proven in [KKMS] for the hyperbolic case.

Recently, it was shown in [RUY] that for any complete minimal surface in $\mathbb{R}^3$ that is symmetric and nondegenerate, there exists a corresponding one-parameter family of CMC 1 surfaces in $\mathbb{H}^3$, called the "cousin" surfaces. (The terms "symmetric" and "nondegenerate" are defined in [RUY].) This property is closely related to the Lawson correspondence

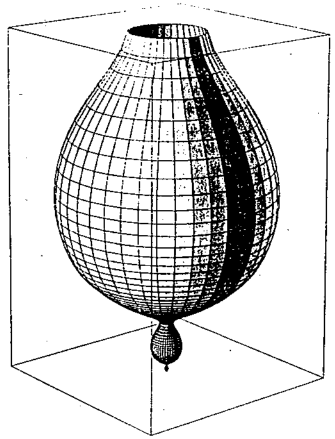

FIGURE 15. An embedded hyperbolic Delaunay surface in the upper half-space model

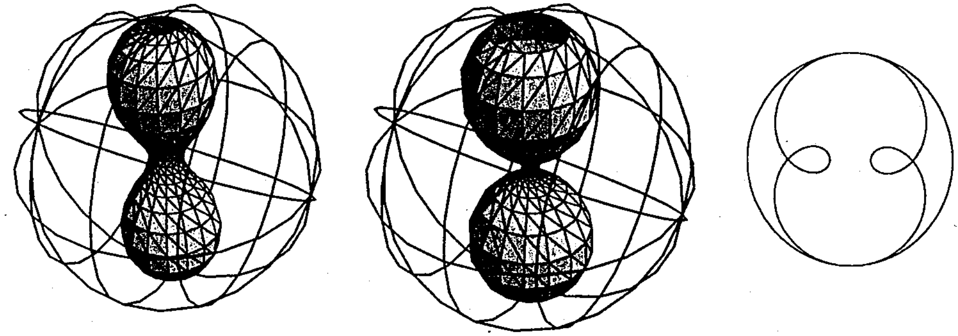

FIGURE 16. Two different embedded members of the one-parameter family of catenoid cousins, and the profile curve of a nonembedded catenoid cousin

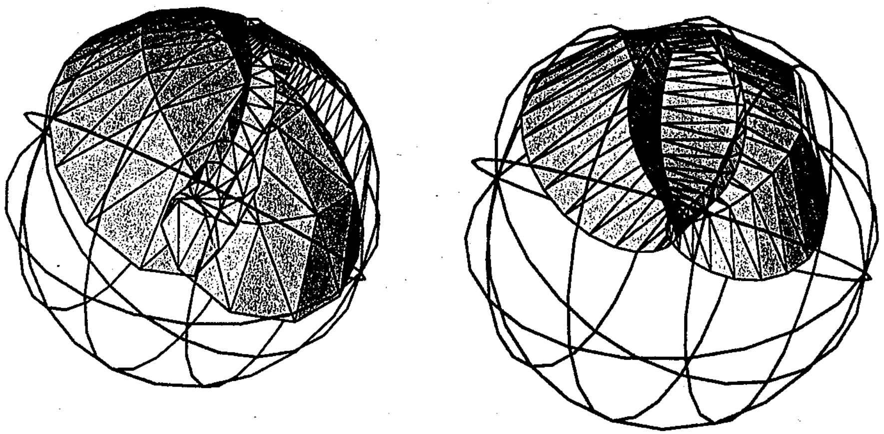

FIGURE 17. Two different members of the one-parameter family of Enneper cousins

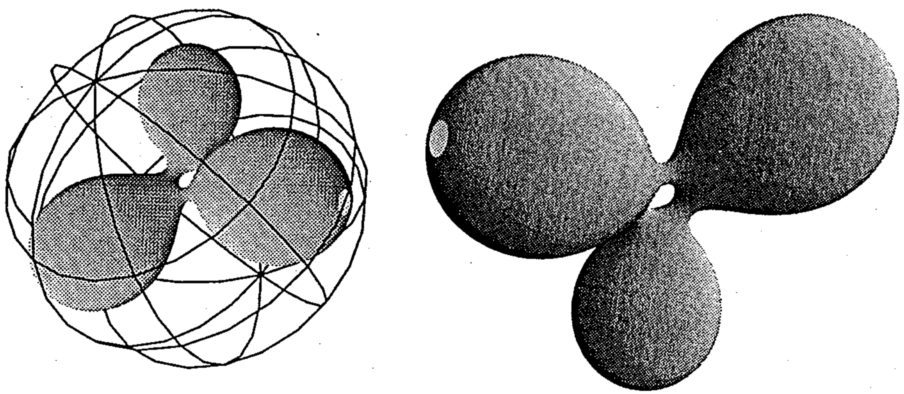

FIGURE 18. A genus 1 trinoid cousin (n-noid cousin with $n = 3$)

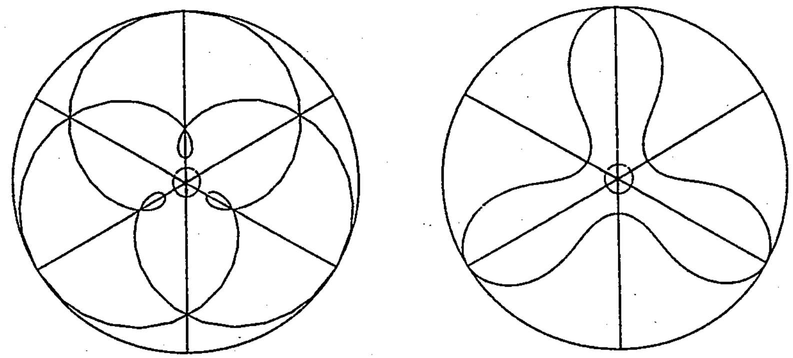

FIGURE 19. Curves in the plane of reflective symmetry of an embedded and non-embedded genus 1 trinoid cousin

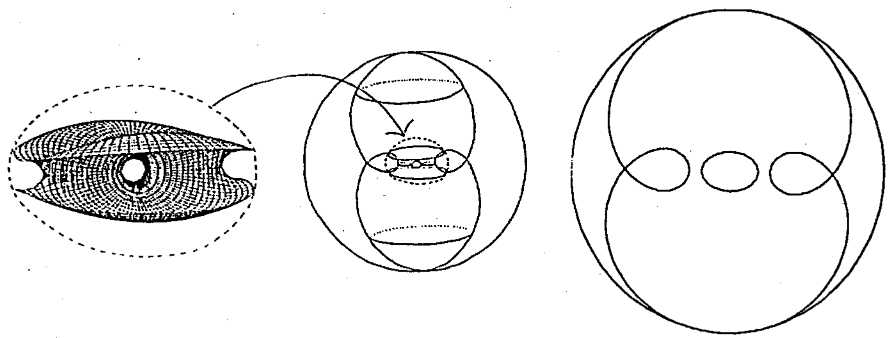

FIGURE 20. A genus 1 catenoid cousin and a schematic of the curve in the plane of reflective symmetry

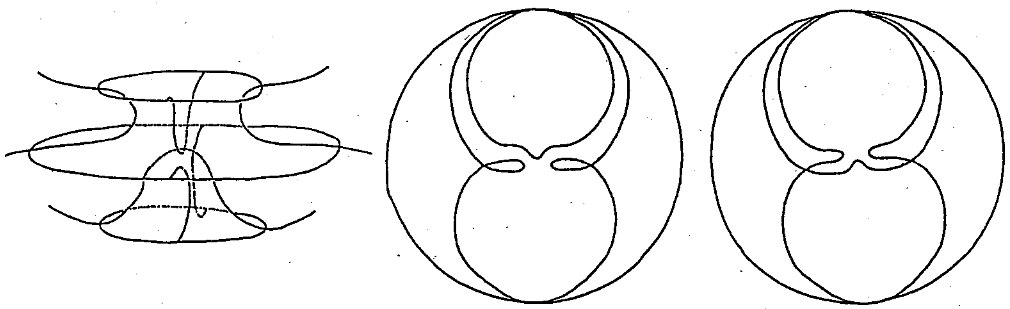

FIGURE 21. The central part of the Costa cousin, and curves in two different reflective symmetry planes

described in the next section. For example, since we have planes and catenoids and Enneper surfaces and Jorge-Meeks n-noids and genus 1 n-noids and Costa surfaces in $\mathbb{R}^3$, we also have CMC 1 horospheres and catenoid cousins (Figure 16) and Enneper cousins (Figure 17) and n-noid cousins and genus 1 n-noid cousins (Figures 18 and 19) and Costa cousins (Figure 21) in $\mathbb{H}^3$.

However, the converse of the [RUY] result does not hold. One can find counterexamples in [UY1]. Another counterexample is the genus 1 catenoid cousin (see Figure 20); there exists a genus 1 catenoid cousin in $\mathbb{H}^3$ [RoSa], but there does not exist a genus 1 catenoid in $\mathbb{R}^3$ [Scn].

10. Bryant representation for CMC 1 surfaces

There is a natural correspondence between minimal surfaces in $\mathbb{R}^3$ and CMC 1 surfaces in $\mathbb{H}^3$. If $x : \mathcal{U} \subset \mathbb{C} \to \mathbb{R}^3$ is a local representation for a minimal surface in $\mathbb{R}^3$ with first and second fundamental forms ds^2 and II, then by the Gauss and Codazzi equations we see that there is a well-defined CMC 1 surface $\tilde{x} : \mathcal{U} \subset \mathbb{C} \to \mathbb{H}^3$ with first and second fundamental forms ds^2 and $II + ds^2$. This is known as Lawson's correspondence. In addition, CMC 1 surfaces in $\mathbb{H}^3$ have a Bryant representation based on a pair of holomorphic functions, similar to the Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ [By].

The Bryant representation can be stated in general form for CMC-c surfaces in $\mathbb{H}^n(-c^2)$ for any real value of c. (One can then produce CMC-1 surfaces in $\mathbb{H}^n(-1)$ by transforming $\mathbb{H}^n(-c^2)$ into $\mathbb{H}^n(-1)$ by homothety.) For simplicity, we will state the Bryant representation only for the case $c = 1$:

Theorem 10.1. *Let Σ be a simply connected and connected Riemann surface with a reference point $z_0 \in \Sigma$, and let α be an $s\ell(2,\mathbb{C})$-valued holomorphic 1-form on Σ. Suppose that α satisfies the following two conditions:*

$$\det \alpha = 0,$$
$$\operatorname{trace}\{\tilde{\alpha}\cdot\alpha^*\} < 0,$$

where $\tilde{\alpha}$ is the cofactor matrix of α and $\alpha^* = \bar{\alpha}^t$. Then there exists a holomorphic immersion $F : \Sigma \to PSL(2,\mathbb{C})$ such that

1) $F(z_0) = id$,
2) $F^{-1}\cdot dF = \alpha$ (Bryant's equation),
3) $f = F \cdot F^* : \Sigma \to \mathbb{H}^3$ is a conformal CMC 1 immersion.

Conversely, every conformal CMC 1 immersion can be described in terms of some F, as above. Moreover, F is unique up to right multiplication by a constant in $SU(2) \subset SL(2,\mathbb{C})$.

Writing the resulting matrix F in the form

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

we define the hyperbolic Gauss map G by $G = \frac{dA}{dC} = \frac{dB}{dD}$. Geometrically, if one considers the CMC 1 surface in the Poincare model of $\mathbb{H}^3$ and at each point p of the surface considers a normal geodesic from p to the virtual boundary of the Poincare model, the endpoint of the geodesic in the virtual boundary stereographically projects to $G(p)$.

If we write the $SL(2,\mathbb{C})$-valued holomorphic 1-form α in the form

$$\alpha = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\omega,$$

then this g is the secondary Gauss map. The Hopf differential is $Q = \omega dg$. The first fundamental form of f is

$$ds^2 = (1+|g|^2)^2|\omega|^2;,$$

and the second fundamental form is

$$II = -Q - \overline{Q} + ds^2 .$$

The "Weierstrass data" in the Bryant representation of CMC 1 surfaces in $\mathbb{H}^3$ corresponds roughly to that of the corresponding minimal surfaces in $\mathbb{R}^3$, for example:
horosphere: $\Sigma = \mathbb{C}, g = 0, \omega = dz$ (like the data for a plane).
Enneper cousin: $\Sigma = \mathbb{C}, g = z, \omega = k dz, k \in \mathbb{R}$ (like the data for Enneper's surface).
catenoid cousin: $\Sigma = \mathbb{C}\setminus\{0\}, g = z, \omega = \frac{k}{z^2}, k \in \mathbb{R}$ (like the data for a catenoid).
trinoid cousin: $\Sigma = (\mathbb{C}\cup\{\infty\})\setminus\{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}, g = z^2, \omega = \frac{k}{(z^3-1)^2}, k \in \mathbb{R}$ (like the data for a Jorge-Meeks trinoid).

In order to describe the period problems for a CMC 1 surface, we first consider a lemma about transformations of the lift F of a CMC 1 surface $f = FF^*$.

Lemma 10.1. *When F is transformed into aFb^{-1} $(a, b \in SL(2,\mathbb{C}))$, we have*

$$g \to \frac{b_{11}g + b_{12}}{b_{21}g + b_{22}}, \; G \to \frac{a_{11}G + a_{12}}{a_{21}G + a_{22}}, \; \text{and } Q \to Q .$$

Proof. Let F be

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

When $F \to Fb^{-1}$, then G becomes

$$\frac{d(b_{22}A - b_{21}B)}{d(b_{22}C - b_{21}D)} = \frac{b_{22}G - b_{21}G\frac{dD}{dC}}{b_{22} - b_{21}\frac{dD}{dC}} = G.$$

So G is left unchanged. When $F \to aF$, G becomes

$$\frac{d(a_{11}A + a_{12}C)}{d(a_{21}A + a_{22}C)} = \frac{a_{11}G + a_{12}}{a_{21}G + a_{22}}.$$

When $F \to aF$, aF is also a solution to the same ordinary differential equation in the second item of Bryant's theorem, so g is left unchanged. When $F \to Fb^{-1}$, then Fb^{-1} is a solution to Bryant's equation, but with g replaced by $(b_{11}g + b_{12})/(b_{21}g + b_{22})$ and ω replaced by $(b_{21}g + b_{22})^2\omega$. Thus the transformation above for g is the correct transformation.

The transformation $F \to aF$ leaves g and ω unchanged, so it leaves Q unchanged. The transformation $F \to Fb^{-1}$ changes Q into $(b_{21}g + b_{22})^2\omega d((b_{11}g + b_{12})/(b_{21}g + b_{22})) = Q$, so it also leaves Q unchanged. □

Remark. As describe in section 8, the matrix a is an isometry of $\mathbb{H}^3$. Hence, given our geometric interpretation for the hyperbolic Gauss map, it is natural to expect that G will change as a changes. However, as a changes, the surface changes by only a congruence, so a essentially leaves the surface unchanged. □

Remark. When b changes, the surface is changed in a non-isometric way. In fact, one can easily see that the surface $f = FF^*$ is left unchanged if and only if $b \in SU(2)$. □

When f is well-defined on Σ, it is clear from the geometric description above that G is well-defined on Σ. However, if Σ is not simply-connected, then g is not necessarily well-defined on Σ. We now consider the $SU(2)$ condition that g must satisfy in order for the surface f to be well-defined on a Σ that is not simply connected.

Definition 10.1. *Let $\alpha(t), t \in [0,1], \alpha(0) = \alpha(1)$ be any loop in Σ. We say that the secondary Gauss map g, considered as a function $g(t)$ of t on the loop $\alpha(t)$, satisfies the* $SU(2)$-condition *on $\alpha(t)$ if $g(1) = (p \cdot g(0) - \bar{q})/(q \cdot g(0) + \bar{p})$ for some*

$$\begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix} \in SU(2) .$$

If g satisfies the $SU(2)$-condition *for all loops $\alpha(t)$, then we say that g satisfies the* $SU(2)$-condition *on Σ.*

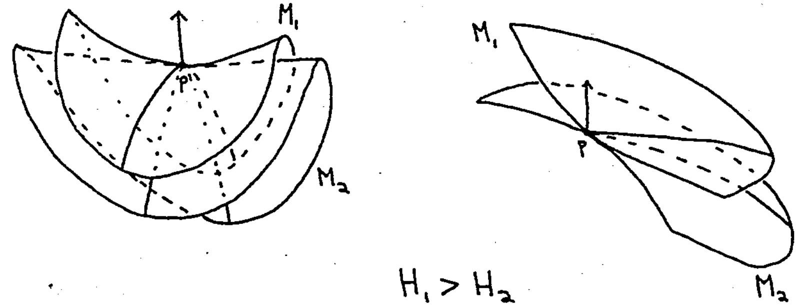

FIGURE 22. The Maximum Principle

It then follows easily from Lemma 10.1 and the second remark following it that we have the following lemma. This lemma below gives us a way to compute the "period problems".

Lemma 10.2. *The surface is well-defined as an immersion of M^2 if and only if g satisfies the $SU(2)$ condition on Σ.*

An interesting property of CMC 1 surfaces in $\mathbb{H}^3$ is that if we change the lift F of the surface $f = FF^*$ to F^{-1} to get the "dual surface" $f^\# = F^{-1}(F^{-1})^*$, the surface $f^\#$ has secondary Gauss map G and hyperbolic Gauss map g (i.e. the two Gauss maps have switched). This property is useful, and has been first exploited in [UY4].

11. The maximum principle for CMC surfaces

The maximum principle is a powerful tool for proving global results about CMC surfaces. For example, one can easily prove the following facts:

- *Every complete minimal surface in $\mathbb{R}^3$ or $\mathbb{H}^3$ is not compact.* To prove this, suppose that M is compact and minimal without boundary, then there exists a geodesic plane P that does not intersect M. Translating P toward M until it makes first contact with M, one has the contradiction that $M = P$, by the maximum principle.
- *Every compact minimal surface in $\mathbb{R}^3$ or $\mathbb{H}^3$ with boundary must lie in the interior of the convex hull of its bdry.* The proof is similar to that of the example just above.
- *The only embedded compact CMC surface in $\mathbb{R}^3$ or $\mathbb{H}^3$ is the sphere.* The proof uses the Alexandrov reflection principle, which is an immediate extension of the maximum principle (see, for example, [KKS], or see the proof of Theorem 12.1 in the next section). Note that the condition "embedded" is really necessary, as the CMC Wente surfaces show.

Many other results have been proven with the maximum principle, among them that any complete connected minimal surface in $\mathbb{R}^3$ with two ends is a catenoid, proven by Schoen [Scn]. In addition, Korevaar, Kusner, Meeks, and Solomon [M], [KKS], [KKMS] have proven that any complete nonminimal finite topology CMC surface with two ends is a Delaunay surface, and any surface of this type with three ends has a plane of reflective symmetry.

In the next section we shall give an application of the maximum principle to show that any complete CMC H graph in $\mathbb{H}^3$ satisfies $H < 1$.

Before stating the maximum principle, we define some terms. Let $M^n(\lambda)$ be an n-dimensional manifold that is conformal to $\mathbb{R}^n$ with conformal factor $\lambda : M^n(\lambda) \to \mathbb{R}^+$. In other words, $M^n(\lambda)$ can be locally described as $\{x = (x_1, ..., x_n) \mid x_i \in \mathbb{R}\}$ with metric $\lambda^2|dx|^2$, where $\lambda > 0$ and $|dx|^2$ is the standard Euclidean metric. We have a notion of a *graph* over any geodesic hyperplane in $M^n(\lambda)$ (see section 12).

If Σ_1 and Σ_2 are two smooth oriented complete hypersurfaces of $M^n(\lambda)$ which are tangent at a point p and have the same oriented normal at p, we say that p is a *point of common tangency* for Σ_1 and Σ_2. Let the common tangent geodesic hyperplane $\mathcal{P}$ through p have the same orientation as Σ_1 and Σ_2 at p. Then, in a neighborhood of p in $\mathcal{P}$, expressing Σ_1 and Σ_2 as graphs $\gamma_x(g_1(x))$ and $\gamma_x(g_2(x))$ over points $x \in \mathcal{P}$, we say that Σ_1 *lies above* Σ_2 near p if $g_1 \geq g_2$. (γ_x is geodesic perpendicular projection to $\mathcal{P}$, as defined in section 12.)

Proposition 11.1. *Suppose the following:*

1) *$M^n(\lambda)$ is either $\mathbb{R}^n$ or $\mathbb{H}^n$.*
2) *Σ_1 and Σ_2 are oriented hypersurfaces in $M^n(\lambda)$.*
3) *Σ_1 and Σ_2 have a point $p \in Int(\Sigma_1) \cap Int(\Sigma_2)$ of common tangency.*
4) *Σ_1 lies above Σ_2 near p.*
5) *There exists some constant H so that the mean curvature of Σ_2 is always at least H and the mean curvature of Σ_1 is always at most H.*

Then Σ_1 and Σ_2 coincide in a neighborhood of p (see Figure 22).

This result is well known [Al], but we include a proof here for the sake of completeness, and because a published proof in the case $M^n(\lambda) = \mathbb{H}^n$ is not so readily available. Some references for the maximum principle in the hyperbolic case are [KKMS], [DCL], and references therein. The proof of the hyperbolic case is essentially like the proof of the Euclidean case ([Scn], [ER]), the primary difference being that the elliptic operator we consider has a non-zero zero-order term in the hyperbolic case.

Proof. Without loss of generality, we may assume $p = \vec{0} \in \{x = (x_1, ..., x_n) \mid x_i \in \mathbb{R}\}$ and $\mathcal{P}$ is perpendicular to the x_n-axis at p. Since the hypersurface Σ_ℓ is locally a graph over $\mathcal{P}$ in $M^n(\lambda)$ near p with respect to the metric $\lambda^2|dx|^2$, we know that in a small neighborhood of p, Σ_ℓ can be represented in Euclidean coordinates as

$$(x_1, ..., x_{n-1}, f_\ell(x_1, ..., x_{n-1}))$$

with $f_\ell(\vec{0}) = 0$ and $Df_\ell(\vec{0}) = 0$, for $\ell = 1, 2$. We may choose orientations for $\mathcal{P}$, Σ_1, and Σ_2 so that all three of these hypersurfaces have an upward-pointing oriented normal vector at p. Then, by the third condition of the proposition, $f_1 \geq f_2$.

Note that with respect to the usual Euclidean metric, a hypersurface of the form $(x_1, ..., x_{n-1}, f(x_1, ..., x_{n-1}))$ has mean curvature

$$H = \frac{1}{(n-1)(1+(\nabla f)^2)^{\frac{3}{2}}} \sum_{i,j=1}^{n-1} f_{ij}(\delta_{ij}(1+(\nabla f)^2) - f_i f_j) .$$

For a graph $(x_1, ..., x_{n-1}, f(x_1, ..., x_{n-1}))$ in Euclidean n-space we define an orthonormal moving frame of vectors $e_1, ..., e_n$ so that $e_1, ...e_{n-1}$ are an oriented orthonormal frame of vectors for the tangent space of the hypersurface, and so that e_n is the unit normal vector to the hypersurface with the upward orientation. We define 1-forms ω^i and ω_i^j by

$$\omega^i(e_j) = \delta_{ij} \ , \quad \nabla e_i = \omega_i^j e_j .$$

Note that ω_i^j are scew symmetric, that is, $\omega_i^j = -\omega_j^i$. Note also that we have the structure equation $d\omega^i = \Sigma_{j=1}^n \omega^j \wedge \omega_j^i$. We then define the mean curvature as $H = \frac{1}{n-1}\Sigma_{i=1}^{n-1} h_{ii}$, where $h_{ij} = \langle \nabla_{e_i} e_j, e_n \rangle = \omega_j^n(e_i)$.

If we now consider the same hypersurface, but with the ambient metric $\lambda^2|dx|^2$, we can define an orthonormal moving frame in the same way as above. We denote the orthonormal vectors and 1-forms and mean curvature in this case by $\hat{e}_i$ and $\hat{\omega}^i$ and $\hat{\omega}_i^j$ and $\hat{h}_{ij}$ and $\hat{H}$. Noting that $\hat{e}_i = \frac{e_i}{\lambda}$ and $\hat{\omega}^i = \lambda\omega^i$, and using that $\hat{\omega}^i \wedge \omega^i = 0$, we see that

$$\hat{\omega}^j \wedge \hat{\omega}_j^i = d\hat{\omega}^i = d(\lambda\omega^i) = d\lambda \wedge \omega^i + \lambda d\omega^i = \lambda_j \omega^j \wedge \omega^i + \lambda\omega^j \wedge \omega_j^i =$$

$$\lambda\omega^j \wedge (\frac{\lambda_j}{\lambda}\omega^i + \omega_j^i) = \hat{\omega}^j \wedge \overbrace{(\frac{\lambda_j}{\lambda}\omega^i - \frac{\lambda_i}{\lambda}\omega^j + \omega_j^i)}^{\text{skew symmetric}} .$$

So we have $\hat{\omega}_j^i = \frac{\lambda_j}{\lambda}\omega^i - \frac{\lambda_i}{\lambda}\omega^j + \omega_j^i$. Thus

$$\hat{h}_{ij} = \hat{\omega}_j^n(\hat{e}_i) = (\frac{\lambda_j}{\lambda}\omega^n - \frac{\lambda_n}{\lambda}\omega^j + \omega_j^n)(\frac{e_i}{\lambda}) = \frac{h_{ij}}{\lambda} - \frac{\lambda_n}{\lambda^2}\delta_{ij} \Longrightarrow \hat{H} = \frac{H}{\lambda} - \frac{\lambda_n}{\lambda^2} .$$

Since $\hat{H}_2 - \hat{H}_1 = \frac{H_2}{\lambda(f_2)} - \frac{\lambda_n(f_2)}{\lambda^2(f_2)} - \frac{H_1}{\lambda(f_1)} + \frac{\lambda_n(f_1)}{\lambda^2(f_1)} \geq 0$, it follows that

$$w_{ij}\frac{\delta_{ij}(1+(\nabla f_2)^2) - (f_2)_i(f_2)_j}{\lambda(f_2)(n-1)(1+(\nabla f_2)^2)^{\frac{3}{2}}} + (f_1)_{ij}\left(\frac{\delta_{ij}(1+(\nabla f_2)^2) - (f_2)_i(f_2)_j}{\lambda(f_2)(n-1)(1+(\nabla f_2)^2)^{\frac{3}{2}}} - \frac{\delta_{ij}(1+(\nabla f_1)^2) - (f_1)_i(f_1)_j}{\lambda(f_1)(n-1)(1+(\nabla f_1)^2)^{\frac{3}{2}}}\right) - \frac{\lambda_n(f_2)}{\lambda^2(f_2)} + \frac{\lambda_n(f_1)}{\lambda^2(f_1)} \geq 0 ,$$

where $w = f_2 - f_1 \leq 0$ with first derivatives $w_j = (f_2)_j - (f_1)_j$ and second derivatives $w_{ij} = (f_2)_{ij} - (f_1)_{ij}$. Defining β_{ij} by

$$\beta_{ij}(x_1, ..., x_{n-1}, u, u_1 = \frac{\partial u}{\partial x_1}, ..., u_{n-1} = \frac{\partial u}{\partial x_{n-1}}) = \frac{\delta_{ij}(1+(\nabla u)^2) - u_i u_j}{\lambda(u)(n-1)(1+(\nabla u)^2)^{\frac{3}{2}}} ,$$

the intermediate value theorem tells us that

$$\beta_{ij}(x_1, ..., x_{n-1}, f_2, (f_2)_1, ..., (f_2)_{n-1}) - \beta_{ij}(x_1, ..., x_{n-1}, f_1, (f_1)_1, ..., (f_1)_{n-1}) =$$

$$(\frac{\partial}{\partial u}\beta_{ij})(x_1, ..., x_{n-1}, cf_2 + (1-c)f_1, (cf_2 + (1-c)f_1)_1, ..., (cf_2 + (1-c)f_1)_{n-1})(f_2 - f_1) +$$

$$\sum_{k=1}^{n-1}(\frac{\partial}{\partial u_k}\beta_{ij})(x_1,...,(cf_2+(1-c)f_1)_{n-1})(f_2-f_1)_k \ ,$$

for some $c = c(i,j) \in [0,1]$. It follows that

$$0 \le \hat{H}_2 - \hat{H}_1 = a_{ij}w_{ij} + (f_1)_{ij}(\tilde{b}_{ij}w + \sum_{k=1}^{n-1}\tilde{b}_{ijk}w_k) - \frac{\lambda_n(f_2)}{\lambda^2(f_2)} + \frac{\lambda_n(f_1)}{\lambda^2(f_1)} \ ,$$

where $\tilde{b}_{ij} = (\frac{\partial}{\partial u}\beta_{ij})(x_1,...,(cf_2+(1-c)f_1)_{n-1})$, and $\tilde{b}_{ijk} = (\frac{\partial}{\partial u_k}\beta_{ij})(x_1,...,(cf_2+(1-c)f_1)_{n-1})$. Note that $a_{ij}, \tilde{b}_{ij}, \tilde{b}_{ijk}$ are all bounded functions. Note also that $a_{ij} \approx \frac{\delta_{ij}}{(n-1)\lambda(f_2)}$ in a small neighborhood of $(x_1,\dots,x_{n-1}) = (0,\dots,0)$, and thus (a_{ij}) is a strictly positive definite $(n-1)\times(n-1)$ matrix in a small neighborhood of the origin.

Since $\lambda_n(u)$ is a function of the variables $x^1,...,x^{n-1},u,u_1,...,u_{n-1}$, we can apply the intermediate value theorem again just as above to conclude that

$$\frac{\lambda_n(f_2)}{\lambda^2(f_2)} - \frac{\lambda_n(f_1)}{\lambda^2(f_1)} = \sum_{k=1}^{n-1}\tilde{\tilde{b}}_k w_k + \tilde{\tilde{b}}w$$

for some bounded functions $\tilde{\tilde{b}}_k$ and $\tilde{\tilde{b}}$, and thus

$$0 \le a_{ij}w_{ij} + (f_1)_{ij}(\tilde{b}_{ij}w + \sum_{k=1}^{n-1}\tilde{b}_{ijk}w_k) - (\sum_{k=1}^{n-1}\tilde{\tilde{b}}_k w_k + \tilde{\tilde{b}}w) \ .$$

Consolidating the zero-order and first-order terms, we have a linear second-order uniformly elliptic operator with bounded coefficients:

$$0 \le a_{ij}w_{ij} + b_j w_j + cw \ .$$

But before we can apply the maximum principle, we need to check that $c \le 0$. Since $M^n(\lambda)$ is either $\mathbb{R}^n$ or $\mathbb{H}^n$, we know that $\lambda \equiv 1$ or $\lambda = 2/(1-|x|^2)$. If $\lambda \equiv 1$, it is clear that $c \equiv 0 \le 0$, so we may assume that $\lambda = 2/(1-|x|^2)$. Note that

$$c = (f_1)_{ij}\tilde{b}_{ij} - \tilde{\tilde{b}} \approx (f_1)_{ij}\frac{\delta_{ij}}{n-1}\frac{-\lambda_u}{\lambda^2}|_{u\approx 0} - \frac{\lambda_{\vec{N},u}\lambda^2 - \lambda_{\vec{N}}2\lambda\lambda_u}{(\lambda(u))^4}|_{u\approx 0}.$$

Note that $\lambda_{\vec{N}} \approx k\lambda_u$ for some positive const k. Hence

$$c \approx -\frac{\sum_i (f_1)_{ii}}{n-1}\frac{\lambda_u}{\lambda^2}|_{u=0} - \frac{k\lambda_{uu}}{\lambda^2}|_{u=0} + \frac{2k\lambda_u^2}{\lambda^3}|_{u=0}.$$

We find that $\lambda \approx 2$, $\lambda_u \approx 0$, $\lambda_{uu} \approx \lambda_{x_n x_n}|_{x_k=0} \approx 4$, and $c \approx -k < 0$. So $c \le 0$.

Since $w \le 0$ in a small open neighborhood of $(x_1,\dots,x_{n-1}) = (0,\dots,0)$ and has a local maximum $w=0$ at $(0,\dots,0)$, it follows from the maximum principle (for example, [PW], pages 61-64, Theorems 5 and 6) that w is identically 0 in a neighborhood of the origin. We conclude that $f_1 = f_2$ near p, and thus Σ_1 and Σ_2 coincide in a neighborhood of p. □

12. CMC Graphs in Hyperbolic Space

Here we consider some questions regarding existence and uniqueness for hypersurfaces that are graphs in hyperbolic n-space – in relation to known results for Euclidean n-space. The result we prove is analogous to the Bernstein theorem for minimal hypersurfaces in Euclidean n-space. We show that any complete hypersurface in hyperbolic n-space with mean curvature at least 1 is not a graph. The hyperspheres show that the condition of mean curvature at least 1 is sharp. The fact that 1 is the critical value in the theorem is natural, since CMC 1 hypersurfaces in hyperbolic n-space have the Lawson correspondence to minimal hypersurfaces in Euclidean n-space. This correspondence justifies refering to this result as a "Bernstein"-type theorem.

We will also note that oriented compact hypersurfaces with boundary of constant mean curvature H which are graphs over a geodesic hyperplane are not necessarily unique with respect to their boundaries – unlike the Euclidean case, where uniqueness does hold.

Bernstein showed that in Euclidean 3-space $\mathbb{R}^3$, the only complete minimal graphs are planes. The same result continues to hold for minimal hypersurfaces in $\mathbb{R}^n$, for $n \le 7$, and does not hold when $n \ge 8$ ([DCL] and references therein).

Consider a totally geodesic hyperplane $\mathbb{H}^{n-1}$ in $\mathbb{H}^n$. The exponential map on the normal bundle gives a global product structure

$$\tau : \mathbb{H}^{n-1} \times \mathbb{R} \to \mathbb{H}^n \ ,$$

where $\tau(x,t) = \gamma_x(t)$, and $\gamma_x(t)$ is the arc-length geodesic emanating from $x = \gamma_x(0) \in \mathbb{H}^{n-1}$ in the oriented normal direction. Then the hypersurfaces $\tau(\mathbb{H}^{n-1} \times \{t\})$ are hyperspheres. This product gives a natural orthogonal projection π from $\mathbb{H}^n$ to the totally geodesic hyperplane $\mathbb{H}^{n-1}$ defined by $\pi(\tau(x,t)) = x$, and thus we have a well-defined notion of a *graph*: We say that a *graph* is a hypersurface which admits a one-to-one orthogonal projection to a totally geodesic hyperplane. If the projection is also surjective, we say the hypersurface is an *entire* graph. Lawson and Do Carmo [DCL] showed that if Σ is a complete CMC H entire graph, then Σ is a hypersphere (a hypersurface of constant distance from a geodesic hyperplane, and thus $|H| < 1$). In regard to complete minimal graphs in $\mathbb{H}^n$, Anderson [An] showed that if we do not assume the graphs to be entire, there are infinitely many different examples of complete minimal graphs.

While Anderson did not require the graphs to be entire, he only considered minimal graphs. And while Do Carmo and Lawson considered graphs of any constant mean curvature, they only considered entire graphs. In the next result we do not assume the graph to be entire, and in regard to curvature, we only assume that the graph has mean curvature (not necessarily constant) at least 1.

Theorem 12.1. *Any complete hypersurface in $\mathbb{H}^n$ with mean curvature at least 1 cannot be a graph.*

In particular, any complete CMC 1 hypersurface in $\mathbb{H}^n$ cannot be a graph, demonstrating a difference from the closely related case of minimal graphs in $\mathbb{R}^n$, as there do exist complete minimal graphs in $\mathbb{R}^n$ (the hyperplanes, for example).

Proof. Assume there exists a complete hypersurface Σ with mean curvature at least 1 that is a graph over a totally geodesic hyperplane in the Poincare model of $\mathbb{H}^n$. By

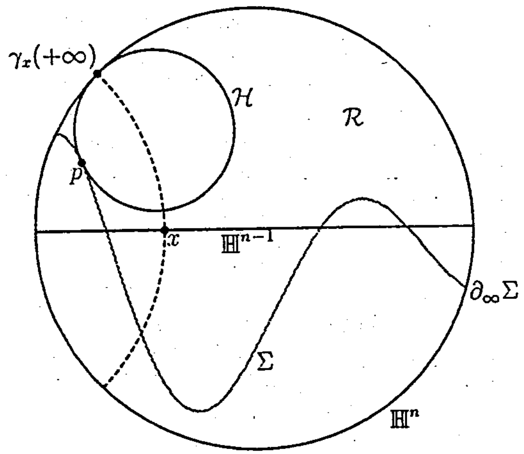

FIGURE 23. The argument in the proof of the Theorem 12.1.

applying an isometry of $\mathbb{H}^n$ if necessary, we may assume the totally geodesic hyperplane is $\{x_n = 0\} \cap B^n$.

Since Σ is a graph, it is embedded. And since Σ is complete and embedded, it separates $\mathbb{H}^n$ into two components. The mean curvature vector on Σ is never zero, so it always points into only one of the two components, and we call this component $\mathcal{R}$.

The projection set $\pi(\Sigma) \subset \{x_n = 0\} \cap B^n$ is open, so there exists an $x \in \{x_n = 0\} \cap B^n$ and an open neighborhood $U \subset \{x_n = 0\} \cap B^n$ of x such that $U \subset \pi(\Sigma)$. Thus for each $y \in U$, the arc-length geodesic $\gamma_y(t)$ through y normal to $\{x_n = 0\} \cap B^n$ intersects Σ at a finite point. It follows that the points $\gamma_y(+\infty)$ and $\gamma_y(-\infty)$ are not contained in $\partial_\infty\Sigma$ (where $\partial_\infty\Sigma$ denotes the asymptotic boundary of Σ in ∂B^n), for all $y \in U$. Reorienting the geodesics if necessary, we may assume that $\{\gamma_y(+\infty) \mid y \in U\}$ is an open set in $\partial\mathcal{R}$. Since $\{\gamma_y(+\infty) \mid y \in U\} \cap \partial_\infty\Sigma$ is empty, the Euclidean distance from $\gamma_x(+\infty)$ to Σ is positive. So there is some horosphere tangent to ∂B^n at $\gamma_x(+\infty)$ with small Euclidean radius that is disjoint from Σ.

By continuously increasing the Euclidean radius of horospheres tangent to ∂B^n at $\gamma_x(+\infty)$, we arrive at the first horosphere $\mathcal{H}$ to have nonempty intersection with Σ. Thus $\mathcal{H}$ and Σ have some finite point p as a common tangency, and $\mathcal{H}$ lies above Σ near p. (See Figure 23.) Since the mean curvature of Σ is at least 1 and the mean curvature of $\mathcal{H}$ is exactly 1, we conclude by the maximum principle that $\Sigma = \mathcal{H}$. But this is a contradiction, since a horosphere is not a graph over a totally geodesic plane. □

Remark. One might consider the same question for graphs over a horosphere. That is, one might ask if any complete CMC 1 hypersurfaces have one-to-one orthogonal projections to a horosphere. This is a natural question, since under the Lawson correspondence,

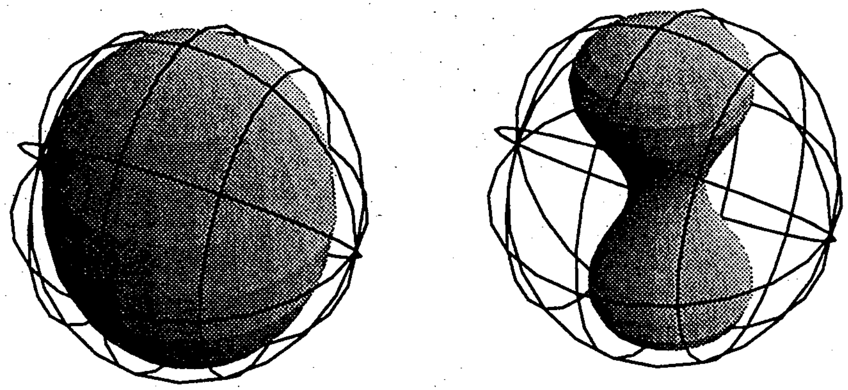

FIGURE 24. Two different catenoid cousins. Note that the left-hand surface is a graph over a horosphere tangent to ∂B^n at $(0,0,1)$.

horospheres in $\mathbb{H}^n$ correspond to hyperplanes in $\mathbb{R}^n$. But in this case one cannot hope for nonexistence or even uniqueness, as the horosphere is certainly a graph over itself, and the catenoid cousins ([By] example 2, [UY1]) are graphs over a horosphere in $\mathbb{H}^3$ when μ (as defined in [By]) is close enough to $-\frac{1}{2}$. (See Figure 24.) □

Remark. (on nonuniqueness of CMC graphs) The maximum principle can often be applied in hyperbolic space the same way that it is applied in Euclidean space to prove corresponding results. For example, suppose that Σ is a compact embedded CMC hypersurface in $\mathbb{R}^n$ with boundary an $(n-2)$-dimensional round sphere in the hyperplane $\{x_n = 0\}$. Suppose also that $\Sigma \subseteq \{x_n \geq 0\}$. The Alexandrov reflection method immediately proves that Σ inherits the planar reflectional symmetries of its boundary. Thus Σ is a compact CMC surface of revolution, and hence Σ is a spherical cap [BEMR], [BE]. One can easily see that a similar argument is valid in the hyperbolic case. Thus, if Σ is a compact embedded CMC hypersurface in $\mathbb{H}^n$ with boundary an $(n-2)$-dimensional geodesic sphere in a geodesic hyperplane $\mathcal{P}$, and if Σ is contained in a half-space bounded by $\mathcal{P}$, then Σ is a portion of a geodesic sphere.

However, there are cases where arguments using the maximum principle cannot be transferred to analogous arguments in the hyperbolic case. For example, suppose that $\Sigma \subseteq \mathbb{R}^n$ is a compact CMC H graph over a region in the hyperplane $\{x_n = 0\}$, where H is determined with respect to the upward pointing normal. Suppose that $\hat{\Sigma} \subseteq \mathbb{R}^n$ is another compact CMC H graph over the hyperplane $\{x_n = 0\}$, with the same value H for constant mean curvature (again determined with respect to the upward pointing normal). Suppose also that the boundaries of these two surfaces coincide, i.e. $\partial\Sigma = \partial\hat{\Sigma}$.

Then it easily follows from the maximum principle that $\Sigma = \hat{\Sigma}$ [Al]. In this sense we can say that a compact CMC H graph in $\mathbb{R}^n$ is unique with respect to its boundary.

In the hyperbolic case, however, a compact CMC H graph in $\mathbb{H}^n$ is not necessarily unique with respect to its boundary. As an example, consider the hypersurfaces

$$\Sigma = \{x_1^2+\ldots+x_{n-1}^2+(x_n-\epsilon)^2 = \frac{1}{4}\}\cap\{x_n \geq 0\} \text{ and } \hat{\Sigma} = \{x_1^2+\ldots+x_{n-1}^2+(x_n+\epsilon)^2 = \frac{1}{4}\}\cap\{x_n \geq 0\},$$

where $\epsilon \in (0, \frac{1}{2})$. Note that $\partial\Sigma = \partial\hat{\Sigma}$ and $\Sigma \neq \hat{\Sigma}$, and that Σ and $\hat{\Sigma}$ both have the same constant mean curvature with respect to the upward orientation. Let $\mathcal{P} = \{x_1^2 + \ldots + x_{n-1}^2 + (x_n+p)^2 = p^2 - 1\} \cap B^n$, with $p > 1$, be the geodesic hyperplane intersecting the x_n-axis perpendicularly at the point $(0, \ldots, 0, \sqrt{p^2-1} - p)$. If ϵ is close to 0 and p is close to 1, then Σ and $\hat{\Sigma}$ are both graphs over $\mathcal{P}$. This shows that uniqueness does not hold in the hyperbolic case. □

13. The Morse index of CMC surfaces

For considerations of index, we will restrict ourselves to the case of minimal surfaces in $\mathbb{R}^3$ and CMC 1 surfaces in $\mathbb{H}^3$, in a moment.

Let $\Phi : M \to M^3(a)$ be an isometric immersion of a 2-dimensional manifold M into a complete simply-connected 3-dimensional manifold $M^3(a)$ with constant sectional curvature a. Let $\vec{N}$ be a unit normal vector field on $\Phi(M)$ (we write $\Phi^*\vec{N}$ simply as $\vec{N}$ defined on M).

Definition 13.1. *A* smooth variation *of Φ is a C^∞ mapping $F : I \times M \to M^3(a)$, $I = (-\epsilon, \epsilon)$ such that*
$\Phi_t = F(t, \cdot) : M \to M^3(a)$ is an immersion,
$\Phi_0 = \Phi$,
$\Phi_t|_{\partial M} = \Phi|_{\partial M}$ for all $t \in I$.

Let $\Phi(t)$ be a smooth variation of Φ, and assume that the variation has compact support. We can assume that the corresponding variation vector field at time $t = 0$ is $u\vec{N}$, $u \in C_0^\infty(M)$. Let $A(t)$ be the area of $\Phi_t(M)$ and H be the mean curvature of $\Phi(M)$. The first variational formula ([L]) is

$$\left.\frac{dA}{dt}\right|_{t=0} = -\int_M \langle nH\vec{N}, u\vec{N}\rangle dA \ ,$$

where $\langle , \rangle$ and dA are the metric and area form on M induced by the immersion Φ. If H is constant, then $A'(0) = -nH\int_M u dA$. Let $V(t)$ be the volume of $\Phi_t(M)$, then $V'(0) = \int_M u dA$. A variation is said to be *volume preserving* if $\int_M u dA = 0$. It follows that $\Phi(M)$ is critical for area amongst all volume preserving variations.

The second variation formula for volume preserving variations ([Che], [Si], [L]) is

$$\left.\frac{d^2A}{dt^2}\right|_{t=0} = \int_M \{|\nabla u|^2 - 2(2a + 2H^2 - K)u^2\} dA \ ,$$

where K is the Gaussian curvature on M. If we are investigating CMC 1 surfaces in $\mathbb{H}^3$, we have $a = -1$ and $H = 1$, so

$$\left.\frac{d^2A}{dt^2}\right|_{t=0} = \int_M \{|\nabla u|^2 + 2Ku^2\} dA \ .$$

This formula is the same for both minimal surfaces in $\mathbb{R}^3 := M^3(0)$ and CMC 1 surfaces in $\mathbb{H}^3 := M^3(-1)$, as one might expect because of the Lawson correspondence.

Definition 13.2. *The index Ind(M) is the maximum possible dimension of a subspace of volume preserving variation functions in $C_0^\infty(M)$ on which $d^2A/dt^2|_{t=0} < 0$.*

Among the known results about the index of complete minimal surfaces in $\mathbb{R}^3$ are the following: A minimal surface has finite index if and only if it has finite total curvature [FC]. The only stable such surface is the plane [CP]. The only such surfaces with index 1 are the catenoid and Enneper's surface [FC]. The index of the Costa surface is 5 [N3], [N4]. The Jorge-Meeks n-noid has index $2n - 3$ [N2]. General lower bounds [Cho] and upper bounds [Ty] have been found for the index of complete minimal surfaces. (Also, although this result is not for minimal surfaces in $\mathbb{R}^3$, it is shown in [BC2] that the only stable complete CMC nonminimal surface in $\mathbb{R}^3$ is the sphere.)

Among the known results about the index of complete CMC 1 surfaces in $\mathbb{H}^3$ are the following: The only stable such surface is the horosphere [Si]. The index of such a surface is finite if and only if it has finite total curvature [CS]. Recently, the author and Levi Lima [LiR] have computed the index of the catenoid cousins and Enneper cousins and other examples.

References

[AA] Aiyama, Akutagawa. Kenmotsu-Bryant type representation formulas for constant mean curvature surfaces in $\mathbb{H}^3(-c^2)$ and $S_1^3(c^2)$, preprint.

[Al] A. D. Alexandrov, Uniqueness theorems for surfaces in the large I, *Vestnik Leningrad Univ. Math.* 11 (1956) 5–17.

[An] M. Anderson, Complete minimal varieties in hyperbolic space, *Invent. Math.* 69 (1982) 477–494.

[BC] J. L. Barbosa, M. do Carmo. Stability of minimal surfaces and eigenvalues of the Laplacian. *Math. Z.* 173 (1980), 13–28.

[BC2] J. L. Barbosa, M. do Carmo. Stability of hypersurfaces with constant mean curvature. *Math. Z.* 185 (1984), 339–353.

[BCE] J. L. Barbosa, M. do Carmo, Eschenburg. Stability of hypersurfaces of CMC in Riemannian manifolds. *Math. Z.* 197, p123–138 (1988).

[BCol] Barbosa- Colares. Minimal surfaces in $\mathbb{R}^3$, Publication of I.M.P.A. (Instituto de Matematica Pura y Aplicada), Monografias de Matematica no. 40.

[Be] P. Berard. Spectral Geometry: Direct and Inverse Problems. Lecture Notes in Math, 1207, Springer-Verlag (1986).

[BCS] P. Berard, M. P. do Carmo, W. Santos. The Index of Constant Mean Curvature Surfaces in Hyperbolic 3-Spaces, preprint.

[BR] J. Berglund, W. Rossman. Minimal Surfaces with Catenoid Ends. *Pacific Journ. Math.* 171(2) (1995), 353–371.

[BE] F. Brito and R. Earp, Geometric configurations of constant mean curvature surfaces with planar boundary, *An. Acad. Brasil Cienc.* 63 (1991), no. 1, 5–19.

)EMR] F. Brito, R. Earp, W. Meeks, and H. Rosenberg, Structure theorems for constant mean curvature surfaces bounded by a planar curve, *Indiana Univ. Math. J.* 40 (1991), no. 1, 333–343.

)y] R. Bryant. Surfaces of Mean Curvature One in Hyperbolic Space. *Astérisque*, 154–155 (1987), p321–347.

)y2] R. Bryant. A duality theorem for Willmore surfaces. *J. of Diff. Geom.* 20, p23–53.

)S] M. P. do Carmo, A. M. Silveira. Index and total curvature of surfaces with constant mean curvature, *Proc. Amer. Math. Soc.* 110, #4, (1990), 1009–1015.

)P] M. P. do Carmo, Peng. Stable complete minimal surfaces in $\mathbb{R}^3$ are planes. *Bull. Am. Math. Soc.*, (NS) 1 (1979), 903–906.

)hGa] C. C. Chen, F. Gackstatter. Elliptische und hyperelliptische Funktionen und vollständige Minimalflächen vom Enneperschen Typ. *Math. Ann.*, 259, p359–369, 1982.

)he] S. S. Chern. Minimal submanifolds in a Riemannian manifold, Dept. of Math. Technical Report 19, Lawrence, Kansas: Univ. of Kansas 1968.

)ho] J. Choe. Index, vision number, and stability of complete minimal surfaces, *Arch. Rat. Mech. Anal.* 109 (1990), 195–212.

)ho2] J. Choe. Enneper hypersurfaces in $\mathbb{R}^{n+1}$, preprint.

)CL] M. P. Do Carmo and H. B. Lawson, On Alexandrov-Bernstein theorems in hyperbolic space. *Duke Math. Journal* Vol. 50 No. 4, (1983) 995–1003.

)R] R. Sa Earp and H. Rosenberg, Some remarks on surfaces of prescribed mean curvature, *Differential Geometry*, Pitman Monographs and Serverys in Pure and Applied Mathematics, 52 (1991) 123–148.

)R2] Earp, Rosenberg. Some structure thms for complete constant mean curvature surfaces with boundary a convex curve, *Proc. of Amer. Math. Soc.*, vol 113, no 4 (1991).

E] J. Eells. The surfaces of Delaunay. *Math. Intelligencer*, 9(1) (1987), 53–57.

EK] N. Ejiri, M. Kotani. Index and flat ends of minimal surfaces, *Tokyo J. of Math.* 16 (1993), 37–48.

FC] D. Fischer-Colbrie. On complete minimal surfaces with finite Morse index in three manifolds, *Invent, math.* 82, 121–132 (1985).

GBP] K. Gross-Braukman, Polthier. Compact constant mean curvature surfaces with low genus, preprint.

G] R. Gulliver. Index and total curvature of complete minimal surfaces, *Proc. Symp. Pure Math.* 44 (1986), 207–211.

GT] D. Gilbarg, N. S. Trudinger. Elliptic Partial Differential Equations of Second Order, Second Edition. Grundlehren der mathematischen Wissenschaften 224, A Series of Comprehensive Studies in Mathematics, Springer-Verlag (1983).

HK] Hoffman, Karcher. Complete embedded minimal surfaces of finite total curvature, survey article (1996).

HKR] D. Hoffman, H. Karcher, and H. Rosenberg. Embedded Minimal Annuli in $\mathbb{R}^3$ Bounded by a Pair of Straight Lines. *Commentarii Mathematici Helvetici*, vol. 66, 599–617, 1991.

HoMe] D. Hoffman and W. H. Meeks. Minimal surfaces based on the catenoid, *Amer. Math. Monthly*, Special Geometry Issue 97(8) (1990), 702–730.

HoMe2] D. Hoffman and W. H. Meeks. The strong halfspace thm for minimal surfaces, *Invent. Math.*, 101, 373–377 (1990).

HoMe3] D. Hoffman and W. H. Meeks. Embedded minimal surfaces of finite topology, *Ann. of Math.*, 131 (1990), 1–34.

HKW] Hoffman, Karcher, Wei. The singly-periodic genus-1 helicoid, MSRI preprint (1996).

JeSe] H. Jenkins and J. Serrin. Variational Problems of Miminal Surface Type II. Boundary Value Problems for the Minimal Surface Equation. *Arch. Rational Mech. Anal.*, vol. 21, 321–342.

[JoMe] L. P. M. Jorge and W. H. Meeks III. The topology of complete minimal surfaces of finite total Gaussian curvature. *Topology*, 22(2):203–221, 1983.

[Kap] Kapouleas. Complete constant mean curvature surfaces in Euclidean 3-space, *Ann. of Math.* 131 (1990), p239–330.

[Ka1] H. Karcher. Embedded Minimal Surfaces Derived From Scherk's Surfaces. *Manuscripta math.* 62, 83–114 (1988).

[Ka2] H. Karcher. The Triply Periodic Minimal Surfaces of Alan Schoen and their Constant Mean Curvature Companions. *Manuscripta math.* 64, 291–357 (1989).

[Ka3] H. Karcher. Construction of Minimal Surfaces. *Surveys in Geometry*, 1–96, University of Tokyo, 1989 (Also: Lecture Notes No. 12, SFB256, Bonn, 1989).

[Ka4] H. Karcher. Construction of Higher Genus Embedded Minimal Surfaces. preprint.

[KP] Karcher, Polthier. Construction of triply periodic minimal surfaces, preprint (1996).

[Kat] S. Kato. Construction of n-end catenoids with prescribed flux, preprint.

[KUY] Kato, Umehara, Yamada. An inverse problem of the flux formula, preprint (1995).

[KUY2] Kato, Umehara, Yamada. General existence of minimal surfaces with prescribed flux, preprint (1996).

[Ke] Kenmotsu. A Weierstrass representation for constant mean curvature surfaces in $\mathbb{R}^3$, *Math. Ann.* (1979).

[Koi] M. Koiso. Symmetry of hypersurfaces of constant mean curvature with symnmnetric boundary, *Math. Z.* 191 (1986), 567–574.

[KKS] N. Korevaar, R. Kusner, and B. Solomon. The structure of complete embedded surfaces with constant mean curvature, J. Differential Geom. 30 (1989), 465–503.

[KKMS] Korevaar, Kusner, Meeks, Solomon. Constant mean curvature surfaces in hyperbolic space, *Amer. J. of Math.* 114, #1 (1992), p1-43.

[Kus] Kusner. Comparison surfaces for the Willmore problem, *Pacific J. of Math.* 138, p317–345.

[KS] Kusner, Schmitt. The spinor representation of minimal surfaces, preprint (1995).

[LaR] R. Langevin, H. Rosenberg. A maximum principle at infinity for minimal surfaces and applications, *Duke Math. J.*, vol 57, no 3 (1988).

[L] H. B. Lawson. Lectures on Minimal Surfaces, Publish or Perish Press, Berkeley, CA 1980.

[LeR] G. Levitt, H. Rosenberg. Symmetry of constant mean curvature hypersurfaces in hyperbolic space, *Duke Math. J.*, vol 52, no 1 (1985).

[LiR] L. Lima, W. Rossman. On the index of constant mean curvature 1 surfaces in hyperbolic space. preprint.

[Lo] F. Lopez. The Classification of Complete Minimal Surfaces with Total Curvature Greater than -12π, *Trans. Amer. Math. Soc.*, 334(1) (1992), 49–74.

[M] W. Meeks. The topology and geometry of embedded surfaces of constant mean curvature, *J. Differential Geom.* 27 (1988), 539–552.

[MW] W. Meeks, White. Minimal surfaces bounded by convex curves in parallel planes, *Comment. Math. Helvetici* 66, (1991), 263–278.

[MY1] W. Meeks and S-T Yau. The existence of embedded minimal surfaces and the problem of uniqueness, Math. Z. 179 (1982), 151–168.

[MY2] W. H. Meeks and S.-T. Yau. The Classical Plateau Problem and the Topology of Three-Dimensional Manifolds. *Topology* Vol. 21, No. 4, 409–442, 1982.

[MESH] J. Hoffman. Software for Constructing Minimal Surfaces Using Weierstrass Data. G.A.N.G. Lab, University of Massachusetss, Amherst, MA 01003.

[MR] Montiel, Ros. Schrödinger operators associated to a holomorphic map, Global Differential Geometry and Global Analysis, Lect. Notes in Math. # 1481, Berlin 1991, 147–174.

[Nad] Nadarashvili, Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces. *Invent. Math.*, 126(3) (1996), 457–466.

[N1] S. Nayatani. On the morse index of complete minimal surfaces in Euclidean space, *Osaka J. Math.* 27 (1990), 441–451.

[N2] S. Nayatani. Lower bounds for the morse index of complete minimal surfaces in Euclidean 3-space, *Osaka J. Math.* 27 (1990), 453–464.

[N3] S. Nayatani. Morse index of complete minimal surfaces, Collection: The Problem of Plateau 181–189, World Sci. Publishing, River Edge, NJ (1992).

[N4] S. Nayatani. Morse index and Gauss maps of complete minimal surfaces in Euclidean 3-space, *Comment. Math. Helv.* 68 (1993), 511–537.

[Ni1] J. C. C. Nitsche. Uber ein verallgemeinertes Dirichletsches Problem fur die Minimalflachengleichung und hebbare Unstetigkeiten ihrer Losungen. *Math. Ann.*, 158, 302–214 (1965).

[Ni2] J. C. C. Nitsche. *Lectures on Minimal Surfaces*, Vol. 1. Cambridge University Press, New York, NY, 1989.

[O] R. Osserman. A Survey of Minimal Surfaces, Dover Publications, New York 1969.

[Po] Polthier. Geometric a priori extimates for hyperbolic minimal surfaces, dissertation (1993).

[P] Perez. Balancing formulae for minimal surfaces in flat 3-manifolds, preprint (1994).

[PR] Perez, Ros. Some uniqueness and nonexistence theorems for embedded minimal surfaces. *Math. Ann.* 295 (1993), p513–525.

[PW] M. Protter, H. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag New York, 1984.

[R] W. Rossman. Minimal Surfaces in $\mathbb{R}^3$ with Dihedral Symmetry Group. *Tohoku Math. J.*, 47 (1995), 31–54.

[RoSa] W. Rossman, K. Sato. Surfaces of constant mean curvature 1 with two ends in hyperbolic space. To appear in *J. Exp. Math.*

[RUY] W. Rossman, M. Umehara, K. Yamada. Irreducible constant mean curvature 1 surfaces in hyperbolic space with positive genus. To appear in *Tohoku Math. J.*.

[Sa] K. Sato. Enneper surfaces of arbitrarily high genus, *Tohoku J. Math.*, 48, 229–246 (1996).

[Scn] R. Schoen. Uniqueness, Symmetry, and Embeddedness of Minimal Surfaces, *Journal of Differential Geometry*, vol. 18, 791–809, 1983.

[Scn2] R. Schoen. Estimates for stable minimal surfaces in three dimensional manifolds, Ann. of Math. Studies, 103, Princeton Univ. Press, 1983.

[Scz] H. A. Schwarz. *Gesammelte mathematische Abhandlungen.* 2 vols, Springer, Berlin, 1890.

[Si] A. M. Silveira. Stability of complete noncompact surfaces with constant mean curvature, *Math. Ann.* 277 (1987), 629–638.

[Sm] A. J. Small. Surfaces of Constant Mean Curvature 1 in $\mathbb{H}^3$ and Algebraic Curves on a Quadric. to appear in *Proc. of A.M.S.*.

[Ty] Tysk. Eigenvalue estimates with applications to minimal surfaces, *Pacific Math. J.* 128 (1987), 361–366.

[UY1] M. Umehara, K. Yamada. Complete Surfaces of Constant Mean Curvature One in the Hyperbolic 3-Space. *Ann. of Math.*, 137 (1993), p611–638.

[UY2] M. Umehara, K. Yamada. A Parametrization of the Weierstrass Formulae and Perturbation of Some Complete Minimal Surfaces of $\mathbb{R}^3$ into the Hyperbolic 3-Space. *J. Reine Angew, Math.*, 432 (1992), p93–116.

[UY3] M. Umehara, K. Yamada. Surfaces of Constant Mean Curvature c in $\mathbb{H}^3(-c^2)$ with Prescribed Hyperbolic Gauss Map. *Math. Ann.*, 304 (1996), 203–224.

[UY4] M. Umehara, K. Yamada. A Duality on CMC Surfaces in Hyperbolic Space, and a Hyperbolic Analogue of the Osserman Inequality. To appear in *Tsukuba J. Math.*.

[W] H. Wente. Counterexample to a conjecture of H. Hopf. *Pacific J. Math.*, 121 (1986), 193–243.

[Xu] Y. Xu. Symmetric Minimal Surfaces. *Pacific J. Math.*, 171(1) (1995), 275–296.

GRADUATE SCHOOL OF MATHEMATICS, KYUSHU UNIVERSITY, HAKOZAKI, FUKUOKA 812-81, JAPAN