Padé method to Painlevé equations

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Abstract. A class of special solutions of Painlevé/Garnier systems arising as the Bäcklund or Schlesinger transformations of the Riccati solutions is known. In the past several years, the corresponding τ-functions have been explicitly computed and expressed as certain specialization of the Schur functions with rectangle shape partitions. In this note, we will give a simple and direct derivation of these solutions. Our method is based on the Padé approximation and its intrinsic relation to iso-monodromy deformations.

Key Words and Phrases: Painlevé equation, Padé approximation, Schur function, Garnier system, iso-monodromy deformation.

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1 Introduction

Various special solutions of Painlevé/Garnier systems were known where the corresponding τ-functions are expressed as certain specialization of the Schur functions with rectangle shape partition. These results have been obtained from the Riccati solutions (see for example [2][8]), via computational method using the Bäcklund or Schlesinger transformations [10][11][12][13][14]. An interesting explanation of these determinant structures (not only for the special solutions but also for generic cases) has recently given in [4][5][6] from the point of view of the Toda equations.

In this note, we will present another derivation of these special solutions of the determinant form based on the Padé approximation. Since there are close connections between the Padé approximation, orthogonal polynomials, Toda equations, Painlevé equations, Bäcklund transformations, continued fractional expansion and so on, the relation between the Padé approximation and the Painlevé equations is naturally expected and, in fact, is well-known (see [9] and references therein). However, our derivation is based only on the simple counting of the exponents, and the fact that those special solutions can be obtained so directly from the Padé approximation were not recognised before. This method gives special solutions for the non-linear differential equations through linear algebraic equations, and bears some resemblance to Date's direct method for soliton equations [1].

This paper is organized as follows. In section 2, we give a Schur function expression of the Padé approximations. The relation between the Padé approximation and the Painlevé/Garnier systems are obtained in section 3. Interpretation of the result from the theory of orthogonal polynomials is discussed in section 4. Finally, section 5 is devoted to explicit examples.

2 Padé approximation

The Padé approximation of a meromorphic function $\psi(x)$ around $x = 0$ is defined by

$$\psi(x) - \frac{P_m(x)}{Q_n(x)} = O(x^{n+m+1}), \tag{1}$$

where $P_m(x)$ and $Q_n(x)$ are polynomials of order m and n respectively. Since zero or pole of $\psi(x)$ at $x = 0$ can be absorbed into a redefinition of P_m or Q_n, we can, and we will, assume that

$$\psi(x) = \sum_{n=0}^{\infty} p_n x^n, \quad p_0 = 1, \quad p_i = 0, \quad (i < 0) \tag{2}$$

near $x = 0$ without loss of generality.

Up to a common normalization factor, the polynomials P_m, Q_n are given by

$$P_m(x) = \sum_{i=0}^{m} s_{(m^n,i)} x^i, \quad Q_n(x) = \sum_{i=0}^{n} s_{((m+1)^i, m^{n-i})}(-x)^i, \tag{3}$$

where s_λ is the Schur function define by $s_{(\lambda_1,\ldots,\lambda_\ell)} = \det(p_{\lambda_i - i + j})_{i,j=1}^{\ell}$. The proof of the formulas (3) is given by the following computation:

$$\begin{aligned}
\psi Q_n &= \psi \begin{vmatrix} 1 & p_{m+1} & \cdots & p_{m+n} \\ x & p_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & p_{m+1} \\ x^n & p_{m-n+1} & \cdots & p_m \end{vmatrix} = \begin{vmatrix} p_m & p_{m+1} & \cdots & p_{m+n} \\ \vdots & \ddots & \ddots & \vdots \\ p_{m-n+1} & \cdots & p_m & p_{m+1} \\ x^n\psi & \cdots & x\psi & \psi \end{vmatrix} \\
&= \sum_{i=0}^{\infty} \begin{vmatrix} p_m & p_{m+1} & \cdots & p_{m+n} \\ \vdots & \ddots & \ddots & \vdots \\ p_{m-n+1} & \cdots & p_m & p_{m+1} \\ p_{i-n} & \cdots & p_{i-1} & p_i \end{vmatrix} x^i = \left(\sum_{i=0}^{m} + \sum_{i=m+n+1}^{\infty} \right) s_{(m^n,i)} x^i \\
&= P_m + O(x^{m+n+1}).
\end{aligned} \tag{4}$$

It would be interesting to note that the polynomials P_m and Q_n can be expressed in terms of single determinant as

$$P_m(x) = x^m s_{(m^{n+1})}\Big|_{p_i \to \sum_{j=0}^{i} x^{-j} p_{i-j}}, \quad Q_n(x) = (-x)^n s_{((m+1)^n)}\Big|_{p_i \to p_i - x^{-1} p_{i-1}}, \tag{5}$$

where the shift in p_i can be interpreted as the action of the free fermion vertex operators (see Appendix of [12] for example).

Though we could not find appropriate reference, the formulas (3) may be classically known. In fact, the deteminantial expression for the Padé interpolation can go back to the work by Jacobi[3].

We note that the relation between the Padé approximation and Painlevé/Garnier equations follows directly from eq.(1) without using the explicit forms (3).

3 From Padé to Painlevé

Let us specialize the function $\psi(x)$ as

$$\psi(x) = (1-x)^{\kappa_1} \prod_{i=1}^{N} (1 - \frac{x}{t_i})^{\theta_i}. \tag{6}$$

The coefficients p_n are given by the Appell-Lauricella function F_D as

$$\begin{aligned} p_n &= \frac{(-\kappa_1)_n}{(1)_n} F_D(-n, -\theta_1, \cdots, -\theta_N, \kappa_1 - n + 1; t_1^{-1}, \cdots, t_N^{-1}), \quad (n \geq 0) \\ F_D(\alpha, \beta_1, \cdots, \beta_N, \gamma; z_1, \cdots, z_N) &= \sum_{m_i \geq 0} \frac{(\alpha)_{|m|} (\beta_1)_{m_1} \cdots (\beta_N)_{m_N}}{(\gamma)_{|m|} (1)_{m_1} \cdots (1)_{m_N}} z_1^{m_1} \cdots z_N^{m_N}, \end{aligned} \tag{7}$$

where $|m| = m_1 + \cdots + m_N$, $(a)_n = a(a+1)\cdots(a+n-1)$.

Our main object is the second order Fuchsian differential equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0, \tag{8}$$

where its fundamental solutions are P_m and ψQ_n. Beside the obvious singularities at $x = 0, 1, t_1, \cdots, t_N$ and ∞, the differential equation (8) may have some apparent singularities, say $\lambda_1, \ldots, \lambda_\nu$. For generic parameters t_i, κ_1 and θ_i, one can assume that these singularities have exponents $(0, 2)$. Then the Riemann scheme is given as follows

$x = 0$	1	t_1	$\cdots$	t_N	λ_1	$\cdots$	λ_ν	∞
0	0	0	$\cdots$	0	0	$\cdots$	0	$-m$
$m+n+1$	κ_1	θ_1	$\cdots$	θ_N	2	$\cdots$	2	$-n-\kappa_\infty$

(9)

where $\kappa_\infty = \kappa_1 + \theta_1 + \cdots + \theta_N$. The number of apparent singularities are determined by the Fuchs relation : (sum of exponents)$= N + \nu + 1$, namely $\nu = N$. This is the key of the construction.

The differential equation (8) corresponding to the scheme (9) with $\nu = N$ takes the form

$$
\begin{aligned}
a_1(x) &= \frac{-(m+n)}{x} + \frac{1-\kappa_1}{x-1} + \sum_{i=1}^{N} \frac{1-\theta_i}{x-t_i} - \sum_{i=1}^{N} \frac{1}{x-\lambda_i}, \\
a_2(x) &= \frac{m(n+\kappa_\infty)}{x(x-1)} - \sum_{i=1}^{N} \frac{t_i(t_i-1)K_i}{x(x-1)(x-t_i)} + \sum_{i=1}^{N} \frac{\lambda_i(\lambda_i-1)\mu_i}{x(x-1)(x-\lambda_i)}.
\end{aligned} \tag{10}
$$

By definition, the iso-monodromy deformation of the Fuchsian equation (8) is the N-th Garnier system,

$$
\frac{\partial \lambda_i}{\partial t_j} = \frac{\partial K_j}{\partial \mu_i}, \quad \frac{\partial \mu_i}{\partial t_j} = -\frac{\partial K_j}{\partial \lambda_i}, \quad (i, j = 1, \ldots, N) \tag{11}
$$

where $K_j(\lambda, \mu, t)$ is the Hamiltonian determined by the non-logarithmic condition of the system (8) at $x = \lambda_i$ (see [2] for example). In the situation we are considering, the monodromy of the solutions $P_m(x), \psi(x)Q_n(x)$ does not depend on parameters t_i. Hence we have proved the following:

Theorem 3.1 *Let $P_m(x)$, $Q_n(x)$ be the Padé approximation of the function $\psi(x)$ in (6). Then, for the generic parameters κ_1, θ_i, t_i, the differential equation (8) with fundamental solutions $P_m(x)$ and $\psi(x)Q_n(x)$ has N apparent singularities, and the positions λ_i and residues μ_i of them give a special solution of the N-th Garnier systems.*

Remark. Though the result (and the proof) may be known among the specialists (see the next section), we hope the above description has at least instructive meaning.

4 Relation to orthogonal polynomials

The Padé approximation method to obtain the Painlevé equations can be viewed as an application of the well-known relation between the orthogonal polynomials and Painlevé type equations (see [9] and references therein).

Here we will recall the relation briefly. We think the following short proof has some meaning complementary to the explicit computational one in [9].

For simplicity, we consider the inner product given by

$$(\phi_1(x),\phi_2(x)) = \int_0^1 \phi_1(x)\phi_2(x)w(x)dx,$$
$$w(x) = x^{\kappa_0}\psi(x), \quad \psi(x) = (1-x)^{\kappa_1}\prod_{i=1}^{N}(1-\frac{x}{t_i})^{\theta_i}. \tag{12}$$

Let $f(x)$ be a polynomial of degree n and put

$$g(x) = \frac{1}{w(x)}\int_0^1 \frac{f(y)}{x-y}w(y)dy. \tag{13}$$

We assume that the first s terms in the expansion of $g(x)$ around $x=\infty$

$$g(x) = \frac{1}{w(x)}\sum_{i=0}^{\infty}\frac{1}{x^{i+1}}(y^i, f(y)) \tag{14}$$

vanishes. Consider the 2nd order differential equation whose fundamental solution is given by f and g. For generic parameters t_i, κ_0,κ_1 and θ_i, the Riemann scheme is given as follows

$x=0$	1	t_1	$\cdots$	t_N	λ_1	$\cdots$	λ_ν	∞
0	0	0	$\cdots$	0	0	$\cdots$	0	$-n$
$-\kappa_0$	$-\kappa_1$	$-\theta_1$	$\cdots$	$-\theta_N$	2	$\cdots$	2	$\kappa_0+\kappa_1+\sum_{i=1}^{N}\theta_i+s+1$

(15)

where $\lambda_1,\ldots,\lambda_\nu$ are apparent singularities. By the Fuchs relation, we have $\nu = N+n-s$. Namely, the maximal orthogonality condition $(x^i,f)=0$ $(i=0,1,\ldots,n-1)$ gives the minimal number of the apparent singularities. This gives the relation between the orthogonality and Painlevé/Garnier equations.

Let us consider the case where $\kappa_0 \in \mathbf{Z}_{<0}$ and

$$(\phi_1(x),\phi_2(x)) = \operatorname*{Res}_{x=0}\Big(\phi_1(x)\phi_2(x)w(x)dx\Big). \tag{16}$$

Putting $\kappa_0 = -m-n-1$, $f(x) = Q_n(x)$ and $g(x) = P_m(x)/\psi(x)$, the eq.(13) becomes

$$P_m(x) = x^{m+n+1}\operatorname*{Res}_{y=0}\Big(\frac{y^{-m-n-1}Q_n(y)}{x-y}\psi(y)dy\Big) = \Big[\psi(x)Q_n(x)\Big]_{\leq m+n}, \tag{17}$$

where $[f(x)]_{\leq k}$ means the degree $\leq k$ part of $f(x)$. This is just the Padé approximation condition (1), and hence, it can be viewed as a special version of orthogonality.

Recently, the connection between the Hankel determinant expression of the τ-function of the Toda equations and the auxiliary linear problems has been clarified [6], and the similar phenomena observed in Painlevé equations [4][5] were recovered from this. The above argument may give another explanation of these results from the point of view of iso-monodromy deformation.

5 Explicit examples

5.1 $P_{\rm VI}$ case

Let us put $\psi(x) = (1-x)^a(1-\frac{x}{t})^b$. The coefficients of the differential equation (8) take the form

$$\begin{aligned} a_1(x) &= \frac{-(m+n)}{x} + \frac{1-a}{x-1} + \frac{1-b}{x-t} - \frac{1}{x-\lambda}, \\ a_2(x) &= \frac{m(n+a+b)}{x(x-1)} - \frac{t(t-1)H}{x(x-1)(x-t)} + \frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}. \end{aligned} \tag{18}$$

where the Hamiltonian H is given by

$$\begin{aligned} t(t-1)H &= \lambda(\lambda-1)(\lambda-t)\mu^2 \\ &+\{(1-b)\lambda(\lambda-1) - (m+n+1)(\lambda-1)(\lambda-t) - a\lambda(\lambda-t)\}\mu \\ &+(a+m+n)m(\lambda-t). \end{aligned} \tag{19}$$

As a result, the function $\lambda = \lambda(t)$ is a solution of the sixth Painlevé equation

$$\begin{aligned} \frac{d^2\lambda}{dt^2} &= \frac{1}{2}\Big(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\Big)\Big(\frac{d\lambda}{dt}\Big)^2 - \Big\{\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\Big\}\frac{d\lambda}{dt} \\ &+ \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\Big\{\alpha - \beta\frac{t}{\lambda^2} + \gamma\frac{t-1}{(\lambda-1)^2} + (\frac{1}{2}-\delta)\frac{t(t-1)}{(\lambda-t)^2}\Big\}, \end{aligned} \tag{20}$$

where

$$\alpha = \frac{(a+b+n-m)^2}{2}, \quad \beta = \frac{(m+n+1)^2}{2}, \quad \gamma = \frac{a^2}{2}, \quad \delta = \frac{b^2}{2}. \tag{21}$$

More precisely, one can determine the solution $\lambda(t)$ in terms of the Schur function as

$$\lambda(t) = \frac{t(m+n+1)}{(m-n-a-b)}\frac{\tau_{m,n}\tau_{m+1,n+1}}{\tau_{m+1,n}\tau_{m,n+1}}, \quad \tau_{m,n} = s_{(m^n)}. \tag{22}$$

To see this, we will show that

$$W := W(P_m, \psi Q_n) = x^{m+n}(1-x)^{a-1}(1-\frac{x}{t})^{b-1}(A+Bx), \tag{23}$$

and determine the coefficient A and B. Here $W(f,g) = f'g - fg'$ is the Wronskian.

From equation (4): $\psi Q_n - P_m = \sum_{i=m+n+1}^{\infty} s_{(m^n,i)} x^i$, we have

$$
\begin{aligned}
W &= W(P_m, \psi Q_n - P_m) \\
&= x^{m+n}\Big\{(m+n+1) s_{(m^n)} s_{(m^n,m+n+1)} + O(x)\Big\} \\
&= x^{m+n}\Big\{(-1)^n (m+n+1) s_{(m^n)} s_{((m+1)^{n+1})} + O(x)\Big\} \\
&= x^{m+n}(1-x)^{a-1}(1-\frac{x}{t})^{b-1}\Big(A + O(x)\Big),
\end{aligned}
\tag{24}
$$

where

$$
A = (-1)^n (m+n+1) \tau_{m,n} \tau_{m+1,n+1}. \tag{25}
$$

On the other hand, expanding around $x = \infty$, we have

$$
\begin{aligned}
W &= \psi\Big\{\Big(\frac{a}{x-1} + \frac{b}{x-t}\Big) P_m Q_n + W(P_m, Q_n)\Big\} \\
&= \psi \frac{x^{m+n+1}}{(x-1)(x-t)}\Big\{(a+b+n-m) s_{(m^{n+1})} (-1)^n s_{((m+1)^n)} + O(\frac{1}{x})\Big\} \\
&= x^{m+n}(1-x)^{a-1}(1-\frac{x}{t})^{b-1}(B + O(\frac{1}{x})),
\end{aligned}
\tag{26}
$$

with

$$
B = (-1)^n \frac{(a+b+n-m)}{t} \tau_{m+1,n} \tau_{m,n+1}. \tag{27}
$$

Then we have the desired result (23) and hence (22).

In this case, the polynomials p_m are the Jacobi polynomials and the τ functions $s_{(m^n)}$ are determinants of them. Corresponding special solutions of $P_{\rm VI}$ equation were obtained before in [10].

5.2 $P_{\rm V}$ case

Let us substitute the $P_{\rm VI}$ variables as

$$
(a, b, t, H) \to (-b - \frac{1}{\epsilon}, \frac{1}{\epsilon}, 1 + \epsilon t, \frac{H}{\epsilon}). \tag{28}
$$

In the limit $\epsilon \to 0$, we have

$$
\begin{aligned}
\psi(x) &= (1-x)^{-b} \exp(\frac{xt}{1-x}), \\
a_1(x) &= \frac{-(m+n)}{x} + \frac{-t}{(x-1)^2} + \frac{b+2}{x-1} - \frac{1}{x-\lambda}, \\
a_2(x) &= \frac{m(n-b)}{x(x-1)} - \frac{tH}{x(x-1)^2} + \frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}.
\end{aligned}
\tag{29}
$$

where

$$tH = m(n-b)(\lambda-1)+(\lambda\mu-m-n)\mu(\lambda-1)^2+b\lambda\mu(\lambda-1)+\mu(\lambda-1)-\lambda\mu t. \quad (30)$$

The position of the singularity $\lambda(t)$ is given by

$$\lambda = \frac{(m+n+1)}{(m-n+b)}\frac{\tau_{m,n}\tau_{m+1,n+1}}{\tau_{m+1,n}\tau_{m,n+1}}, \quad (31)$$

and satisfy the fifth Painlevé equation

$$\frac{d^2\lambda}{dt^2} = \Big(\frac{1}{2\lambda}+\frac{1}{\lambda-1}\Big)\Big(\frac{d\lambda}{dt}\Big)^2-\frac{1}{t}\frac{d\lambda}{dt}+\frac{(\lambda-1)^2}{t^2}\Big(\alpha\lambda+\frac{\beta}{\lambda}\Big)+\gamma\frac{\lambda}{t}+\delta\frac{\lambda(\lambda+1)}{\lambda-1}, \quad (32)$$

with parameter

$$\alpha = \frac{(b+m-n)^2}{2}, \quad \beta = -\frac{(m+n+1)^2}{2}, \quad \gamma = b, \quad \delta = -\frac{1}{2}. \quad (33)$$

The polynomials p_m are the Laguerre polynomials. The corresponding Schur function solutions were obtained in [10][11].

Note that the above system is equivalent to the standard Hamiltonian system for $P_{\rm V}$ through the canonical transformation $(\mu, \lambda, t, H) \to (p, q, t, H_{\rm V})$:

$$\begin{aligned} &\lambda = 1-\frac{1}{q}, \quad \mu = q(q(p+t)-m), \\ &H_{\rm V} = (b+m-n)p-(1+b+2m)pq-(b+m)qt+p(p+t)q(q-1). \end{aligned} \quad (34)$$

5.3 $P_{\rm IV}$ case

Let us further redefine the $P_{\rm V}$ variables as

$$(b, t, x, \lambda, \mu, H, \tau_{m,n}) \to (\frac{2}{\epsilon^2}, \frac{2t}{\epsilon}-\frac{2}{\epsilon^2}, \epsilon x, \epsilon\lambda, \frac{\mu}{\epsilon}, -m+(\frac{H}{2}-mt)\epsilon, \epsilon^{-mn}\tau_{m,n}). \quad (35)$$

In the limit $\epsilon \to 0$, we have

$$\begin{aligned} \psi(x) &= \exp(2tx-x^2), \\ a_1(x) &= 2(x-t)+\frac{-(m+n)}{x}-\frac{1}{x-\lambda}, \\ a_2(x) &= -2m+\frac{H}{x}+\frac{\lambda\mu}{x(x-\lambda)}. \end{aligned} \quad (36)$$

where

$$H = \lambda\mu(-\mu-2\lambda+2t)+(m+n+1)\mu+2m\lambda. \quad (37)$$

The position of the singularity $\lambda(t)$ is given by

$$\lambda = \frac{(m+n+1)}{2}\frac{\tau_{m,n}\tau_{m+1,n+1}}{\tau_{m+1,n}\tau_{m,n+1}}, \tag{38}$$

and satisfy the fourth Painlevé equation

$$\frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\Big(\frac{d\lambda}{dt}\Big)^2 + 6\lambda^3 - 8t\lambda^2 + 2(t^2-\alpha)\lambda + \frac{\beta}{\lambda}, \tag{39}$$

with parameter

$$\alpha = -m+n, \quad \beta = -\frac{(m+n+1)^2}{2}. \tag{40}$$

The polynomials p_m are the Hermite polynomials. The corresponding Schur function solutions were obtained in [12].

5.4 Garnier case

As we have seen in section 3, the weight function (6) gives special solutions of N-Garnier system where the Appell-Lauricella F_D functions and their determinants play the role of the τ-functions. Such kinds of solutions were obtained by Tsuda using the Bäcklund transformations and Toda equations [13][14](see also [8]). However, the explicit identification of these solutions, as in subsection 5.1 for $N = 1$ case, is a remaining problem for $N \geq 2$. To do this, we need the bi-rational change of variables from the canonical coordinate (λ_i, μ_i) to their symmetric functions (q_i, p_i) [2][7].

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