Geometry of elliptic Painlevé equation and its Lax formalism

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2008/Dec/20 at Kyushu Univ.

Abstract.

A geometric formulation of Lax pairs for the elliptic Painlevé equation is presented.

[Ref: arXiv:0811.1796]

Aim.

The Lax formulation of the Painlevé equation is important problem. For discrete cases, it has been studied by Jimbo-Sakai, Boalch, Arinkin-Borodin, Rains $\cdots$. Explicit construction of the Lax pair is, however, a very difficult problem in particular for the elliptic case. We will develop a geometric method to construct the Lax equations explicitly.

Plan.

1. Differential Painlevé equations
2. Discrete Painlevé equations
3. Examples of the Lax equations
4. Lax formalism of elliptic Painlevé equations (New)

1. Differential Painlevé equations

We will review the geometric aspects of the classical Painlevé equation P_{VI}.

The sixth Painlevé equation P_{VI}:

$$\frac{d^2q}{dt^2} = \frac{1}{2}\Big(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\Big)\Big(\frac{dq}{dt}\Big)^2 - \Big(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\Big)\frac{dq}{dt}$$
$$+\frac{q(q-1)(q-t)}{t^2(t-1)^2}\Big\{\alpha - \beta\frac{t}{q^2} + \gamma\frac{t-1}{(q-1)^2} + (\frac{1}{2} - \delta)\frac{t(t-1)}{(q-t)^2}\Big\}.$$

Hamiltonian form:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

$$H = \frac{1}{t(t-1)}\Big[q(q-1)(q-t)p^2 + \Big\{(a_1 + 2a_2)(q-1)q$$
$$+ a_3(t-1)q + a_4 t(q-1)\Big\}p + a_2(a_1+a_2)(q-1)\Big],$$

$$\alpha = \frac{a_1^2}{2}, \qquad \beta = \frac{a_4^2}{2}, \qquad \gamma = \frac{a_3^2}{2}, \qquad \delta = \frac{a_0^2}{2}.$$

Homogeneous coordinates $(x : y : z)$:

$$q = \frac{z}{z - x}, \quad p = \frac{y(z - x)}{xz}$$

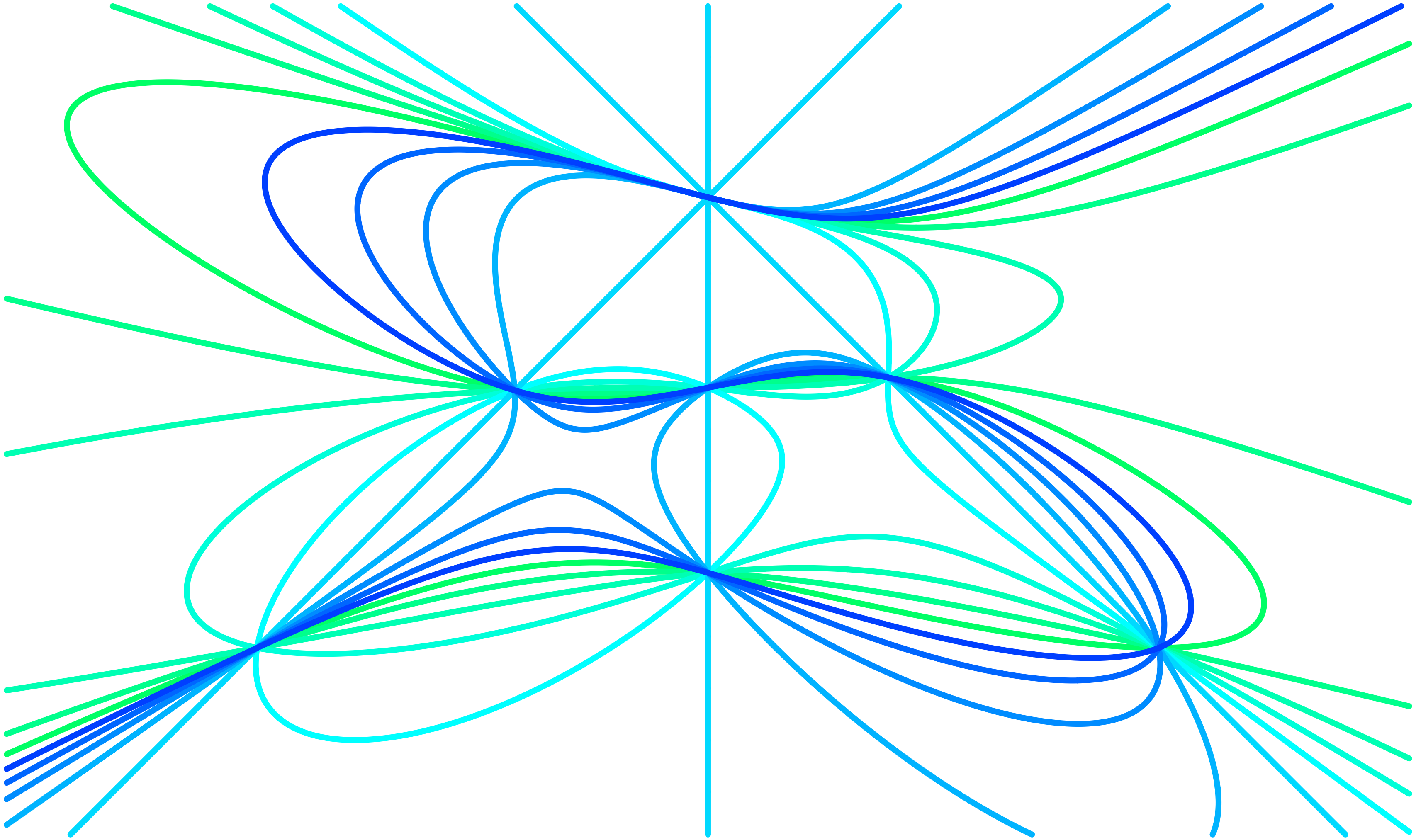
The curve $H = \mu$ becomes the *cubic pencil*:

$$F(x, y, z) + \mu G(x, y, z) = 0,$$

$$F = -(t-1)y^2 z + a_3(t-1)yz^2 - a_4 t x^2 y + a_2(a_1 + a_2)x^2 z$$
$$+txy^2 + (a_1 + 2a_2 + a_3 - a_3 t + a_4 t)xyz,$$

$$G = t(t-1)xz(z - x).$$

$H = \mu$ curves:



Intersections of $F = 0$ and $G = 0$:

$$\begin{array}{lll} (0:0:1), & (1,-a_2,1), & (1,0,0), \\ (0,a_3,1), & (1,-a_1-a_2,1), & (1,a_4,0), \end{array}$$

and

$$((t-1)\varepsilon : 1 : t\varepsilon - a_0 t\varepsilon^2), \quad (1,\varepsilon,\varepsilon^2)$$

Vanishing condition at these 9 points $\Rightarrow$ $H = \mu$ curves.

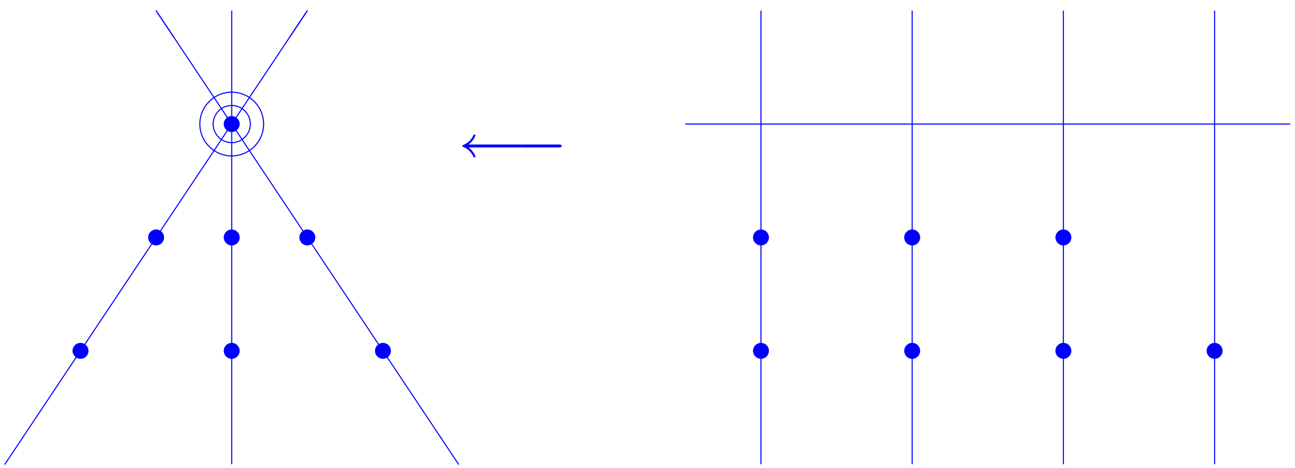
(autonomous case)

This gives a geometric characterization of H.

[Kajiwara-Masuda-Noumi-Ohta-Y, FE **48** (2005) 147-160]

Space of initial conditions [Okamoto]:

9 points blown-up of $\mathbb{P}^2 \setminus \{$divisors $D_4^{(1)}\}$.



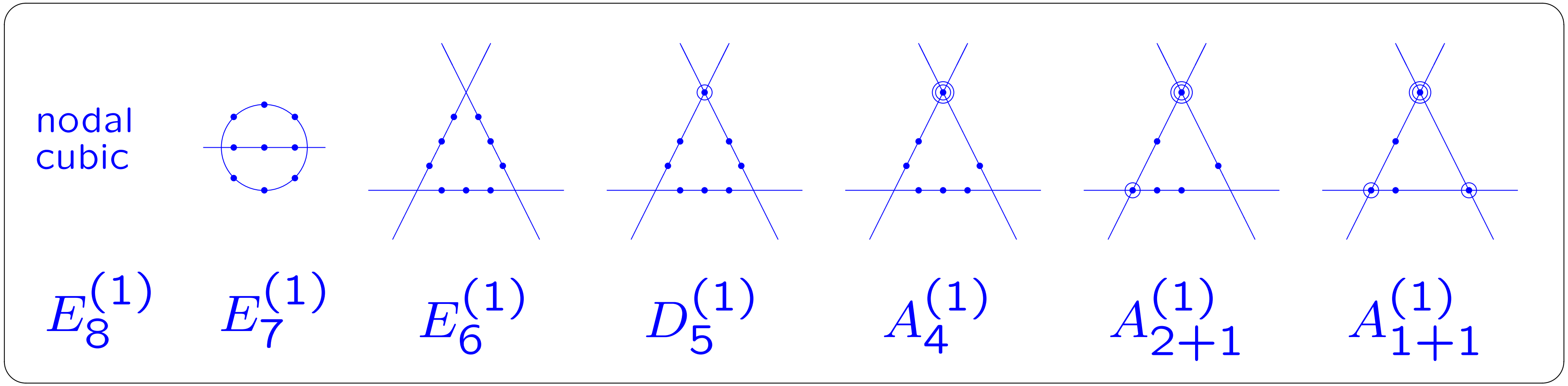
2. Discrete Painlevé equations

We will recall the discrete Painlevé equations and their geometric formulation.

The second order discrete Painlevé equations [H.Sakai].

$$
\begin{array}{lccccccccccccccccc}
\text{Ell.} & E_8^{(1)} & & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & \mathbb{Z} & & \\
 & & & & & & & & & & & & & \nearrow & & & & \\
\text{Mul.} & E_8^{(1)} & \to & E_7^{(1)} & \to & E_6^{(1)} & \to & D_5^{(1)} & \to & A_4^{(1)} & \to & A_{2+1}^{(1)} & \to & A_{1+1}^{(1)} & \to & A_1^{(1)} & \to & \mathcal{D}_6 \\
 & & & & & & & & & & & & & & & & & \\
\text{Add.} & E_8^{(1)} & \to & E_7^{(1)} & \to & E_6^{(1)} & \to & D_4^{(1)} & \to & A_3^{(1)} & \to & A_{1+1}^{(1)} & \to & A_1^{(1)} & \to & \mathbb{Z}_2 & & \\
 & & & & & & & & & & \searrow & & \searrow & & & \downarrow & & \\
 & & & & & & & & & & & A_2^{(1)} & \to & A_1^{(1)} & \to & 1 & &
\end{array}
$$

(Mul.) q-Painlevé equations:



(Ell.) Elliptic Painlevé equation:

Cubic curve passing through 9 points on $\mathbb{P}^2$.

Curve of degree (2,2) passing through 8 points on $\mathbb{P}^1 \times \mathbb{P}^1$.

q-P_{VI} equation: [Jimbo-Sakai(96)]

$$T : (f, g, a_i, b_i) \mapsto (\dot{f}, \dot{g}, \dot{a_i}, \dot{b_i}),$$

$$\dot{f} f = \frac{(\dot{g} - b_1)(\dot{g} - b_2)}{(\dot{g} - b_3)(\dot{g} - b_4)} a_3 a_4,$$

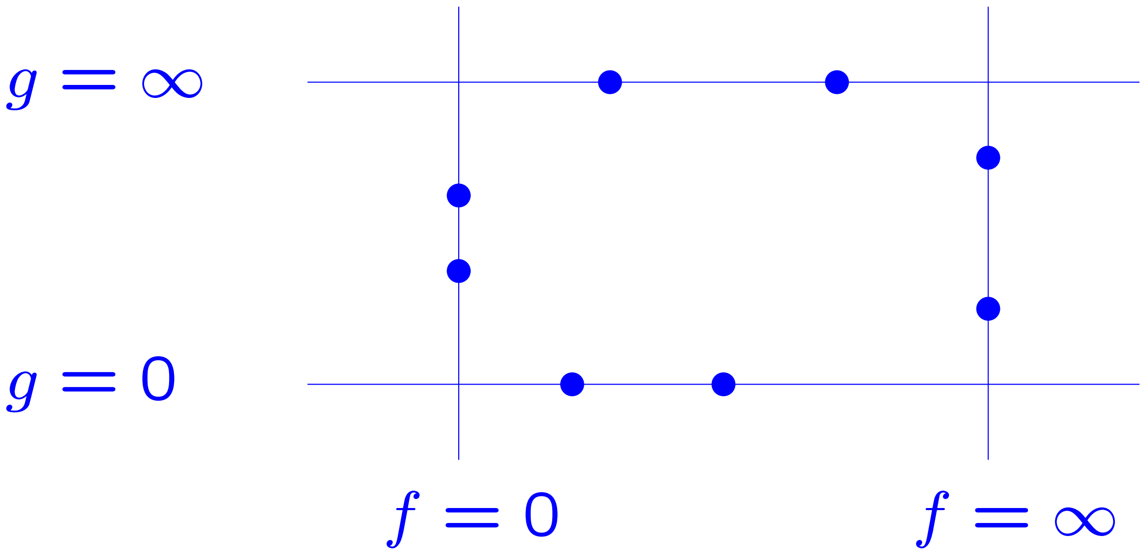
$$\dot{g} g = \frac{(f - a_1)(f - a_2)}{(f - a_3)(f - a_4)} b_3 b_4,$$

$$\begin{pmatrix} \dot{a_1}, & \dot{a_2}, & \dot{a_3}, & \dot{a_4} \\ \dot{b_1}, & \dot{b_2}, & \dot{b_3}, & \dot{b_4} \end{pmatrix} = \begin{pmatrix} q a_1, & q a_2, & a_3, & a_4 \\ q b_1, & q b_2, & b_3, & b_4 \end{pmatrix},$$

$$q = \frac{a_3 a_4 b_1 b_2}{a_1 a_2 b_3 b_4}.$$

$D_5^{(1)}$-symmetry, $A_3^{(1)}$-configuration.

$$(f,g) = (0, b_1/q),\ (0, b_2/q),\ (\infty, b_3),\ (\infty, b_4),$$
$$(a_1, 0),\ (a_2, 0),\ (a_3, \infty),\ (a_4, \infty).$$



Elliptic Painlevé equation ($E_8^{(1)}$-symmetry) on $\mathbb{P}^1 \times \mathbb{P}^1$

Example. $\dot{P} = T_{12}(P)$

Parameters $P_1, \ldots, P_8$: On the fixed curve C_0 of degree (2,2) passing through $P_1, \ldots, P_8$,

$$\dot{P}_1 + P_2 + \cdots + P_8 = 0, \quad \dot{P}_1 + \dot{P}_2 = P_1 + P_2,$$
$$\dot{P}_i = P_i, \quad (i \neq 1, 2).$$

Dependent variable P: On the moving curve C of degree (2,2) passing through $P_2, \ldots, P_8, P$,

$$\dot{P}_1 + \dot{P} = P_2 + P.$$

Explicit form of the equation :

$$\dot{f} = \frac{F_1(f,g)}{F_0(f,g)}, \qquad \dot{g} = \frac{G_1(f,g)}{G_0(f,g)},$$

where $\lambda F_0 + \mu F_1 = 0$ [or $\lambda G_0 + \mu G_1 = 0$] is the pencil of rational curves of degree (5,4) [or (4,5)] with base points at $(P_1, P_2, \ldots, P_8)$ of multiplicity $(0, 4, 2, 2, \ldots, 2)$.

3. Examples of the Lax equations

Examples of Lax equations for P_{VI}, q-P_{VI}, q-$E_6^{(1)}$ and their geometry are discussed.

Lax pair for P_{VI}: [R.Fuchs(1907)]

$$\frac{\partial^2 y}{\partial z^2} + \Big(\frac{1-a_4}{z} + \frac{1-a_3}{z-1} + \frac{1-a_0}{z-t} - \frac{1}{z-q}\Big)\frac{\partial y}{\partial z}$$
$$+\Big\{\frac{a_2(a_1+a_2)}{z(z-1)} - \frac{t(t-1)H}{z(z-1)(z-t)} + \frac{q(q-1)p}{z(z-1)(z-q)}\Big\}y = 0,$$

$$\frac{\partial y}{\partial t} + \frac{z(z-1)(q-t)}{t(t-1)(q-z)}\frac{\partial y}{\partial z} + \frac{zp(q-1)(q-t)}{t(t-1)(z-q)}y = 0.$$

Monodromy preserving deformation on $\mathbb{P}^1 \setminus \{0, 1, t, \infty\}$.

The Lax pair for q-P_{VI} [Jimbo-Sakai(96)]

$$\Big\{\frac{(a_1-z)(a_2-z)}{a_1a_2(z-f)}T_z-\Big(c_0+c_1z+\frac{c_2z}{z-f}+\frac{c_3z}{z-qf}\Big)+\frac{a_1a_2(z-qa_3)(z-qa_4)}{b_3b_4q^2(z-qf)}T_z^{-1}\Big\}y=0,$$

$$\Big\{qgT_z-a_1a_2+z(z-f)T^{-1}\Big\}y=0.$$

Where $y=y(z)$, $T_zy=y(qz)$, $Ty=\dot{y}(z)$ and

$$c_0=-\frac{a_1a_2}{f}\Big(\frac{1}{b_1}+\frac{1}{b_2}\Big),\quad c_1=\frac{1}{q}\Big(\frac{1}{b_3}+\frac{1}{b_4}\Big),$$

$$c_2=\frac{(f-a_1)(f-a_2)}{qfg},\quad c_3=\frac{(f-a_3)(f-a_4)g}{b_3b_4f}.$$

Other cases? Experiments using Padé approach.

$f(z) \to \dfrac{N(z)}{D(z)} \to$ differential (or difference) equations whose fundamental solution is $\{N(z), f(z)D(z)\}$.

Appropriate input function $f(z)$ gives Lax equations.

$$\frac{(1/c_1, c_2/a_2)_j}{(1/a_2, a_1/c_1)_j} = \frac{P_m(q^{-j})}{Q_n(q^{-j})}, \qquad (j = 0, 1, \ldots, m+n)$$

$$P_m(z) = \sum_{i=0}^{m} A_i \frac{(z)_i}{(qa_2 z)_i}, \qquad Q_n(z) = \sum_{i=0}^{n} B_i \frac{(z)_i}{(qc_1 z)_i}.$$

A version of Padé interpolation with prescribed poles and zeros [Zhedanov-Spridonov].

$\longrightarrow$ Lax pair for q-$E_6^{(1)}$-Painlevé equation:

$$\Big\{K_1(T_z-1)+K_2+K_3(T_z^{-1}-1)\Big\}y=0,$$
$$\Big\{C_1T_{a_2}+C_2+C_3(T_z^{-1}-1)\Big\}y=0.$$

$$K_1=(1-z)(c_2-a_2qz)(1-c_1q^{n+1}z)(1-a_2q^{m+1}z)(z-f),$$
$$K_2=z(1-q^m)(1-a_2q)(\bigcirc+\bigcirc z+\bigcirc z^2),$$
$$K_3=(1-a_2z)(a_1-c_1z)(1-a_2qz)(1-q^{m+n})(qz-f),$$
$$C_1=\bigcirc(a_1-c_2)(1-a_2q)(c_2-a_2qz)(1-a_2q^{m+1}z)(z-f),$$
$$C_2=(1-a_2qz)(\bigcirc+\bigcirc z+\bigcirc z^2),$$
$$C_3=a_2q(1-c_2q^n)(1-a_2z)(a_1-c_1z)(1-q^{m+n}z)(1-a_2qz).$$

Where $\bigcirc$ are some rational functions in f,g variables.

The coefficients $\bigcirc$ are complicated functions of f, g

$\simeq$ The coefficient H is a complicated function of p, q

Recall that H has a geometric characterization.

Question:

Can we characterize these coefficients $\bigcirc$ (or the Lax equation itself) by some geometric conditions?

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Recall that H has a geometric characterization.

Question:

Can we characterize these coefficients $\bigcirc$ (or the Lax equation itself) by some geometric condition?

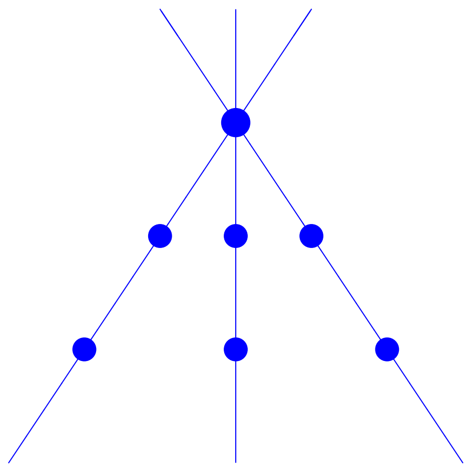
Yes!

P_{VI} case: The Lax equation

$$\bigcirc y'' + \bigcirc y' + \bigcirc y = 0$$

is characterized as the nodal curve of degree 4 in $\mathbb{P}^2$ passing through the 9+3+2 points:

$$(0:0:1),\quad (1:-a_2:1),\quad (1:0:0),$$
$$(0:a_3:1),\ (1:-a_1-a_2:1),\ (1:a_4:0),$$
$$\Big((t-1)\varepsilon : 1 : t\varepsilon - a_0 t\varepsilon^2\Big)_{(\varepsilon^3=0)},$$
$$\Big((z-1)\varepsilon : 1 : z\varepsilon + z\varepsilon^2\Big)_{(\varepsilon^3=0)},$$
$$\Big(\frac{1}{z+\varepsilon} : \frac{y'(z+\varepsilon)}{y(z+\varepsilon)} : \frac{1}{z+\varepsilon-1}\Big)_{(\varepsilon^2=0)}.$$



q-P_{VI} case: The Lax equation

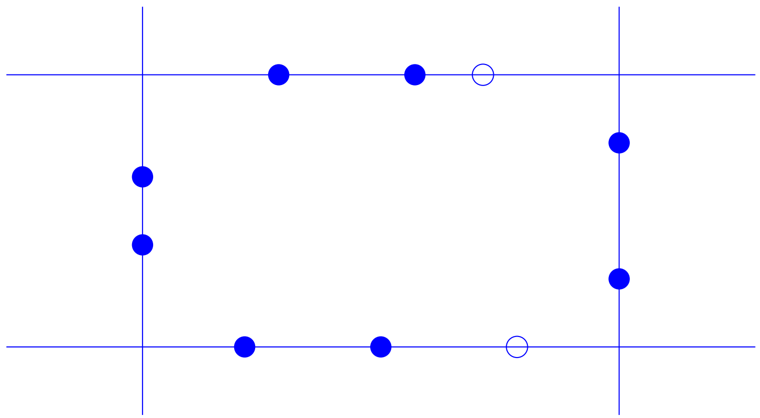
$$\Big\{ \bigcirc T_z + \bigcirc + \bigcirc T_z^{-1} \Big\} y = 0$$

is characterized as the degree (3,2) curve in $\mathbb{P}^1 \times \mathbb{P}^1$ passing through the 8+2+2 points:

$$(0, b_1/q), (0, b_2/q), (\infty, b_3), (\infty, b_4),$$
$$(a_1, 0), (a_2, 0), (a_3, \infty), (a_4, \infty),$$

$$(z, \infty), \ \ (z/q, 0),$$

$$\Big(z, \frac{a_1 a_2}{q} \frac{y(z)}{y(qz)}\Big), \ \ \Big(\frac{z}{q}, \frac{a_1 a_2}{q} \frac{y(z/q)}{y(z)}\Big),$$



q-$E_6^{(1)}$ case: The Lax equation

$$\Big\{\bigcirc T_z + \bigcirc + \bigcirc T_z^{-1}\Big\} y = 0$$

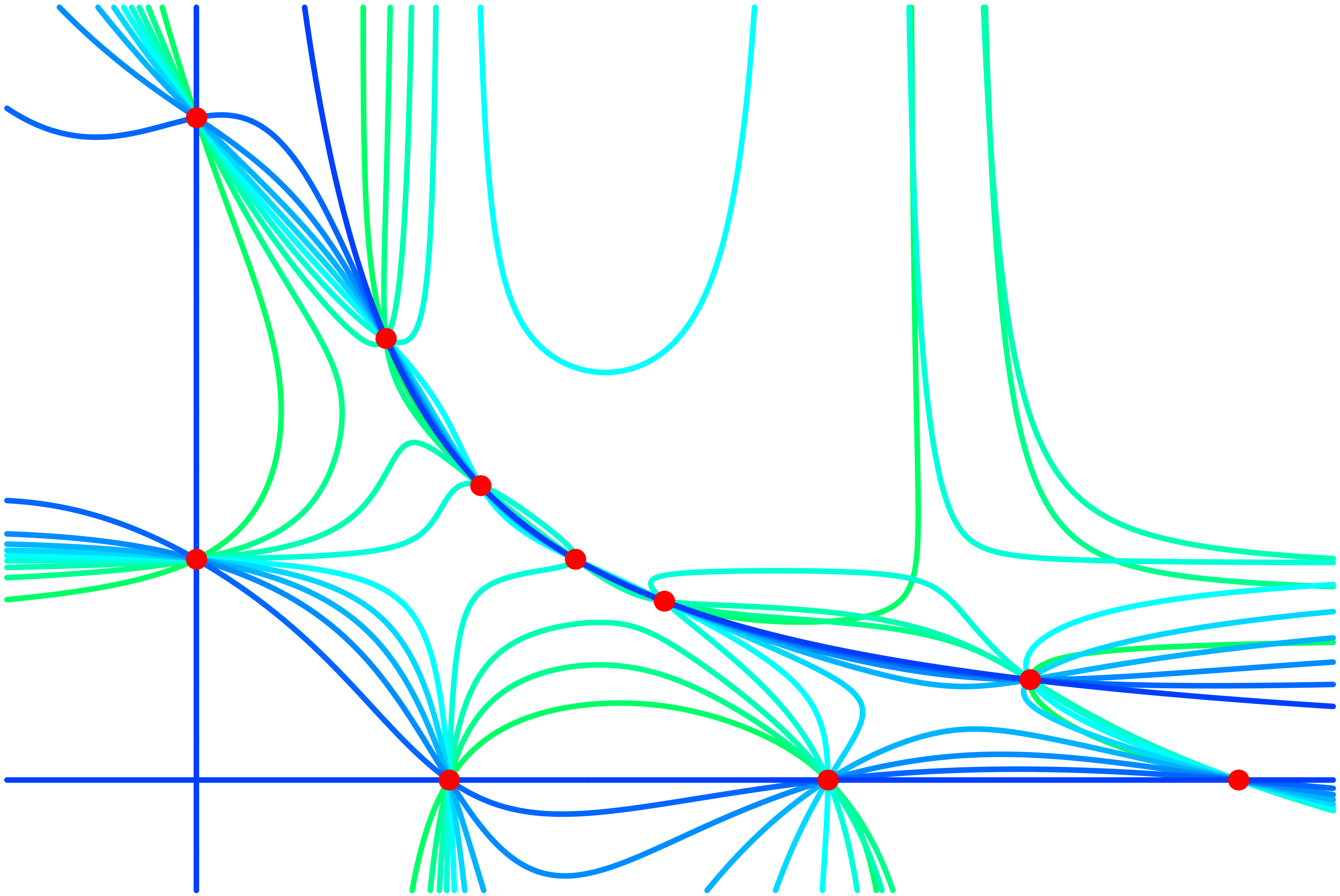
is characterized as the degree $(3,2)$ curve passing through 8+2+2 points on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$(q^{-m-n}, 0), \quad (\frac{a_1}{c_1}, 0), \quad (0, a_2), \quad (0, \frac{a_1 a_2}{c_2}),$$
$$\Big(u, \frac{1}{u}\Big), \quad u = q^{-1},\ a_2 q^m\ \ c_1 q^n,\ \frac{a_2}{c_2},$$
$$(qz, 0), \quad \Big(\frac{1}{z}, z\Big),$$

and

$$(u, G_u), \quad \frac{y(u,t)}{y(u/q,t)} = \frac{(1-a_2 u)G_u}{a_2(1-uG_u)}, \quad u = z,\ qz.$$

q-$E_6^{(1)}$ Lax curves:



Experiments using the Padé approach,

$\longrightarrow$ similar structure for q-$E_7^{(1)}$, q-$E_8^{(1)}$.

Taking a hint from these, one can formulate the general Lax equations including the elliptic case.

4. Lax formalism of elliptic Painlevé equation

Lax equations for the elliptic Painlevé equation are described explicitly in terms of the point configurations.

- $C_0 : \varphi_{2,2}(f,g) = 0$: Elliptic curve on $\mathbb{P}^1 \times \mathbb{P}^1$.

- $P_z = (f_z, g_z)$: Parametrization of C_0. $z \in \mathbb{C}/\Gamma$ is the Jacobian parameter which plays the role of the independent variable of the Lax equation.

- $P_i = (f_i, g_i) = P_{u_i}$ $(i = 1, \ldots, 8)$: 8 points on C_0.

$$\delta = \sum_{i=1}^{8} u_i \ (\neq 0).$$

- Involution: $P = (f,g) \leftrightarrow P^* = (f, g^*)$ on C_0.

- We will consider the time evolution $T = T_{12}$ such that

$$T(u_1) = u_1 - \delta, \quad T(u_2) = u_2 + \delta, \quad T(u_i) = u_i \quad (i \neq 1, 2).$$

Definition. The Lax equations

$$L_1 = \{\bigcirc T_z + \bigcirc + \bigcirc T_z^{-1}\} y = 0,$$
$$L_2 = \{\bigcirc T + \bigcirc + \bigcirc T_z^{-1}\} y = 0,$$

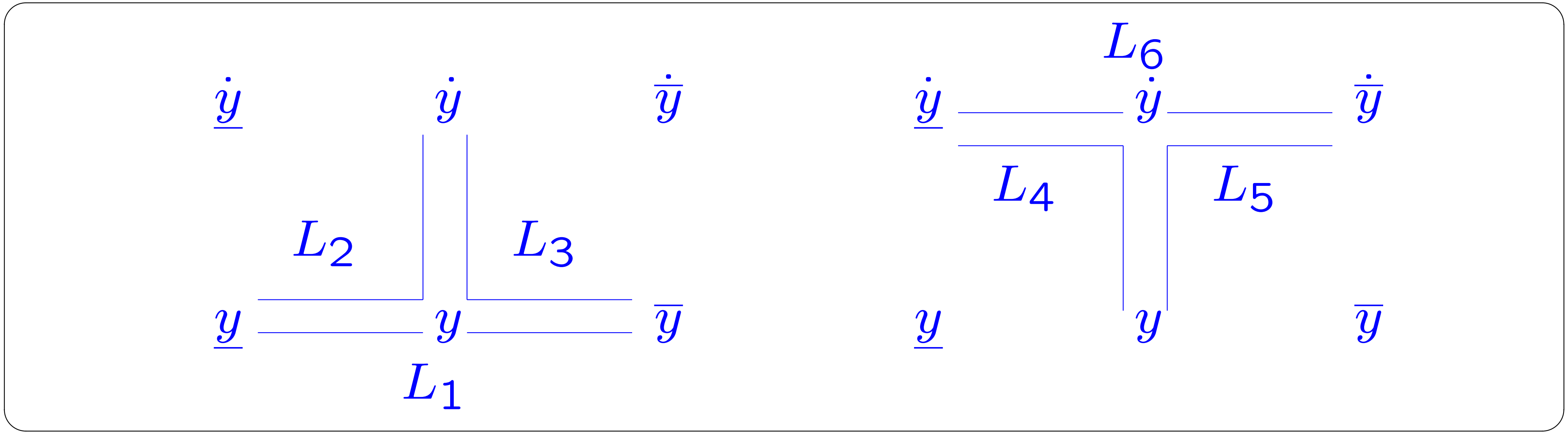
are defined as the curves on $\mathbb{P}^1 \times \mathbb{P}^1$ of degree (3,2) and passing through the following 11(+1) points:

$$L_1: \; P_1, \ldots, P_8, P_z, (P^*_{z-\delta}), \qquad Q_z, Q_{z-\delta},$$
$$L_2: \; P_2, \ldots, P_8, P_{z+u_1-u_2}, P^*_{z-\delta}, (P_2), \; Q_2, Q_{z-\delta},$$

where

$$Q_z \quad : f = f_z, \quad (g - g_z) y(z) = (g - g_z^*) y(z+\delta),$$
$$Q_{z-\delta} : f = f_{z-\delta}, \; (g - g_{z-\delta}) y(z-\delta) = (g - g^*_{z-\delta}) y(z),$$
$$Q_2 \quad : f = f_2, \quad (g - g_2) y(z) = (g - g_2^*) T y(z).$$

From equations $L_1 = 0$, $L_2 = 0$, one can derive other 3-term equations $L_3 = 0, \cdots, L_6 = 0$:



The last one L_6 is the 3 term equation for $\dot{y} = Ty$:

$$L_6 = \left\{ \bigcirc T_z + \bigcirc + \bigcirc T_z^{-1} \right\} \dot{y} = 0,$$

which should be compared with $L_1 = 0$.

The equation L_6 is too huge (of degree (7,6)). However, by analysing its geometric characterization, we can prove

Theorem(Compatibility). The equation $L_6 = 0$ is equivalent to the time evolution $T(L_1) = 0$.

This means that the huge equation $L_6 = 0$ shrinks down to $L_1 = 0$, when it is written in terms of $\dot{f}, \dot{g}$.

The proof of the Theorem is based on some classical (antique) algebraic geometry of plane curves. (arXiv:0811.1796)

Concluding Remark.

• It is known that the geometry (the Okamoto space) determines the Painlevé equation itself (Takano's theory). Sakai showed that this is also true for discrete cases. Now, we can say that the geometry knows also the Lax equations.

• The Lax equation has been a source of various non-trivial results for the Painlevé equations. Since our result is concrete enough, it may serve for further study. It will be interesting if we can say anything about the solutions y or (f, g) from the geometry.

Thank you.