Data for the paper
"Shift operators of the Dotsenko-Fateev equation
and its higher order versions"
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arXiv xxxx.xxxx, math.CA
doi number will be listed later

Explanation of data listed

For the Fuchsian differential equations treated in the paper, we list the equations and the shift operators in the form that the data is readable by the help of the mathematical software Maple.

Definition of the differential equations

The equations we treated in the paper are

$$H_6, G_6, E_6, SE_6, E_5, SE_5, E_4, SE_4, ST_4, E_3, SE_3, E_{43}, E_{32}, E_2,$$

where E_2 is the Gauss hypergeometric equation, which is related to all others; For their mutual relations, we refer to the paper [1].

We summarize the defintion of the differential equations. The text-file **HDFequations.txt** lists the maple forms of the differential equations.

- $H_6 = H_6(e1, \ldots, e9, T10) = T0 + T1 * dx + T2 * dx^2 + T3 * dx^3$

```
T0:=(z+s+2)*(z+s+1)*(z+s)*(z+e7)*(z+e8)*(z+e9):
T1:=(z+s+2)*(z+s+1)*B1:
T2:=(z+s+2)*B2:
T3:=-(z+3-e1)*(z+3-e2)*(z+3-e3):
B1:=T13*z^3+T12*z^2+T11*z+T10:
B2:=T23*z^3+T22*z^2+T21*z+T20:
```

  where s is determined by the Fuchs relation $e1 + \cdots + e9 + 3s + 9 = 15$. The coefficients Tij, except $T10$ are polynomials in $e = (e1, \ldots, e9)$, and $T10$ is the accessory parameter. $G_6 = G_6(e, a)$ is a specialization of H_6, where $T10$ is a polynomial $T10(e, a)$ in e with a set a of parameters. $E_6 = E_6(e) = G_6(e, 0)$. The operators $B1$, $B2$ and the polynomials Tij are given in the file **HDFequations.txt**.

- SE_6 is a specialization of E_6 with the condition

$$e1 - 2e2 + e3 = e4 - 2e5 + e6 = e7 - 2e8 + e9.$$

  It is parameterized by (a, b, c, g, p, q, r) by the relations

```
e1:=p+r+1:
e2:=a+c+p+r+2:
e3:=2a+2c+g+p+r+3:
e4:=q+r+1:
e5:=b+c+q+r+2:
e6:=2b+2c+g+q+r+3:
e7:=-2c-p-q-r-1:
e8:=-a-b-2c-p-q-r-g-2:
e9:=-2a-2b-2c-p-q-r-g-3:
s:=-r:
```

The equation H_5 $(j = 5, 4, 3)$ has one accessory parameter. $G_j(e, a)$ $(j = 5, 4, 3)$ is defined as above, and E_j $(j = 5, 4, 3)$ is defined by $G_j(e, 0)$ $(j = 5, 4, 3)$. In the following we tabulate only E_j and SE_j $(j = 5, 4, 3)$.

- The equation $E_5 = E_5(e1, \ldots, e8) = x * Pn + P0 + P1 * dx + P2 * dx^2$ is the quotient of E_6 with the condition $e9 = 0$ divied by dx on the right.

```
Pn:=(z-r+1)*(z-r+2)*(z-r+3)*(z+e7+1)*(z+e8+1):
P0:=(z-r+1)*(z-r+2)*B1(e9=0):
P1:=(z-r+2)*B2(e9=0):
P2:=-(z+3-e1)*(z+3-e2)*(z+3-e3):
```

where $r = (e1 + \cdots + e8 - 6)/3$.

- SE_5 is a specialization of E_5 with the condition

$$e1 - 2e2 + e3 = e4 - 2e5 + e6 = e7 - 2e8.$$

It is parameterized by (a, b, c, g, p, q) as

```
e1:= -2*a - 2*b - 2*c - g - q - 2:
e2:= -a - 2*b - c - g - q - 1:
e3:= -2*b - q:
e4:= -2*a - 2*b - 2*c - g - p - 2:
e5:= -b  - 2*a- c - g - p - 1:
e6:= -2*a - p:
e7:= 2*a + 2*b + g + 2:
e8:= a + b + 1:
```

- $E_4 = E_4(e1, \ldots, e7) = Q_0 + Q_1 * dx + Q_2 * dx^2$ is defined as

```
Q0:=(z+e5)*(z+e6)*(z+e7)*(z+e8):
Q1:=-2*z^3+Q12*z^2+Q11*z+Q10:
Q12:=e1+e2-e5-e6-e7-e8-5:
Q11:=3*(e1+e2)-e1*e2+e3*e4-e5*e6-e5*e7-e5*e8-e6*e7-e6*e8-e7*e8-8:
Q2:=(z-e1+2)*(z-e2+2):
```

where $e8$ is determined by the Fuchs raltion $e1 + e2 + \cdots + e7 + e8 = 4$ and $Q10$ is given in **HDFequations.txt**. The equation is written also as

$$E_4 := x^2 * (x-1)^2 dx^4 + p3 * dx^3 + p2 * dx^2 + p1 * dx + p0 :$$

where

```
p3:= x*(x-1)*((-t11-t12+10)*x+t11-5):
p2:= (-3*t11-3*t12+t23+19)*x^2 +(5*t11+t12-t21+t22-t23-19)*x+4-2*t11+t21:
p1:= (t3-t11-t12+t23+5)*x+Q10:
p0:= e5*e6*e7*e8:
```

Refer to **HDFequations.txt** for $t11$, $t12$, ...

- SE_4 is a specialization of E_4 with the condition

$$e1 - 2e2 + 1 = e3 - 2e4 + 1 = e5 - 2e6 + e7 + e8, \quad e1 + \cdots + e7 + e8 = 4.$$

It is parameterized by (a, b, c, g, u) as

```
e1:=2+2a+2c+g-u:
e2:=1+a+c-u:
e3:=2+2b+2c+g-u:
e4:=1+b+c-u:
e5:=u+1:
e6:=-1-a-2c-g-b+u:
e7:=-2c+u:
e8:=-2-2a-2b-2c-g+u:
```

- $\mathrm{ST}_4 = \mathrm{ST}_4(e1,\dots,e6) = V0 + V1 * dx + V2 * dx^2$ is given as

```
V0 := (z+s+1)*(z+s)*(z+e5)*(z+e6):
V1 := (z+s+1)*(-2*z^2+(e1+e2-e5-e6-4)*z
        +1/4*((e6-e5)^2-(e3-e4)^2+(e1-e2)^2+2*(e1+e2-3)*(e5+e6+1)-1)):
V2 := (z+2-e1)*(z+2-e2):
```

  where s is determined by the Fuchs relation $e1 + e2 + e3 + e4 + e5 + e6 + 2 * s + 3 = 6$.

- ${}_4E_3(a0, a1, a2, a3; b1, b2, b3) := (z+a0)*(z+a1)*(z+a2)*(z+a3)-(z+b1)*(z+b2)*(z+b3)*dx$ is the generalized hypergeometric equation of rank 4.

- $E_3 = E_3(e1,\dots,e6) = x * Sn + S0 + S1 * dx$ is defined as

```
Sn:=(z+e5)*(z+e6)*(z+e7):
S0:= -2*z^3+(2*e1+2*e2+e3+e4-3)*z^2
   +(-e1*e2+(e3-1)*(e4-1)-e5*e6-(e5+e6)*e7)*z+a00,\\
S1:=(z-e1+1)*(z-e2+1):
```

  where $e7$ is determined by the relation $e1+e2+e3+e4+e5+e6+e7 = 3$ and $a00$ is the accessory parameter defined as

```
54*a00:= -4*(e1+e2-e3-e4)^3-27*e5*e6*e7+9*(e1+e2-e3-e4)*(e5*e6+e5*e7+e6*e7-2)
    +9*e1*e2*(e1+e2-1)+18*(e1+e2-1)*(e3^2+e3*e4+e4^2)
    -9*e3*e4*(e3+e4-1)-18*(e3+e4-1)*(e1^2+e1*e2+e2^2):
```

- SE_3 is a specialization of E_3 with the condition

$$2e1 - e2 = 2e3 - e4 = -e5 + 2e6 - e7.$$

  It is parameterized by (a, b, c, g) as:

```
e1:=a+c+1:
e2:=2e1+g:
e3:=b+c+1:
e4:=2e3+g:
e5:=-2c:
e6:=-(a+b+2c+g+1):
e7:=2e6+g-e5:
```

  The accessory parameter turns out to be

$$a00 = c * (2 * a + 2 * c + 1 + g) * (2 * a + 2 * b + 2 * c + 2 + g).$$

- ${}_3E_2(a0, a1, a2; b1, b2) := (z + a0) * (z + a1) * (z + a2) - (z + b1) * (z + b2) * dx$ is the generalized hypergeometric equation of rank 3.

- $E_2 := E_2(e1, e2, e3, e4) = E(a, b, c) = (z + a) * (z + b) - (z + c) * dx$ is the Gauss hypergeometric equation: We used the parameters $(e1, e2, e3, e4)$ defined as $e1 = 1 - c$, $e2 = c - a - b$, $e3 = a$, and $e4 = b$, where $e1 + e2 + e3 + e4 = 1$.

Shift operators

We review definitions of shift operators and explain the text-files for such operators.

Given a differential operator $E(a)$ with parameter a of order n, suppose a shift operator P_{a+} (*resp.* P_{a-}) exists, which is an operator of order lower than n sending $\mathrm{Sol}(E(a))$ to $\mathrm{Sol}(E(a+1))$ (*resp.* $\mathrm{Sol}(E(a-1))$), we have a shift relation such as

$$E(a + 1) \circ P_{a+}(a) = Q_{a+}(a) \circ E(a) \quad (\textit{resp. } E(a - 1) \circ P_{a-}(a) = Q_{a-}(a) \circ E(a)),$$

where $Q_{a\pm}$ are some operators of the same order of $P_{a\pm}$.

In the following, we list the operators $P_{a\pm}$ and $Q_{a\pm}$ for each equation and each shift.

For the Gauss equation $E_2 = E(a, b, c)$, the following shift operators are classically known:

$$
\begin{aligned}
P_{a+} &= x * dx + a, & Q_{a+} &= x * dx + a + 1, \\
P_{a-} &= x(x-1) * dx + a + bx - c, & Q_{a-} &= x(x-1) * dx + a + bx - c + x - 1, \\
P_{c+} &= (x-1) * dx + a + b - c, & Q_{c+} &= P_{c+}, \\
P_{c-} &= x * dx + c - 1, & Q_{c-} &= P_{c-}.
\end{aligned}
$$

For the equation E_4, we find only a simple shift operator, which are mentioned in the paper. For the equation E_3, we could not find a shift operator.

For other equations, refer to the list of shift operators given in

H6PQ.txt	for H_6,
G6PQ.txt, G6memo.txt	for G_6,
E6PQ.txt	for E_6,
SE6PQ.txt	for SE_6,
E5PQ.txt	for E_5,
SE5PQ.txt	for SE_5,
SE4PQ.txt	for SE_4,
ST4PQ.txt	for ST_4,
SE3PQ.txt	for SE_3.