CONVERGENCE AND CHARACTER SPECTRA OF COMPACT SPACES

An infinite subset A of a space X is said to converge to the point $p \in X$ (in symbols: $A \to p$) if for every neighbourhood U of p we have $|A \setminus U| < |A|$. The convergence spectrum $cS(p, X)$ is then defined as

$$cS(p, X) = \{|A| : A \to p\},$$

moreover the convergence spectrum of X is

$$cS(X) = \cup\{cS(p, X) : p \in X\}.$$

The character $\chi(p, X)$ is the smallest cardinal of a neighbourhood base of p in X, the character spectrum $\chi S(p, X)$ is then defined as

$$\chi S(p, X) = \{\chi(p, Y) : p \in Y \subset X\} \setminus \{1\},$$

moreover

$$\chi S(X) = \cup\{\chi S(p, X) : p \in X\}$$

is the character spectrum of X. If X is compact (T_2) then we have $\chi S(p, X) \subset cS(p, X)$ and hence $\chi S(X) \subset cS(X)$.

If X is first countable then clearly $\kappa \in cS(X)$ implies $\mathrm{cf}(\kappa) = \omega$. In 1998 Arhangel'skii and Buzyakova asked if first countable compacta are the only ones satisfying $\mathrm{cf}(\kappa) = \omega$ for all $\kappa \in cS(X)$. This is still open (in ZFC).

We call X an AB-compactum if $\mathrm{cf}(\kappa) = \omega$ for all $\kappa \in \chi S(X)$. Then we have

- if $2^\omega < \aleph_\omega$ then any AB-compact X is first countable (if CH holds then already $\omega_1 \notin \chi S(X)$ suffices);
- it is consistent to have an AB-compact X with e.g.

$$\chi S(X) = \{\omega\, ,\, \aleph_\omega\}.$$

We can show that the cardinality of an AB-compactum is at most $2^{<\mathbf{c}}$, and $\omega_1 \notin cS(X)$ for a compact X implies $\chi(X) \le \mathbf{c}$, hence $|X| \le 2^{\mathbf{c}}$. However, it is consistent that there are arbitrarily large compact spaces whose character spectrum omits ω_1. If X is compact with $\chi(X) > \mathbf{c}$ then either ω_1 or $\mathbf{c}$ belongs to $\chi S(X)$.