Algebraic Combinatorics on Symmetric edge polytopes

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Symmetric edge polytopes

$\mathbf{e}_1, \ldots, \mathbf{e}_n$: the standard basis for $\mathbb{R}^n$
G : a finite simple graph on $[n] := \{1, \ldots, n\}$
$E(G)$: the edge set of G

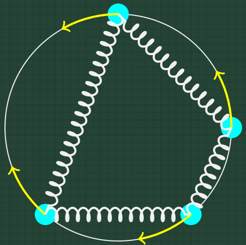
Definition

The symmetric edge polytope or (PV-type) adjacency polytope $\mathcal{A}_G$ of G is the lattice polytope which is the convex hull of

$$\{\pm(\mathbf{e}_i - \mathbf{e}_j) \in \mathbb{R}^n : \{i, j\} \in E(G)\}.$$

Recently, we have focused on computing the volume of $\mathcal{A}_G$.

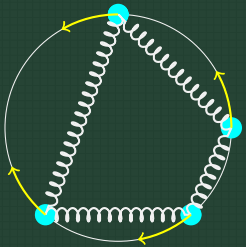
Kuramoto Model



Synchronization phenomena for network of interconnected oscillators is modeled by a wighted graph $G = (V, E, A)$, where

- vertices $V = \{1, \ldots, n\}$ (oscillators with natural frequencies ω_i)
- edges E (connectivity)
- weights $A = \{a_{i,j}\}$ for each $\{i, j\} \in E$ (coupling strengths between coupled oscillators)

Kuramoto Model



For $i = 1, \dots, n$,

$$\frac{d\theta_i}{dt} = \omega_i - \sum_{j \in N_G(i)} a_{i,j} \sin(\theta_i - \theta_j),$$

where

- $\theta_i \in [0, 2\pi)$ is the phase angle of the ith oscillator,
- $N_G(i)$ is the set of neighbors of the ith osillator.

$(\theta_1, \dots, \theta_n)$: frequency synchronization configuration $\underset{\text{def}}{\Longleftrightarrow} \forall i, \frac{d\theta_i}{dt} = 0$

Kuramoto Model

Fix $\theta_n = 0$.

Problem

Given $\omega_1, \ldots, \omega_{n-1} \in \mathbb{R}$ and a wighted graph of n nodes, what is the maximal number $M(G)$ of real roots the system of $n-1$ nonlinear equations

$$\omega_i - \sum_{j \in N_G(i)} a_{i,j} \sin(\theta_i - \theta_j) = 0, \text{ for } i = 1, \ldots, n-1$$

could have?

Theorem (Baillieul–Byrnes, 1982)

$$M(G) \leq \binom{2(n-1)}{n-1}$$

Normalized volume of $\mathcal{A}_G$

What is $\binom{2(n-1)}{n-1}$?

If G is a graph with n vertices, then

$$\mathrm{Vol}(\mathcal{A}_G) \leq \mathrm{Vol}(\mathcal{A}_{K_n}) = \binom{2(n-1)}{n-1},$$

where K_n is a complete graph with n vertices and $\mathrm{Vol}(\cdot)$ is the normalized volume of a lattice polytope.

Theorem (Chen, 2017)

$$M(G) \leq \mathrm{Vol}(\mathcal{A}_G).$$

Normalized volume of $\mathcal{A}_G$

Problem

Compute $\mathrm{Vol}(\mathcal{A}_G)$ *or give a formula for* $\mathrm{Vol}(\mathcal{A}_G)$ *in terms of* G.

Theorem (Higashitanai–Jochemko–Michałek, 2019)

Let $K_{a,b}$ *be a complete bipartite graph with* a *and* b *vertices* ($a \leq b$). *Then*

$$\mathrm{Vol}(\mathcal{A}_{K_{a,b}}) = \sum_{i=0}^{a-1} \binom{2i}{i}\binom{a-1}{i}\binom{b-1}{i} 2^{a+b-2i-1}$$

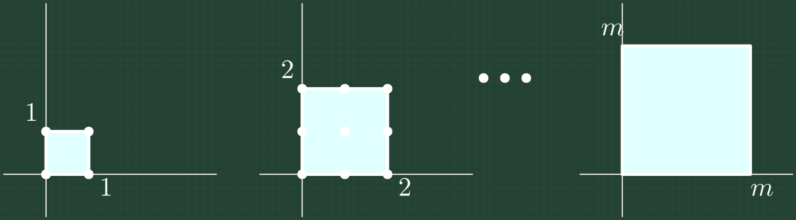
How do we get this volume from information of $K_{a,b}$?

Ehrhart theory

$\mathcal{P} \subset \mathbb{R}^n$: a lattice polytope of dimension d

$m\mathcal{P} = \{m\mathbf{x} : \mathbf{x} \in \mathcal{P}\}$: the mth dilated polytope of $\mathcal{P}$

$L_{\mathcal{P}}(m) := |m\mathcal{P} \cap \mathbb{Z}^n|$: the Ehrhart polynomial of $\mathcal{P}$



$L_{\mathcal{P}}(1) = 4$ $\quad$ $L_{\mathcal{P}}(2) = 9$ $\quad$ $L_{\mathcal{P}}(m) = (m+1)^2 = m^2 + 2m + 1$

Theorem (Ehrhart)

$L_{\mathcal{P}}(m)$ is a polynomial in m of degree d.

Ehrhart h^*-polynomials

We consider the generating series of $L_{\mathcal{P}}(m)$

$$\sum_{m\geq 0} L_{\mathcal{P}}(m)t^m = \frac{h^*(\mathcal{P},t)}{(1-t)^{d+1}}$$

$$h^*(\mathcal{P},t) = h^*_d t^d + h^*_{d-1}t^{d-1} + \cdots + h^*_1 t + h^*_0$$

$h^*(\mathcal{P},t)$ is a polynomial of degree $\leq d$, called the h^*-polynomial of $\mathcal{P}$.

All the coefficients have a combinatorial interpretation.
In particular,

- each $h^*_i \geq 0$ (Stanley)
- $h^*_0 = 1, h^*_1 = |\mathcal{P}\cap \mathbb{Z}^n| - (d+1)$ and $h^*_d = |\mathrm{relint}(\mathcal{P})\cap \mathbb{Z}^n|$

Moreover,

- $1 + h^*_1 + \cdots + h^*_d =$ Vol$(\mathcal{P})$: the normalized volume of $\mathcal{P}$

Palindromic polynomials and γ-polynomials

$f(t)=\sum_{i=0}^{d} a_i t^i \in \mathbb{Z}_{>0}[t]$: a palindromic polynomial
i.e., $a_i = a_{d-i}$ for any $1 \leq i \leq \lfloor d/2 \rfloor$

Then there exists a unique expression

$$f(t)=\sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1+t)^{d-2i}$$

$\gamma(t) := \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i \in \mathbb{Z}[t]$ is called the γ-polynomial of $f(t)$.

For example,

$$\begin{aligned} f(t) &= 1+6t+11t^2+6t^3+t^4 \\ &= (1+t)^4+2t+5t^2+2t^3 \\ &= (1+t)^4+2t(1+t)^2+t^2 \end{aligned}$$

$$\gamma(t)=1+2t+t^2$$

Properties of palindromic polynomials

We consider the following properties of palindromic polynomials:

(RR) $f(t)$ is real-rooted if all roots of $f(t)$ are real.

(GP) $f(t)$ is γ-positive if $\gamma_i \geq 0$ for all i.

(UN) $f(t)$ is unimodal if $a_0 \leq \cdots \leq a_k \geq \cdots \geq a_d$ with some k.

In general,

$$\text{(RR)} \Rightarrow \text{(GP)} \Rightarrow \text{(UN)}$$

If $f(t)$ is γ-positive, then

$$f(t) \text{ is real-rooted} \iff \gamma(t) \text{ is real-rooted}$$

For example, $f(t) = 1 + 6t + 11t^2 + 6t^3 + t^4$ is real-rooted
because $\gamma(t) = 1 + 2t + t^2$ is real-rooted.

Unimodality of $h^*(\mathcal{P}_G, t)$

A lattice polytope $\mathcal{P}$ is called Gorenstein if $h^*(\mathcal{P}, t)$ is palindromic. Especially, a Gorestein polytope with a unique relative interior lattice point is called reflexive.

Theorem (Bruns–Römer)

If $\mathcal{P}$ is Gorenstein and has a regular unimodular triangulation, then $h^*(\mathcal{P}, t)$ is unimodal.

Theorem (Matsui et al.)

$\mathcal{P}_G$ is reflexive and has a regular unimodular triangulation. In particular, $h^*(\mathcal{A}_G, t)$ is palindromic and unimodal.

When is $h^*(\mathcal{A}_G, t)$ γ-positive or real-rooted?

Complete bipartite graph

For a Gorenstein polytope $\mathcal{P}$, let $\gamma(\mathcal{P}, t)$ be the γ-polynomial of $h^*(\mathcal{P}, t)$.

Theorem (Higashitanai–Jochemko–Michałek, 2019)

Let $K_{a,b}$ be a complete bipartite graph with a and b vertices ($a \leq b$). Then

$$\gamma(\mathcal{A}_{K_{a,b}}, t) = \sum_{i=0}^{a-1} \binom{2i}{i} \binom{a-1}{i} \binom{b-1}{i} t^i.$$

Furthermore, $\gamma(\mathcal{A}_{K_{a,b}}, t)$ is real-rooted.

Perfectly matchable set polynomials

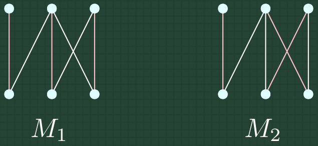
G : a graph on $[n]$

- A k-matching of G is a set of k pairwise non-adjacent edges of G.
- $m_G(k)$: the number of k-matchings of G
- $g(G,t) = \sum_{k\geq 0} m_G(k)t^k$: matching generating polynomial
- A k-matching of G is called a perfect matching if $2k = n$.
- A subset $S \subset [n]$ is called a perfectly matchable set if the induced subgraph of G on S has a perfect matching.
- $pm_G(k)$: the number of perfectly matchable sets with cardinality $2k$
- $p(G,t) = \sum_{k\geq 0} pm_G(k)t^k$: perfectly matchable set polynomial

Remark

Clearly, $pm_G(k) \leq m_G(k)$.

Example of k-matchings and perfectly matchable sets



3-matchings

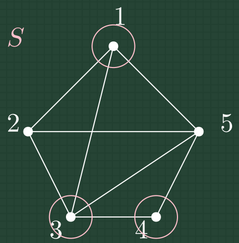
M_1 and M_2 are different 3-matchings.
M_1 and M_2 give a same perfectly matchable set.

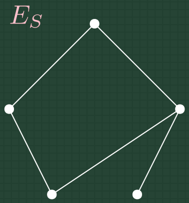
Cuts of graphs

G : a graph on $[n]$
Given a subset $S \subset [n]$,

$$E_S := \{e \in E(G) : |e \cap S| = 1\} \text{ : a cut of G.}$$

We identify E_S with the subgraph of G on the vertex set $[n]$ and the edge set E_S. In particular, E_S is a bipartite graph.



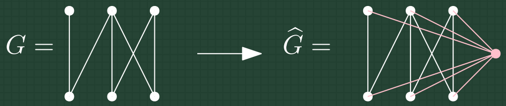


$\mathrm{Cut}(G)$: the set of all cuts of G.
Note that $|\mathrm{Cut}(G)| = 2^{n-1}$.

Suspension graphs

G : a graph on $[n]$
Let $\widehat{G}$ be the suspension of G, i.e., the connected graph on $[n+1]$ whose edge set is

$$E(\widehat{G}) = E(G) \cup \{\{i, n+1\} : i \in [n]\}.$$



Theorem (Ohsugi-T, 2021a)

Let G be a graph on $[n]$. Then one has

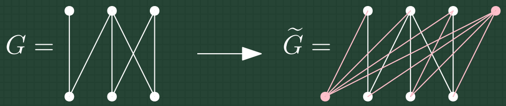
$$\gamma(\mathcal{A}_{\widehat{G}}, t) = \frac{1}{2^{n-1}} \sum_{H \in \mathrm{Cut}(G)} p(H, 4t).$$

In particular, $h^*(\mathcal{A}_{\widehat{G}}, t)$ is γ-positive.

Interior polynomials and perfectly mathable set polynomials

G : a bipartite graph with a bipartition $V_1 \cup V_2 = [n]$
Let $\widetilde{G}$ be the connected bipartite graph on $[n+2]$ whose edge set is

$$E(\widetilde{G}) = E(G) \cup \{\{n+1, n+2\}\} \cup \{\{i, n+1\} : i \in V_1\} \cup \{\{j, n+2\} : j \in V_2\}$$



Theorem (Ohsugi-T, 2021a)

Let G be a bipartite graph. Then one has

$$\gamma(\mathcal{A}_{\widetilde{G}}, t) = \gamma(\mathcal{A}_{\widehat{G}}, t).$$

In particular, $h^*(\mathcal{A}_{\widetilde{G}}, t)$ is γ-positive.

γ-positivity of $h^*(\mathcal{A}_G, t)$

Theorem (Ohsugi–T, 2021a)

$h^*(\mathcal{A}_G, t)$ is γ-positive if one of the following

- $G = \widehat{H}$ *for some graph* H *(complete graphs);*
- $G = \widetilde{H}$ *for some bipartite graph* H *(complete bipartite graphs);*
- G *is a cycle;*
- G *is an outerplaner bipartite graph.*

Conjecture (Ohsugi–T, 2021a)

$\gamma(\mathcal{A}_G, t)$ is γ-positive for any graph G.

Theorem (D'Alì–Kubitzke–Köhne–Venturello, 2023)

For any G, $\gamma_1, \gamma_2 \geq 0$.

Real-rootedness of $h^*(\mathcal{A}_G, t)$

Example

We consider a cycle C_n of length n. Then

$$\gamma(\mathcal{A}_{C_n}, t) = \sum_{i=0}^{(n-1)/2} \binom{2i}{i} t_i.$$

When $n = 5$, $\gamma(\mathcal{A}_{C_5}, t) = 1 + 2t + 6t^2$. Hence $\gamma(\mathcal{A}_{C_5}, t)$ is not real-rooted. Therefore, $h^*(\mathcal{A}_{C_5}, t)$ is not real-rooted.

Problem

When is $h^*(\mathcal{A}_G, t)$ real-rooted?

Wheel graphs

A wheel graph W_n is $\widehat{C_n}$.

Theorem (D'Alì–Delucchi–Michałek, 2022)

For $n \geq 3$, one has

$$\gamma(\mathcal{A}_{W_n}, t) = \begin{cases} (1+\sqrt{3})^n + (1-\sqrt{3})^n & \text{if } n \text{ is odd,} \\ (1+\sqrt{3})^n + (1-\sqrt{3})^n - 2 & \text{otherwise.} \end{cases}$$

Theorem (Ohsugi–T, 2021b)

For $n \geq 3$, one has

$$\gamma(\mathcal{A}_{W_n}, t) = \begin{cases} \frac{(1+\sqrt{1+8t})^n + (1-\sqrt{1+8t})^n}{2^t} & \text{if } n \text{ is odd,} \\ \frac{(1+\sqrt{1+8t})^n + (1-\sqrt{1+8t})^n}{2^n} - 2t^{\frac{n}{2}} & \text{otherwise.} \end{cases}$$

In particular, $\gamma(\mathcal{A}_{W_n}, t)$ is real-rooted.

Cactus graphs

A graph is called cactus if each edge belongs to at most one cycle.
For example, a cycle is cactus.
We consider the suspension of a cactus graph.

Theorem (Ohsugi-T, 2021b)

Let G be a cactus graph. Then

$$\gamma(\mathcal{A}_{\widehat{G}}, t) = g(G, 2t) + \sum_{R \in \mathcal{R}'_2(G)} (-2)^{c(R)} g(G - R, 2t)\, t^{\frac{|E(R)|}{2}},$$

where $\mathcal{R}'_2(G)$ is the set of all subgraphs of G consisting of vertex-disjoint even cycles, and $c(R)$ is the number of the cycles of R.

μ-polynomials

$\alpha(G,t) := \sum_{k\geq 0}(-1)^k m_k(G) t^{n-2k}$: the matching polynomial of G
Note that $\alpha(G,t) = t^n g(G, -t^{-2})$ and it is real-rooted.

Definition

Assume that G has r cycles $C_1, \ldots, C_r$. Let $\mathrm{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$ be a vector. Then the μ-polynomial of G is

$$\mu(G, \mathrm{s}, t) = \alpha(G,t) + \sum_{R \in \mathcal{R}_2(G)} (-2)^{c(R)} \alpha(G - R, t) \prod_{C_i \subset R} s_i.$$

Theorem (Gutman–Polansky, 1981)

Let G be a cactus graph. If $|s_i| \leq 1$ for all $1 \leq i \leq r$, then $\mu(G, \mathrm{s}, t)$ is real-rooted.

Theorem (Ohsugi–T, 2021b)

Let G be a cactus graph. Then $\gamma(\mathcal{A}_{\widehat{G}}, t)$ is real-rooted.

Connection to hypergeometric function?

From the explicit formulas for $\gamma(\mathcal{A}_{K_a}, t)$ and $\gamma(\mathcal{A}_{K_{a,b}}, t)$, we can notice the following:

$$\gamma(\mathcal{A}_{K_a}, t) = {}_2F_1\left(\frac{1-a}{2}, 1-\frac{a}{2}; 1; 4t\right)$$

$$\gamma(\mathcal{A}_{K_{a,b}}, t) = {}_3F_2\left(\frac{1}{2}, -a+1, -b+1; 1, 1; 4t\right)$$

Theorem (Driver–Jordaan, 2002)

${}_2F_1\left(\frac{1-a}{2}, 1-\frac{a}{2}; 1; 4t\right)$ *is real-rooted.*

Theorem (Driver–Jordaan–Martínez-Finkelshtein, 2007)

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Thank you for your attentions!