Intersection theory for regular holonomic GKZ systems

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August 18, 2023
Hypergeometric School 2023

Introduction

We study GKZ hypergeometric systems (A-hypergeometric systems) from the view point of Euler-type integral representations of solutions. We can use the intersection theory of twisted homology and cohomology groups.

Contents

- Hypergeometric integrals and GKZ systems
- Cycles corresponding to series solutions (by Matsubara-Heo)
- Their intersection numbers
- Applications
  - Signature of monodromy invariant Hermitian forms (Beukers-Verschoor's conjecture)
  - (Twisted period relations)

Gauss hypergeometric function

hypergeometric series

$$ {}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, $$

$$ (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. $$

integral representation

$$ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 x^{a-1}(1-x)^{c-a-1}(1-zx)^{-b} dx. $$

differential equation We set $\theta_z = z\frac{d}{dz}$.

$$ \Big(\theta_z(\theta_z + c - 1) - z(\theta_z + a)(\theta_z + b)\Big) \bullet {}_2F_1 = 0. $$

Hypergeometric integral

$$\int_0^1 x^{a-1}(1-x)^{c-a-1}(1-zx)^{-b}dx$$

(complex powers of polynomials (of degree 1))

$\downarrow$ generalization

$$\int_\Gamma h_1(x)^{-\gamma_1}\cdots h_k(x)^{-\gamma_k}\cdot x^c\ \frac{dx_1\wedge\cdots\wedge dx_d}{x_1\cdots x_d}$$

$$\left(\begin{array}{l} z^{(l)}=(z_1^{(l)},\ldots,z_{N_l}^{(l)}),\ c={}^t(c_1,\ldots,c_d),\\ h_l(x)=h_l(z^{(l)};x)=\sum_{j=1}^{N_l} z_j^{(l)}x^{\boldsymbol{a}^{(l)}(j)}\in\mathbb{C}[x_1^{\pm1},\ldots,x_d^{\pm1}],\\ \boldsymbol{a}^{(l)}(j)\in\mathbb{Z}^{d\times 1}\end{array}\right)$$

Example — ${}_2F_1$

$k=2$, $d=1$, $N_1=N_2=2$.

$$\boldsymbol{a}^{(1)}(1)=0,\quad \boldsymbol{a}^{(1)}(2)=1,\quad \boldsymbol{a}^{(2)}(1)=0,\quad \boldsymbol{a}^{(2)}(2)=1.$$

We set $(z_1^{(1)}, z_2^{(1)} \mid z_1^{(2)}, z_2^{(2)}) = (z_1, z_2 \mid z_3, z_4)$.

$$h_1(x) = h_1(z^{(1)};x) = z_1^{(1)} \cdot x^{\boldsymbol{a}^{(1)}(1)} + z_2^{(1)} \cdot x^{\boldsymbol{a}^{(1)}(2)} = z_1 + z_2 x,$$
$$h_2(x) = h_2(z^{(2)};x) = z_3 + z_4 x,$$

$$\int_\Gamma (z_1+z_2x)^{-\gamma_1}(z_3+z_4x)^{-\gamma_2}\ x^c\ \frac{dx}{x} \qquad \left(x = -\frac{z_1}{z_2}t\right)$$
$$= (-1)^c z_1^{-\gamma_1+c} z_2^{-c} z_3^{-\gamma_2} \int_{\Gamma'} (1-t)^{-\gamma_1}\left(1-\frac{z_1z_4}{z_2z_3}t\right)^{-\gamma_2} t^c \frac{dt}{t}$$

This corresponds to the integral representation of ${}_2F_1\left(c, \gamma_2, 1-\gamma_1+c; \frac{z_1z_4}{z_2z_3}\right)$.

$k=2,\ d=2,\ N_1=N_2=3.$

$$\boldsymbol{a}^{(1)}(1)=\begin{pmatrix}0\\0\end{pmatrix},\ \boldsymbol{a}^{(1)}(2)=\begin{pmatrix}1\\0\end{pmatrix},\ \boldsymbol{a}^{(1)}(3)=\begin{pmatrix}0\\1\end{pmatrix},$$
$$\boldsymbol{a}^{(2)}(1)=\begin{pmatrix}0\\0\end{pmatrix},\ \boldsymbol{a}^{(2)}(2)=\begin{pmatrix}-1\\0\end{pmatrix},\ \boldsymbol{a}^{(2)}(3)=\begin{pmatrix}0\\-1\end{pmatrix}.$$

We set $(z_1^{(1)},z_2^{(1)},z_3^{(1)}\mid z_1^{(2)},z_2^{(2)},z_3^{(2)})=(z_1,z_2,z_3\mid z_4,z_5,z_6)$.

$$h_1(x)=z_1\cdot x^{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)}+z_2\cdot x^{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}+z_3\cdot x^{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}=z_1+z_2x_1+z_3x_2,$$
$$h_2(x)=z_4\cdot x^{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)}+z_5\cdot x^{\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)}+z_6\cdot x^{\left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right)}=z_4+\frac{z_5}{x_1}+\frac{z_6}{x_2}.$$

$$\rightsquigarrow\int_\Gamma(z_1+z_2x_1+z_3x_2)^{-\gamma_1}\left(z_4+\frac{z_5}{x_1}+\frac{z_6}{x_2}\right)^{-\gamma_2}x_1^{c_1}x_2^{c_2}\frac{dx_1\wedge dx_2}{x_1x_2}$$

For a "cycle" Γ,

$$\int_\Gamma h_1(z^{(1)};x)^{-\gamma_1}\cdots h_k(z^{(k)};x)^{-\gamma_k}\cdot x^c\ \frac{dx_1\wedge\cdots\wedge dx_d}{x_1\cdots x_d}$$

is a solution to the GKZ system $M_A(\delta)$
(w.r.t. variables $z = \left(z^{(1)}|z^{(2)}|\cdots|z^{(k)}\right)$), where

$$A = \left(\begin{array}{ccc|ccc|c|ccc} 1 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 & & 0 & \cdots & 0 \\ \hline & & & & & & \ddots & & & \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 1 \\ \hline & A_1 & & & A_2 & & \cdots & & A_k & \end{array}\right)$$

$$(A_l = (\boldsymbol{a}^{(l)}(1),\dots,\boldsymbol{a}^{(l)}(N_l)) \in \mathbb{Z}^{d\times N_l})$$

$\delta = {}^t(\gamma_1,\dots,\gamma_k,c_1,\dots,c_d) \in \mathbb{C}^{(k+d)\times 1}$: generic.

GKZ hypergeometric system (A-hypergeometric system)

$A \in \mathbb{Z}^{n\times N}$ & $\delta \in \mathbb{C}^{n\times 1} \rightsquigarrow$ GKZ system $M_A(\delta)$

(a system of differential equations)

Fact ([Hotta, 1998])

For our A, $M_A(\delta)$ is regular holonomic.

$$\Delta_A = \text{(the convex hull of the column vectors of } A) \subset \mathbb{R}^n,$$
$$\mathrm{Vol}(A) = \text{(the normalized volume of } \Delta_A) \in \mathbb{N}$$

Fact ([GZK, 1989])

If δ is "generic", then $\mathrm{rank}(M_A(\delta)) = \mathrm{Vol}(A)$.

For a generic $z \in \mathbb{C}^N$, $Sol_{M_A(\delta),z}$ denotes the space of the local solutions around z. We have $\dim Sol_{M_A(\delta),z} = \mathrm{Vol}(A)$.

Example — ${}_2F_1$

$$\boldsymbol{a}^{(1)}(1) = 0,\ \boldsymbol{a}^{(1)}(2) = 1 \longrightarrow A_1 = (0\ 1)$$
$$\boldsymbol{a}^{(2)}(1) = 0,\ \boldsymbol{a}^{(2)}(2) = 1 \longrightarrow A_2 = (0\ 1)$$

$$A = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{l} k = 2,\ d = 1,\ n = k + d = 3 \\ A_1 = A_2 = (0\ 1), \\ (z_1^{(1)}, z_2^{(1)} \mid z_1^{(2)}, z_2^{(2)}) = (z_1, z_2 \mid z_3, z_4) \end{array}\right)$$

$$\delta = {}^t(\gamma_1, \gamma_2, c)$$

$$\int_\Gamma (z_1 + z_2 x)^{-\gamma_1} (z_3 + z_4 x)^{-\gamma_2}\ x^c\ \frac{dx}{x}$$ is a solution to $M_A(\delta)$.

$\mathrm{Vol}(A) = 2.$

If $v = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \in \mathbb{C}^{N\times 1}$ satisfies $Av = -\delta$, then the series

$$\varphi_v(z) = z^v \sum_{u \in \mathrm{Ker}_{\mathbb{Z}} A} \frac{1}{\Gamma(u+v+1)} z^u$$

$$\left(z^v = z_1^{v_1} \cdots z_N^{v_N}, \quad \Gamma(u+v+1) = \prod_{j=1}^{N} \Gamma(u_j + v_j + 1) \right)$$

is a (formal) solution to $M_A(\delta)$.

Fact ([GZK, 1989])

T : a regular triangulation of Δ_A
$\rightsquigarrow$ We can take $v_1, \ldots, v_r \in \mathbb{C}^{N\times 1}$ ($r = \mathrm{Vol}(A)$) such that $\{\varphi_{v_i}\}_{i=1}^{r}$ form a basis of $Sol_{M_A(\delta),z}$.

Example — ${}_2F_1$

$\{(123),(234)\}\cdots$ a regular triangulation of Δ_A

$$A=\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 \end{array}\right) \qquad \delta=\begin{pmatrix}\gamma_1\\ \gamma_2\\ c\end{pmatrix}=:\begin{pmatrix}\delta_1\\ \delta_2\\ \delta_3\end{pmatrix}$$

$$v=\begin{pmatrix}-\delta_1+\delta_3\\ -\delta_3\\ -\delta_2\\ 0\end{pmatrix}(\leftrightarrow (123)),\quad v'=\begin{pmatrix}0\\ -\delta_1\\ -\delta_1-\delta_2+\delta_3\\ \delta_1-\delta_3\end{pmatrix}(\leftrightarrow (234))$$

satisfy $Av=Av'=-\delta$.

$$\varphi_v=\frac{z_1^{-\delta_1+\delta_3}z_2^{-\delta_3}z_3^{-\delta_2}\cdot{}_2F_1\left(\delta_3,\delta_2,1-\delta_1+\delta_3;\frac{z_1z_4}{z_2z_3}\right)}{\Gamma(1-\delta_3)\Gamma(1-\delta_2)\Gamma(1-\delta_1+\delta_3)},$$

$$\varphi_{v'}=\frac{z_1^{-\delta_1+\delta_3}z_2^{-\delta_3}z_3^{-\delta_2}\cdot\left(\frac{z_1z_4}{z_2z_3}\right)^{\delta_1-\delta_3}{}_2F_1\left(\delta_1,\delta_1+\delta_2-\delta_3,1+\delta_1-\delta_3;\frac{z_1z_4}{z_2z_3}\right)}{\Gamma(1-\delta_1)\Gamma(1-\delta_1-\delta_2+\delta_3)\Gamma(1+\delta_1-\delta_3)}$$

$\varphi_v,\varphi_{v'}$ correspond to "well-known" solutions around $\frac{z_1z_4}{z_2z_3}=0$.

Twisted (co)homology group

$$\int_{\Gamma} h_1(x)^{-\gamma_1} \cdots h_k(x)^{-\gamma_k} \cdot x^c \, \frac{dx}{x} \qquad \left(\frac{dx}{x} = \frac{dx_1 \wedge \cdots \wedge dx_d}{x_1 \cdots x_d} \right)$$

The "cycle" Γ (twisted cycle) can be regarded as an element of

twisted homology group $H_d(X;\Phi)$

$\fallingdotseq \big\{$(domain of integration $\subset X(\leftarrow x$-space))

$$\& \left(\text{the branch of multi-val. func. } \Phi(x) = \prod_{l=1}^{k} h_l(x)^{-\gamma_l} \cdot x^c \right) \Big\}.$$

It is dual to

twisted cohomology group $H^d(X;\nabla)$

$= \big($ de-Rham cohomology w.r.t. $\nabla = d + d\log\Phi\wedge \qquad (d = d_x) \big)$.

The pairing between them gives hypergeometric integrals.

Fact ([GKZ, 1990])

$$H_d(X;\Phi) \xrightarrow{\sim} Sol_{M_A(\delta),z};\ \Gamma \longmapsto \int_{\Gamma} \Phi \, \frac{dx}{x}$$

Because of $H_d(X;\Phi) \xrightarrow{\sim} Sol_{M_A(\delta),z}$, there exists a cycle such that

$$\varphi_v(z) \qquad = \int_\Gamma \Phi \, \frac{dx}{x} = \int_\Gamma \prod_{i=1}^{k} h_l(z^{(l)};x)^{-\gamma_l} \cdot x^c \, \frac{dx}{x}.$$

(power series sol.)

$\longrightarrow$ How to construct such cycles?

Fact ([Matsubara-Heo; arXiv:1904.00565])

T: a regular triangulation of $\Delta_A \rightsquigarrow \{\varphi_{v_i}\}_{i=1}^r$: a basis of $Sol_{M_A(\delta),z}$
$\Longrightarrow$ We can construct cycles $\{\Gamma_i\}_{i=1}^r$ such that

$$\int_{\Gamma_i} \prod_{l=1}^{k} h_l(z^{(l)};x)^{-\gamma_l} \cdot x^c \, \frac{dx}{x} = \varphi_{v_i}(z).$$

An explicit construction is given.

Intersection forms

By replacing Φ with $\Phi^{-1} = 1/\Phi$, we consider
the twisted homology $H_d(X;\Phi^{-1})$ and
the twisted cohomology $H^d(X;\nabla^-)$ ($\nabla^- = d - d\log\Phi\wedge$).

Intersection form ([Kita-Yoshida, 1994], [Cho-Matsumoto, 1995])

- homology $I_h\colon H_d(X;\Phi) \times H_d(X;\Phi^{-1}) \to \mathbb{C}$
  $\cdots\cdots$ topological int. num. + information of branches.
- cohomology $I_c\colon H^d(X;\nabla) \times H^d(X;\nabla^-) \to \mathbb{C}$
  $\cdots\cdots$ taking the exterior product & integrating it on X.

Theorem 1 ([G.-Matsubara-Heo; 2022, Theorem 2.6])

The intersection matrix $(I_h(\Gamma_i, \Gamma_j))_{i,j=1,\dots,r}$ is explicitly evaluated.

Applications

- Signatures of invariant Hermitian forms
  (Beukers-Verschoor's conjecture)
- Twisted period relations (quadratic relations of HG series)

Monodromy, Invariant Hermitian form

By considering the analytic continuation of the solutions, we have the monodromy representation

$$\pi_1(\mathbb{C}^N - (\text{singular locus}), z) \to GL(Sol_{M_A(\delta),z})$$
$$\rho \mapsto \rho_* \text{ (: analytic conti. along } \rho).$$

For $\delta \in \mathbb{R}^{n\times 1}$, there exist a monodromy invariant Hermitian form (unique up to constant):

$$(\bullet,\bullet) : Sol_{M_A(\delta),z} \times Sol_{M_A(\delta),z} \to \mathbb{C},$$
$$(\rho_* f, \rho_* g) = (f, g) \quad (\forall \rho \in \pi_1).$$

We consider its signature.

Beukers-Verschoor's conjecture

T: a regular triangulation of Δ_A,
$\rightsquigarrow \{\varphi_{v_i}\}_{i=1}^{r}$: a basis of $Sol_{M_A(\delta),z}$
$(v_i = {}^t(v_{i1}, \ldots, v_{iN}) \in \mathbb{R}^{N\times 1} \leftrightarrow \sigma \in T \ (\sigma \subset \{1, \ldots, N\})\ )$

Beukers-Verschoor's conjecture (cf. [Verschoor; 2022])

The signature of the monodromy invariant Hermitian form is given by

$$\left\{ \operatorname{sgn}\left(\prod_{j\in\sigma} \sin \pi v_{ij} \right) \right\}_{i=1,\ldots,r}$$

Example — ${}_2F_1$

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$v = \begin{pmatrix} -\delta_1 + \delta_3 \\ -\delta_3 \\ -\delta_2 \\ 0 \end{pmatrix} \quad (\leftrightarrow (123) \in T)$$

$$\Rightarrow Av = Av' = \begin{pmatrix} -\delta_1 \\ -\delta_2 \\ -\delta_3 \end{pmatrix}.$$

$$v' = \begin{pmatrix} 0 \\ -\delta_1 \\ -\delta_1 - \delta_2 + \delta_3 \\ \delta_1 - \delta_3 \end{pmatrix} \quad (\leftrightarrow (234) \in T)$$

We set

$$\alpha := \sin \pi(-\delta_1 + \delta_3) \cdot \sin \pi(-\delta_3) \cdot \sin \pi(-\delta_2),$$
$$\alpha' := \sin \pi(-\delta_1) \cdot \sin \pi(-\delta_1 - \delta_2 + \delta_3) \cdot \sin \pi(\delta_1 - \delta_3).$$

Then the signature is given by $(\mathrm{sgn}(\alpha), \mathrm{sgn}(\alpha'))$.

Theorem 2 ([G.-Matsubara-Heo; 2022, Theorem 2.12])

The signature of the monodromy invariant Hermitian form is given by

$$\left\{ \operatorname{sgn} \left(\prod_{j \in \sigma} \sin \pi v_{ij} \right) \right\}_{i=1,\ldots,r}$$

Proof.

$$H_d(X;\Phi) \overset{\sim}{\longleftrightarrow} Sol_{M_A(\delta),z}$$

$$\text{(intersection form)} \longleftrightarrow \text{(inv. Herm. form)}$$

We can compute the eigenvalues of the intersection matrix in Theorem 1. □

Twisted period relations

Proposition

$\{\Gamma_j^{\pm}\}_{j=1}^r$: bases of $H_d(X;\Phi^{\pm 1})$
$\{\psi_i^{\pm}\}_{i=1}^r$: bases of $H^d(X;\nabla^{\pm})$

$$H=\Big(I_h(\Gamma_i^+,\Gamma_j^-)\Big)_{i,j=1,\dots,r},\quad C=\Big(I_c(\psi_i^+,\psi_j^-)\Big)_{i,j=1,\dots,r},$$

$$\Pi_+=\Big(\int_{\Gamma_j^+}\Phi\psi_i^+\Big)_{i,j=1,\dots,r},\quad \Pi_-=\Big(\int_{\Gamma_j^-}\Phi^{-1}\psi_i^-\Big)_{i,j=1,2}.$$

We have

$$C=\Pi_+\ {}^tH^{-1}\ {}^t\Pi_-.$$

cohomology intersection number $=$ sum of product of HG series

By Theorem 1, we obtain an explicit form of RHS.

Summary

- A hypergeometric integral (an integral of the product of complex powers of Laurent polynomials) is a solution to a GKZ system.
- Matsubara-Heo constructed cycles (domains of integration) which correspond to series solutions to the GKZ system.
- We evaluate their intersection numbers.
- By using our intersection matrix, we can prove Beukers-Verschoor's conjecture.

Thank you for your kind attention!

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