DELIGNE-MOSTOW-TERADA CLASSIFICATION, K3 SURFACES, AUTOMORPHIC FORMS, JACOBI-THOMAE IDENTITY (HYPERGEOMETRIC SCHOOL 2022.9.18)

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1. Introduction

This is a text for the hypergeometric school. In this text, we try to present uniform method of treat a period map to complex ball arising from Appell-Lauricella hypergeometric function. We treat important classes classified by Terada-Deligne-Mostow and Mostow. We try to give a course related to the following topics. We appologize that only references of the topics (5) and (6) are mentioned.

(1) Appell-Lauricella hypergeometric function. Period map to a complex ball.
(2) Deligne-Mostow-Terada classfication
(3) Automorphic function Theta function of curves with cyclic group action
(4) Individual results
(5) Period of K3 surface with cyclic group action.
(6) Thomae's formula

We will see that only several cases are known in detail. In the list of Terada-Deligne-Mostow and Mostow, there are many cases to be done, including period maps for non-principal Prym variety.

1.1. Preliminary to Hodge structure and its polarizations.

In this subsection, we introduce the definition of Hodge structures and its polarization. Let A be a ring contained in $\mathbf{R}$. We may assume that $A=\mathbf{Z}$, $\mathbf{Q}$ or $\mathbf{R}$.

Definition 1.1. *A pair $H=(H_A,F^\bullet)$ consisting of*

(1) *A lattice H_A of finite type over A, and*
(2) *A decreasing filtration $F^\bullet$ on $H_{\mathbf{C}}$*

satisfying the condition

$$H_{\mathbf{C}}=\oplus_{i+j=n}(F^i\cap\overline{F^j})$$

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is called a A-Hodge structure of weight i. Here $H_{\mathbf{C}} = H_A \otimes_A \mathbf{C}$ and the conjugation is that with respect to $H_{\mathbf{R}} = H_A \otimes_A \mathbf{R}$. The subspace $H^{ij} = F^i \cap \overline{F^j}$ is called the Hodge (i,j)-component of H.

Let S be a real algebraic group defined by

$$S = \{(a,b) \in \mathbf{R}^2 \mid a^2 + b^2 \neq 0\}.$$

An element $z = \alpha + \beta i$ in $\mathbf{C}^\times$ defines an $\mathbf{R}$-valued point (α, β) of S, which is denoted by z. By this correspondence the set $S(\mathbf{R})$ of $\mathbf{R}$-valued points of S is identified with $\mathbf{C}^\times$. For an element $z = \alpha + \beta i \in \mathbf{C}^\times = S(\mathbf{R})$, $\alpha - \beta i$ in $S(\mathbf{R})$ is denoted by $\overline{z}$. The condition (2) is equivalent to the existence of the action ρ of S on $H_{\mathbf{R}}$ such that

$$\rho(z) \mid H^{ij} = \text{the multiplication of } z^i \overline{z^j}$$

on the scalar extension $H_{\mathbf{C}} = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$.

1.2. **Polarization of Hodge structure.** Let $H = (H_{\mathbf{Q}}, F^i)$ be a polarized Hodge strucutre of weight n. The Hodge decomposition defines an action $\rho : S \to Aut(H_{\mathbf{R}})$. The action of $C = \rho(i)$ is called the Weil operator of $H_{\mathbf{R}}$.

Definition 1.2. *A $\mathbf{Q}$-valued bilinear form $\langle\ \ \rangle$ on $H_{\mathbf{Q}}$ is called a polarization if*

(1) *for $x \in H^{i,j}$, $y \in H^{k,l}$, $\langle x, y\rangle = 0$ if $i + k \neq n$, and*
(2) *$\langle x, C(y)\rangle$ is symmetric and positive definite on $H_{\mathbf{R}}$.*

Here the scalar extension of the pairing $\langle\ \ \rangle$ to $H_{\mathbf{R}}$ and $H_{\mathbf{C}}$ are also denoted by $\langle\ \ \rangle$.

Definition 1.3. *This definition is a little bit different from [Weil2]. The value of the pairing is in $\mathbf{Q}$ not in $\mathbf{Q}(-n)$.*

Since we have

$$\langle x, y\rangle = \langle C(x), C(y)\rangle = \langle y, C^2(x)\rangle = (-1)^n\langle y, x\rangle,$$

the pairing is symmetric if n is even and skew symmetric if n is odd.

1.2.1. *The weight one case.* We consider the case where the weight $n = 1$. In this case, the pairing $\langle\ \ \rangle$ is non-degenerate skew symmetric form. Therefore there exists a symplectic base A_i, B_i satsifying

$$\langle A_i, B_j\rangle = \delta_{ij}, \langle A_i, A_j\rangle = \langle B_i, B_j\rangle = 0.$$

We can chosse a basis $\{\omega_i\}$ of H^{10} such that

$$\langle \omega_i, B_j\rangle = \delta_{ij}. \tag{1.1}$$

This base is called a noralized basis of H^{10}.

Proposition 1.4. *We set*

$$\langle \omega_i, A_j\rangle = \tau_{ij} \tag{1.2}$$

and set $\tau = (\tau_{ij})_{ij}$. Then

(1) *The matrix τ is a symmetric matrix.*
(2) *The matrix $\mathrm{Im}(\tau)$ is a positive definite matrix.*

Proof. By (1.1) and (1.2), we have

$$\omega_i = A_i - \sum_k \tau_{ik} B_k$$

Since $\langle \omega_i, \omega_j \rangle = 0$, we have

$$0 = \langle A_i - \sum_k \tau_{ik} B_k, A_j - \sum_l \tau_{jl} B_l \rangle = -\tau_{ji} + \tau_{ij}$$

$H_{\mathbf{R}}$ is generated by $\{\gamma_i = \omega_i + \overline{\omega_i}\}$. Since $C(\gamma_i) = i\omega_i - i\overline{\omega_i}$, we have

$$\begin{aligned}\langle \gamma_i, C(\gamma_j) \rangle =& \langle \omega_i, -i\overline{\omega_j} \rangle + \langle \overline{\omega_i}, i\omega_j \rangle \\ =& \langle A_i - \sum_k \tau_{ik} B_k, -iA_j + i \sum_k \overline{\tau_{jk}} B_k \rangle \\ &+ \langle A_i - \sum_k \overline{\tau_{ik}} B_k, iA_j - i \sum_k \tau_{jk} B_k \rangle \\ =& i\overline{\tau_{ji}} - i\tau_{ij} - i\tau_{ji} + i\overline{\tau_{ij}} \\ =& -2i(\tau_{ij} - \overline{\tau_{ij}}) = 2\,\mathrm{Im}(\tau_{ij}).\end{aligned}$$

Therefore $\mathrm{Im}(\tau)$ is positive definite. □

Definition 1.5. *We define the Siegel upper half space $\mathfrak{S}_g$ of degree g by*

$$\mathfrak{S}_g = \{\tau \in M_g \mid {}^t\tau = \tau, \mathrm{Im}(\tau) \text{ is positive definite}\}.$$

2. Appell Lauricella hypergeometric function

Appell-Lauricella hypergeometric function is defined as the following integral:

$$F(\lambda_1, \dots, \lambda_d) = \int_1^\infty x^{-\mu_0}(1-x)^{-\mu_1} \prod_{i=1}^d (x - \lambda_i)^{-\mu_i} dx$$

We assume that $0 < \mu_i < 1$. We define μ_∞ as the order of the pole of the integrand at infinity. Then we have

$$\sum \mu_i + \mu_\infty = 2.$$

We assume that $0 < \mu_\infty < 1$. The function F satisfies a linear differential equation of rnak $d+1$, which is called Appell-Lauricella hypergeometric equation.

The function F is "a period of cohomology" of a curve C. Let C be a covering of $\mathbf{P}^1$ defined by

$$C : y^n = x^{m_0}(1-x)^{m_1} \prod_{i=2}^{d+1} (x - \lambda_i)^{m_i} \tag{2.1}$$

where n is the smallest common denominator of μ_i and $m_i = -n\mu_i$. On the curve C the cyclic group μ_n of n-the roots of unity. An element $\zeta \in \mu_n$ acts on C by

$$g_\zeta : y \mapsto \zeta^{-1} y$$

By Hodge theory, the cohomology $H^1(C, \mathbf{Z})$ is equipped with a natural Hodge structures:

$$H^1(C, \mathbf{Z}) \otimes \mathbf{C} = H^{01}(C) \oplus H^{10}(C), \quad H^{10} \simeq H^0(C, \Omega^1), \quad H^{01} \simeq H^1(C, \mathcal{O})$$

The action of μ_n induces an action of μ_n on $H^{10}(C)$ and $H^{01}(C)$. We set

$$\begin{aligned} H^1(C, \mathbf{C})(\chi) &= \{v \in H^1(C, \mathbf{C}) \mid g_\zeta^* v = \zeta v\} \\ H^{01}(C)(\chi) &= \{v \in H^{01}(C) \mid g_\zeta^* v = \zeta v\} \\ H^{10}(C)(\chi) &= \{v \in H^{10}(C) \mid g_\zeta^* v = \zeta v\} \end{aligned}$$

Then we have

$$H^1(C, \mathbf{C})(\chi) = H^{01}(C)(\chi) \oplus H^{10}(C)(\chi)$$

and by holomorphic Lefschetz theorem, we have

$$\begin{aligned} \dim_{\mathbf{C}} H^{10}(C)(\chi) &= \sum \mu_i - 1 = 2 - 1 = 1, \\ \dim_{\mathbf{C}} H^{01}(C)(\chi) &= \sum (1 - \mu_i) - 1 = d + 3 - 2 - 1 = d. \end{aligned} \tag{2.2}$$

2.0.1. *Moduli spaces and a marked curve.* Let $\mathcal{M}$ be the naive moduli space

$$\mathcal{M} = \{(\lambda_1, \dots, \lambda_d) \in \mathbf{C}^d \mid \lambda_i \neq \lambda_j.\}$$

A reference curve C_0 is a curve corresponding to a chosen base point of the moduli space. Let $K = \mathbf{Q}(\mu_n)$ be a subfield of $\mathbf{C}$ generated by μ_n over $\mathbf{Q}$. Then $\dim_{\mathbf{Q}} K = \varphi(n)$, where $\varphi(n)$ is the Eular function of n. Let χ be the natural inclusion $\mu_n \to \mathbf{C}^\times$, which defines a character of μ_n. For an element $t \in (\mathbf{Z}/n\mathbf{Z})^\times$, we have another character $\chi^t : \mu_n \to \mathbf{C}^\times$. Then we have an isomorphism

$$H^1(C, K)(\chi^t) \otimes_K \mathbf{C} \simeq H^1(C, \mathbf{C})(\chi^t).$$

We define the primitive part of $H^1_{\text{prim}}(C, \mathbf{Q})$ by the following $\mathbf{Q}$-subvector space of $H^1(C, \mathbf{Q})$:

$$H^1_{\text{prim}}(C, \mathbf{Q}) = \Big(\oplus_{t \in (\mathbf{Z}/n\mathbf{Z})^\times} H^1(C, K)(\chi^t) \Big) \cap H^1(C, \mathbf{Q}).$$

Similarly, we define the prmitive part $H_1^{\text{prim}}(C, \mathbf{Q})$ of the homology $H_1(C, \mathbf{Q})$. We define the lattice $H_1^{\text{prim}}(C, \mathbf{Z})$ of $H_1^{\text{prim}}(C, \mathbf{Q})$ by

$$H_1^{\text{prim}}(C, \mathbf{Z}) = H_1^{\text{prim}}(C, \mathbf{Q}) \cap H_1(C, \mathbf{Z}).$$

A polarization of $H_1^{\text{prim}}(C, \mathbf{Z})$ is induced from that of $H_1(C, \mathbf{Z})$. Note that the polarization on $H_1^{\text{prim}}(C, \mathbf{Z})$ is not necessarily principal unless $H_1^{\text{prim}}(C, \mathbf{Q}) = H_1(C, \mathbf{Q})$.

For a moduli problem of curves with the above structures, we often choose a typical curve C_0, which is called a reference curve. For a reference curve C_0, we have a lattice $H_{1,\mathbf{Z}}^{\text{prim}} = H_1^{\text{prim}}(C_0, \mathbf{Z})$ with the polarization and the action of μ_n.

Definition 2.1. (1) *Let C be a curve C which is a covering of $\mathbf{P}^1$. An isomorphism*

$$m : H_1^{\mathrm{prim}}(C, \mathbf{Z}) \simeq H_{1,\mathbf{Z}}^{\mathrm{prim}}$$

compatible with the polarization and the μ_n-action is called a marking of C.

(2) *A pair (C, m) of a curve C and its marking is called a marked curve. Let $\mathcal{M}_{\mathrm{mk}}$ be the set of the marked curves. It is called the moduli space of marked curves.*

The K-vector space $H_K^1(\chi)$ and $\mathbf{C}$-vector space $H_{\mathbf{C}}^1(\chi)$ are defined as follows:

$$m : H_K^1(\chi) = H^1(C_0, K)(\chi), H_{\mathbf{C}}^1(\chi) = H^1(C_0, \mathbf{C})(\chi).$$

Definition 2.2. *We introduce a Hermatian form h on $H^1(C, \mathbf{C})(\chi)$ and $H_{\mathbf{C}}^1(\chi)$ by*

$$h(\xi, \eta) = 2\langle \xi, \overline{\eta} \rangle (\overline{\omega} - \omega). \tag{2.3}$$

We set $q(\xi) = h(\xi, \xi)$.

A marking m induces an isomorphism

$$m : H^1(C, K)(\chi) \simeq H_K^1(\chi)$$

compatible with the polarizations and the μ_n-actions.

2.1. **Period map for a marked curve.**

2.1.1. *Hermitian form on $H_{\mathbf{C}}^1(\chi)$.* Let (C, m) be a marked curve. Then we have a sequence of homomorphisms:

$$H^{10}(C)(\chi) \subset H^1(C, K)(\chi) \otimes_K \mathbf{C} \xrightarrow{m \otimes \mathbf{C}} H_K^1 \otimes_K \mathbf{C} \simeq H_{\mathbf{C}}^1. \tag{2.4}$$

To define a period map to a complex ball using this homomorphisms, we use a hermatian form q on $H_{\mathbf{C}}^1$

Let σ be a generator of μ_n and $\omega = \chi(\sigma)$. We assume that $Im(\omega) > 0$. We set

$$H_{\mathbf{R}}^1(\chi, \overline{\chi}) = H_{\mathbf{R}}^1 \cap \left(H_{\mathbf{C}}^1(\chi) \oplus H_{\mathbf{C}}^1(\overline{\chi}) \right)$$

Then the map

$$\iota : H_{\mathbf{C}}^1(\chi) \to H_{\mathbf{R}}^1(\chi, \overline{\chi}) : \xi \mapsto x = \xi + \overline{\xi}$$

is an isomorphism of $\mathbf{R}$-vector spaces. We introduce an action of $\mathbf{R}[\sigma]$ on $H_{\mathbf{R}}^1(\chi, \overline{\chi})$ by

$$\sigma(\xi + \overline{\xi}) = \omega\xi + \overline{\omega}\overline{\xi}.$$

The isomorphism ι is compatible with the action of $\mathbf{C} = \mathbf{R}[\omega]$ and $\mathbf{R}[\sigma]$. By simple computation, we have the following proposition.

Proposition 2.3. (1) *Via the isomorphism ι, the Hermitian form h is identified with the Hermitian form on $H_{\mathbf{R}}^1(\chi, \overline{\chi})$ defined by*

$$h^*(x, y) = \langle x, (\sigma - \sigma^{-1})y \rangle + \langle x, y \rangle (\overline{\omega} - \omega)$$

(2) *Since* $\langle \xi, \eta \rangle = \langle \overline{\xi}, \overline{\eta} \rangle = 0$ *for* $\xi, \eta \in H^1_{\mathbf{C}}(\chi)$, *the space* $H^{10}(C)(\chi)$ *is positive definite and* $H^{01}(C)(\chi)$ *is negarive definite subspace of* $H^1(C, \mathbf{C})(\chi)$. *The space* $H^{10}(C)(\chi)$ *and* $H^{01}(C)(\chi)$ *is orthogonal complement to each other.*

Proof. (1) We have

$$\begin{aligned} h^*(x,y) =& \langle \xi + \overline{\xi}, (\sigma - \sigma^{-1})(\eta + \overline{\eta}) \rangle + \langle \xi + \overline{\xi}, \eta + \overline{\eta} \rangle (\overline{\omega} - \omega) \\ =& \langle \xi + \overline{\xi}, \omega\eta + \overline{\omega\eta})) \rangle - \langle \xi + \overline{\xi}, \overline{\omega}\eta + \omega\overline{\eta} \rangle \\ &+ \langle \xi + \overline{\xi}, \eta + \overline{\eta} \rangle (\overline{\omega} - \omega) \\ =& 2\langle \xi, \overline{\eta} \rangle (\overline{\omega} - \omega) \end{aligned}$$

(2) Since $\omega - \overline{\omega} = r\mathbf{i}$ with $r > 0$, for $\xi \in H^{10}(\chi)$ and $x = \xi + \overline{\xi}$,

$$h^*(x,x) = \langle x, (\sigma - \sigma^{-1})x \rangle + \langle x, x \rangle (\overline{\omega} - \omega) = r\langle x, Cx \rangle > 0$$

where C is the Weil operator. If $\xi \in H^{01}(\chi)$, then $h^*(x,x) = -r\langle x, Cx \rangle < 0$. $\square$

Via the homomorphism

$$H^{10}(C)(\chi) \subset H^1(C,K)(\chi) \otimes \mathbf{C} \xrightarrow{m \otimes \mathbf{C}} H^1_{\mathbf{C}}(\chi)$$

the space $H^{10}(C)(\chi)$ is a one dimentional subspace in $H^1_{\mathbf{C}}(\chi)$, which defines a point $\pi(C,m)$ in $P(H^1_{\mathbf{C}}(\chi)) \simeq \mathbf{P}^d$. By the positivity of Proposition 2.3, the point $\pi(C,m)$ is contained in the domain $D \subset \mathbf{P}^d$ defined by

$$D = \{[v] \in \mathbf{P}(H^1_{\mathbf{C}}(\chi)) \mid q(v) > 0\}.$$

Here D is the symmetric domain associated to the unitary group $U(H^1_{\mathbf{C}}(\chi), q) = U(1,d)$ of V_C defined by

$$U(H^1_{\mathbf{C}}(\chi), q) = \{g \in \mathrm{Aut}(H^1_{\mathbf{C}}(\chi)) \mid q(\xi) = q(g(\xi)) \text{ for all } \xi \in H^1_{\mathbf{C}}(\chi)\}$$

It is easy to see that D is a complex ball. Thus we have a map

$$\pi : \mathcal{M}_{mk} \to D : (C,m) \mapsto \pi(C,m),$$

which is called the period map. Thus we have the following diagram

$$\begin{array}{ccccc} \mathcal{M}_{\mathrm{mk}} & \longrightarrow & D & \longrightarrow & \mathbf{P}^d \\ \downarrow & & & & \\ \mathcal{M} & & & & \end{array}$$

2.2. **Deligne-Mostow-Terada classification.** Deligne and Mostow gives a sufficient condition such that $\mathcal{M}_{mk} \to D$ is almost an isomorphism.

Theorem 2.4 (Deligne-Mostow-Terada). *If the condition*

$$\text{(INT)} : (1 - \mu_i - \mu_j)^{-1} \in \mathbf{Z} \text{ for any } i \neq j \text{ and } \mu_i + \mu_j < 1$$

is satisfied, then $\pi : \mathcal{M}_{\mathrm{mk}} \to D$ *is almost isomorphic. Moreover, suitable compactification induces an isomorphism. Actually* π *is an isomorphism by adding stable locus.*

Here is the list of indexes μ satisfying the above theorem. Let N be the number of branch locus. We consider the case $N > 4$. Schwarz triangular group is classified as $N = 4$. The following lists are from [Td], [DM].

(1) $N = 5$

TABLE 1. $N = 5$

type no	d	m_0	m_1	m_2	m_1	m_2
1	3	2	1	1	1	1
2	4	2	2	2	1	1
3	4	3	2	1	1	1
4	5	2	2	2	2	2
5	6	3	3	2	2	2
6	6	3	3	3	2	1
7	6	4	3	2	2	1
8	6	5	2	2	2	1
9	8	4	3	3	3	3
10	8	5	5	2	2	2
11	8	6	3	3	3	1
12	9	4	4	4	4	2
13	10	7	4	4	4	1
14	12	5	5	5	5	4
15	12	6	5	5	4	4
16	12	6	5	5	5	3
17	12	7	5	4	4	4
18	12	7	6	5	3	3
19	12	7	7	4	4	2
20	12	8	5	5	3	3
21	12	8	5	5	5	1
22	12	8	7	3	3	3
23	12	10	5	3	3	3
24	15	8	6	6	6	4
25	18	11	8	8	8	1
26	20	14	11	5	5	5
27	24	14	9	9	9	7

(2) $N = 6$

Table 2. $N=6$

type no	d	m_0	m_1	m_2	m_3	m_4	m_5
1	3	1	1	1	1	1	1
2	4	2	2	1	1	1	1
3	4	3	1	1	1	1	1
4	6	3	2	2	2	2	1
5	8	3	3	3	3	3	1
6	12	5	5	5	3	3	3
7	12	7	5	3	3	3	3

(3) $N=7$

Table 3. $N=7$

type no	d	m_0	m_1	m_2	m_3	m_4	m_5	m_6
1	4	2	1	1	1	1	1	1

(4) $N=8$

Table 4. $N=8$

type no	d	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7
1	4	1	1	1	1	1	1	1	1

Moreover Mostow relaxed the above condition to the "half integral conditions" to get wider class. (See the list of [Mo].)

3. Arithmetic case and construction of Automorphic forms

In the Deligne-Mostow classifiction, there are criterion of the arithmeticity of the monodory group.

Definition 3.1. *We define a group $U(H^{\mathrm{prim}}_{1,\mathbf{Z}})$ as the set of elements $g \in \mathrm{Aut}(H^{\mathrm{prim}}_{1,\mathbf{Z}})$ preserving the polarizations $\langle \ \ \rangle$ and μ_n-actions.*

Proposition 3.2. *If the following condition is satisfied, then the group $U(H^{\mathrm{prim}}_{1,\mathbf{Z}})$ is covolume finite discrete subgroup of $U(H^{\mathrm{prim}}_{1,\mathbf{C}})$.*

(Arith)*: For all $t \in (\mathbf{Z}/d\mathbf{Z})^\times - \{1,-1\}$, we have*

$$\sum \langle t\mu_i \rangle - 1 = 0$$

or

$$\sum \langle -t\mu_i \rangle - 1 = d+1.$$

A group with the above properties is called arithmetic lattice.

In the following, we treat the cases of arithmetic lattices. By transpose of the action, the group $U(H^{\mathrm{prim}}_{1,\mathbf{Z}})$ acts on the domain D.

Definition 3.3. *A lattice L containg $H_{1,\mathbf{Z}}^{\text{prim}}$ such that $\# \mid L/H_{1,\mathbf{Z}}^{\text{prim}} \mid < \infty$ is called a superlattice of $H_{1,\mathbf{Z}}^{\text{prim}}$. If the restriction of the polarization to a super lattice is integral and principal, it is called a principally polarized superlattice. Two super lattices L_1, L_2 are associated if there exists $g \in U(H_{1,\mathbf{Z}}^{\text{prim}} \otimes \mathbf{Q})$ such that $g(H_{1,\mathbf{Z}}^{\text{prim}}) = H_{1,\mathbf{Z}}^{\text{prim}}$ and $gL_1 = L_2$. For any $H_{1,\mathbf{Z}}^{\text{prim}}$, there exists a principally polarized superlattice. Even though $H_{1,\mathbf{Z}}^{\text{prim}}$ is stable under the action of μ_n, a principally polarized superlattice may not by stable under the action of μ_n.*

Problem 3.4. *Are there μ_n-stable principally polarized superlattice ? This question is trivial for $H_1^{\text{prim}}(C, \mathbf{Z}) = H_1(C, \mathbf{Z})$. We have an affirmative answer for* $(1/4^8)$.

3.1. **Modular embedding for μ_n-action and Theta functions.** In this section, we assume that

$$\deg_{\mathbf{Q}} \mathbf{Q}(\zeta_n) = 2 \quad \text{and} \quad H^1(C, \mathbf{Q}) = H^1_{\text{prim}}(C, \mathbf{Q}). \tag{3.1}$$

For a reference curve C_0, we set $H_{\mathbf{Z}}^1 = H^1(C_0, \mathbf{Z})$, etc. In this case, we have

$$H^1 \otimes K = H_K^1(\chi) \oplus H_K^1(\overline{\chi}).$$

In (2.3), we introduced a skew symmetric form $h(v, w)$ on $H^1(C, K)(\chi)$. Then by Proposition 2.3, the space $H^{10}(C)(\chi)$ is one dimensional and positive definite. The spaces $H^{10}(C)(\chi)$ and $H^{01}(C)(\chi)$ are orthogonal complement to each other.

3.2. **Modular embedding.** On the dual lattice $H_{1,\mathbf{Z}}$ a polarization is induced by that of $H_{\mathbf{Z}}^1$. Let A_i, B_i be a symplectic basis of $H_{1,\mathbf{Z}}$, i.e. a basis satisfying

$$\langle A_i, B_j \rangle = \delta_{ij}, \langle A_i, A_j \rangle = \langle B_i, B_j \rangle = 0.$$

A base $\{\omega_i\}$ of $F^1 H_{\mathbf{C}}^1 = H^{10}$ is normalized if

$$\langle \omega_i, B_j \rangle = \delta_{ij}$$

We set $\langle \omega_i, A_j \rangle = \tau_{ij}$. Then we have

$$\begin{pmatrix} A_1 \\ \vdots \\ B_g \end{pmatrix} (\omega_1, \dots \omega_g) = \begin{pmatrix} \tau \\ I_g \end{pmatrix}$$

By Proposition 1.4, $\tau = (\tau_{ij})_{ij}$ is an element of Siegel upper space $\mathfrak{S}_g$.

Let $[v_1]$ be an element in $\mathbf{P}(H_{\mathbf{C}}^1(\chi))$ such that $h(v_1, v_1) > 0$. The element v_1 determines a polarized Hodge structure on $H_{\mathbf{Z}}^1$ with an action of μ_n such that v_1 is a generator of $H^{10}(\chi)$. The Hodge structure is given as follows.

(1) Under the hermitian form $h(\ ,\ )$ on $H_{\mathbf{C}}^1(\chi)$, $H^{01}(\chi)$ is defined as the space vertical to v_1.
(2) The subspace H^{10} is defined as the dirct sum of $H^{10}(\chi)$ and $\overline{H^{01}(\chi)} = H^{10}(\overline{\chi})$.

By this correspondence, we have a embedding of symmetric domain

$$D \to \mathfrak{S}_g. \tag{3.2}$$

Now we give the explicit formula of the above embedding. The action of the generator ζ of the covering transformation group induces the action σ_* and σ^* on the space H_1 and $H^1_{\mathbf{C}}$. For a differential form $\omega \in H^1_{\mathbf{C}}$ and $\Gamma \in H_1$, we have

$$\int_{\sigma_*\Gamma} \omega = \int_{\Gamma} \sigma^* \omega.$$

We define a matrix

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sp(2g, \mathbf{Z})$$

by

$${}^t(\sigma_* A_1, \dots, \sigma_* A_g, \sigma_* B_1, \dots, \sigma_* B_g) = \sigma\ {}^t(A_1, \dots, A_g, B_1, \dots, B_g)$$

where $Sp(2g, \mathbf{Z})$ is an element satisfying

$$\sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

An element $\eta \in H^1_{\mathbf{C}}$ is identified with $v = {}^t(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g) \in \mathbf{C}^{2g}$ by

$$v = {}^t(\int_{A_1} \eta, \dots, \int_{A_g} \eta, \int_{B_1} \eta, \dots, \int_{B_g} \eta) = \begin{pmatrix} v_A \\ v_B \end{pmatrix}$$

Under this identification, the cup product is given by

$$\langle v, u \rangle = {}^t v_A u_B - {}^t v_B u_A. \tag{3.3}$$

Since $H^1_{\mathbf{C}} = H^1_{\mathbf{C}}(\chi) + H^1_{\mathbf{C}}(\overline{\chi})$, $(\sigma - \omega)(\sigma - \overline{\omega}) = 0$. Therefore $H^1_{\mathbf{C}}(\chi)$ is equal to the image of $\sigma - \overline{\omega}$. Therefore a general element of $H^1_{\mathbf{C}}(\chi)$ is given by

$$\begin{pmatrix} \alpha - \overline{\omega} & \beta \\ \gamma & \delta - \overline{\omega} \end{pmatrix} \begin{pmatrix} u' \\ 0 \end{pmatrix} = \begin{pmatrix} (\alpha - \overline{\omega}) u' \\ \gamma u' \end{pmatrix} = \begin{pmatrix} (\alpha - \overline{\omega}) \gamma^{-1} u_B \\ u_B \end{pmatrix}$$

By a similar computation, we have

$$u_A = \begin{cases} (\alpha - \overline{\omega}) \gamma^{-1} u_B & (u \in H^1_{\mathbf{C}}(\chi)) \\ (\alpha - \omega) \gamma^{-1} u_B & (u \in H^1_{\mathbf{C}}(\overline{\chi})) \end{cases}$$

Thus, we have an isomorphism:

$$\mathbf{C}^g \to H^1(\chi) : u_B \mapsto \begin{pmatrix} (\alpha - \overline{\omega}) \gamma^{-1} u_B \\ u_B \end{pmatrix} \tag{3.4}$$

Under the isomorhism (3.4), the Hermitian fom h on $H^1_{\mathbf{C}}(\chi)$ is transformed to the Hermitian form h_B on $\mathbf{C}^g$. By (3.3) and (3.4) we have

$$h_B(u_B, v_B) = {}^t u_B H^* \overline{v_B}$$

where

$$H^* = 2(\overline{\omega} - \omega) \Big({}^t((\alpha - \overline{\omega}) \gamma^{-1}) - (\alpha - \omega) \gamma^{-1} \Big)$$

Let ψ be a generator of $H^{10}(\chi)$. Under the decomposition $H^{10} = H^{10}(\chi) \oplus H^{10}(\overline{\chi})$, $\xi \in H^{10}$ is expressed as

$$\xi = t\psi + \overline{\phi} \quad (\psi \in H^{10}(\chi), \phi \in H^{01}(\chi)).$$

Since $\phi = \overline{(\xi - t\psi)}$ and $h(\psi, \phi) = 0$, we have

$$\begin{aligned} 0 =& h(\psi, \overline{(\xi - t\psi)}) = h_B(\psi_B, \overline{(\xi_B - t\psi_B)}) \\ =& h_B(\psi_B, \overline{\xi_B}) - t h_B(\psi_B, \overline{\psi_B}) \\ =& {}^t\psi_B H^* \xi_B - t\ {}^t\psi_B H^* \psi_B \\ t =& \frac{{}^t\psi_B H^* \xi_B}{{}^t\psi_B H^* \psi_B} \end{aligned}$$

Therefore the $H^{10}(\overline{\chi})$ and $H^{10}(\overline{\chi})$ components $t\psi$ and $\overline{\phi}$ are given by

$$t\psi = \frac{{}^t\psi_B H^* \xi_B}{{}^t\psi_B H^* \psi_B}\psi, \quad \overline{\phi} = \xi - \frac{{}^t\psi_B H^* \xi_B}{{}^t\psi_B H^* \psi_B}\psi$$

Therefore,

$$t\psi_A = (\alpha - \overline{\omega})\gamma^{-1} \frac{{}^t\psi_B H^* \xi_B}{{}^t\psi_B H^* \psi_B}\psi_B, \quad \overline{\phi}_A = (\alpha - \omega)\gamma^{-1}(\xi_B - \frac{{}^t\psi_B H^* \xi_B}{{}^t\psi_B H^* \psi_B}\psi_B)$$

and ξ_A can be computed by ξ_B by the formula:

$$\begin{aligned} \xi_A =& (\alpha - \overline{\omega})\gamma^{-1} \frac{{}^t\psi_B H^* \xi_B}{{}^t\psi_B H^* \psi_B}\psi_B + (\alpha - \omega)\gamma^{-1}(\xi_B - \frac{{}^t\psi_B H^* \xi_B}{{}^t\psi_B H^* \psi_B}\psi_B) \\ =& (\alpha - \omega)\gamma^{-1}\xi_B + (\omega - \overline{\omega})\gamma^{-1} \frac{{}^t\psi_B H^* \xi_B}{{}^t\psi_B H^* \psi_B}\psi_B \\ =& \left[(\alpha - \omega)\gamma^{-1} + (\omega - \overline{\omega})\gamma^{-1} \frac{\psi_B\ {}^t\psi_B H^*}{{}^t\psi_B H^* \psi_B}\right]\xi_B \end{aligned}$$

Therefore we have the following proposition.

Proposition 3.5. *The map* (3.2) *is given by*

$$\tau = (\alpha - \omega)\gamma^{-1} + (\omega - \overline{\omega})\gamma^{-1} \frac{\psi_B\ {}^t\psi_B H^*}{{}^t\psi_B H^* \psi_B}$$

3.3. **Theta function and Abel-Jacobi map.** As for the theta function, we refer [I].

3.3.1. *Igusa transformation formula of theta function.*

Definition 3.6. *For $\epsilon', \epsilon'' \in \mathbf{Q}^g$, we define the theta function $\vartheta_{\epsilon'\epsilon''}(\tau, z)$ for $\tau \in \mathfrak{S}_g$ and $z \in \mathbf{C}^g$ by*

$$\vartheta_{\epsilon'\epsilon''}(\tau, z) = \sum_{m \in \mathbf{Z}^g} \mathbf{e}(\frac{1}{2}(m + \epsilon')\tau\ {}^t(m + \epsilon') + (m + \epsilon')\ {}^t(z + \epsilon''))$$

We define the theta characteristic $\vartheta_{\epsilon\epsilon'}(\tau)$ by

$$\vartheta_{\epsilon'\epsilon''}(\tau) = \vartheta_{\epsilon'\epsilon''}(\tau, 0).$$

Since $\mathrm{Im}(\tau)$ *is positive definite, the above summations convege.*

Theorem 3.7 (Quasi-periodicity)**.** *We have the following relations.*

(1) *For $\delta', \delta'' \in \mathbf{Q}^g$, we have*

$$\vartheta_{\epsilon',\epsilon''}(z+\tau\delta'+\delta'',\tau) = \mathbf{e}(-\frac{1}{2}\delta'\tau\ {}^t\delta' - \delta'\ {}^t(z+\delta'') - \delta'\ {}^t\epsilon'')\ \vartheta_{\epsilon'+\delta',\epsilon''+\delta''}(z,\tau)$$

(2) *For $m', m' \in \mathbf{Z}^g$, we have*

$$\vartheta_{\epsilon'+m',\epsilon''+m''}(z,\tau) = \mathbf{e}(\epsilon'\ {}^t m'')\vartheta_{\epsilon',\epsilon''}(z,\tau)$$

(3) *For $m', m' \in \mathbf{Z}^g$, we have*

$$\vartheta_{\epsilon',\epsilon''}(z+m'',\tau) = \mathbf{e}(\epsilon'\ {}^t m'')\vartheta_{\epsilon',\epsilon''}(z,\tau)$$

$$\vartheta_{\epsilon',\epsilon''}(z+\tau m',\tau) = \mathbf{e}(-\epsilon''\ {}^t m')\underbrace{\mathbf{e}(-\frac{1}{2}m'\tau\ {}^t m' - m'\ {}^t z)}_{\textit{independent of } \epsilon',\epsilon''}\vartheta_{\epsilon',\epsilon''}(z,\tau)$$

Elements A_i, B_i in the homology group is a linear form on the space of differential forms H^{10} by setting

$$A_i(\omega_j) = \int_{A_i}\omega_j, B_i(\omega_j) = \int_{B_i}\omega_j$$

Here we use the following identification:

$$\varsigma : \begin{array}{ccc} (H^{10})^* & \to & \mathbf{C}^g \\ \psi & \mapsto & (z_1,\dots,z_g) = (\psi(\omega_1),\dots,\psi(\omega_g)) \end{array}$$

$$\iota : \begin{array}{ccc} H_{1,\mathbf{Z}} & \to & \tau\mathbf{Z}^g \oplus \mathbf{Z}^g \\ \sum_i(\alpha_i A_i + \beta_i B_i) & \mapsto & \tau\alpha + \beta \end{array}$$

Theorem 3.8. *If $n\epsilon', n\epsilon', n\delta', n\delta' \in \mathbf{Z}$, then*

$$\left(\frac{\vartheta_{\epsilon',\epsilon''}(z,\tau)}{\vartheta_{\delta',\delta''}(z,\tau)}\right)^n$$

is a rational function on the abelian variety

$$J = (H^{10})^*/H_1 = \mathbf{C}^g/(\tau\mathbf{Z}^g + \mathbf{Z}^g).$$

To analyse the zero of theta function, we user theta trnasformation formula ([I]). To avoid the complexity, in this section H^1_{prim} is principally polarized. Non-principal case is considered later. Let

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be an element in $Sp(2g,\mathbf{Z})$ and $m = (m', m'')$ an element in $\frac{1}{2n}(\mathbf{Z}^g)^2$. We set

$$\sigma^\# z = z(\gamma\tau+\delta)^{-1}, \quad \sigma^\#\tau = (\alpha\tau+\beta)(\gamma\tau+\delta)^{-1},$$

$$\sigma^\# m = m\sigma^{-1} + \frac{1}{2}((\gamma^t\delta)_0, (\alpha^t\beta)_0).$$

Proposition 3.9. *Under the above notation, we have*

$$\vartheta_{\sigma^\# m}(\sigma^\# z, \sigma^\# \tau) = \mathbf{e}(\frac{1}{2} z(\gamma\tau+\delta)^{-1}\gamma^t z)\det(\gamma\tau+\delta)^{\frac{1}{2}}\vartheta_m(z,\tau)\cdot u$$

where $\mid u \mid = 1$.

3.4. **Transformation for cyclic action and Abel-Jacobi map.**

3.4.1. *Cyclic action on Prym variety.* We apply the above formula to the situation of Section 3.2. Let $\eta_1, \dots, \eta_g$ be a basis of H^{10} such that

$$\sigma^*\eta_1 = \omega\eta_1, \sigma^*\eta_2 = \overline{\omega}\eta_2, \dots, \sigma^*\eta_g = \overline{\omega}\eta_g.$$

The unnormalized period matrices Ω_A, Ω_B are defined by

$$\begin{pmatrix} \Omega_A \\ \Omega_B \end{pmatrix} = \begin{pmatrix} A_1 \\ \vdots \\ B_g \end{pmatrix} \begin{pmatrix} \eta_1, \dots \eta_g \end{pmatrix}$$

Therefore, we have

$$\begin{aligned} \sigma \begin{pmatrix} \Omega_A \\ \Omega_B \end{pmatrix} =& \sigma\ {}^t(A_1, \dots, A_g, B_1, \dots, B_g) \begin{pmatrix} \eta_1, \dots \eta_g \end{pmatrix} \\ =&\ {}^t(A_1, \dots, A_g, B_1, \dots, B_g) \begin{pmatrix} \sigma^*\eta_1, \dots \sigma^*\eta_g \end{pmatrix} \\ =&\ {}^t(A_1, \dots, A_g, B_1, \dots, B_g) \begin{pmatrix} \eta_1, \dots \eta_g \end{pmatrix} R(\zeta) \\ =& \begin{pmatrix} \Omega_A R(\zeta) \\ \Omega_B R(\zeta) \end{pmatrix} \end{aligned}$$

where $R(\zeta) = \mathrm{Diag}(\omega, \overline{\omega}, \dots, \overline{\omega})$. We compute the automorphic factor for the theta function. Since $\tau = \Omega_A \Omega_B^{-1}$, we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \tau \\ I_g \end{pmatrix} = \begin{pmatrix} \Omega_A R(\zeta)\Omega_B^{-1} \\ \Omega_B R(\zeta)\Omega_B^{-1} \end{pmatrix}$$

and

$$\gamma\tau + \delta = \chi(\zeta), \quad (\alpha\tau+\beta)(\gamma\tau+\delta)^{-1} = \tau$$

where $\chi(\zeta) = \Omega_B R(\zeta)\Omega_B^{-1}$.

Proposition 3.10. *Under the situation of Section 3.2, we have*

$$\vartheta_{\sigma^\# m}(z\chi(\zeta)^{-1}, \tau) = \mathbf{e}(\frac{1}{2} z(\gamma\tau+\delta)^{-1}\gamma^t z)\det(R(\zeta))^{\frac{1}{2}}\vartheta_m(z,\tau)\cdot u$$

Proof. Since $\gamma\tau + \delta = \chi(\zeta)$ we have

$$\zeta^\# z = z\chi(\zeta)^{-1}, \quad \zeta^\# \tau = \tau$$

□

4. Abel-Jacobi map

In this section, we consider the case where $H_1^{\text{prim}} = H_1$. We consider the case where (3.1) and the equation of the cyclic covering is given as (2.1). Then the spaces $H^{10}(\chi)$ and $H^{10}(\overline{\chi})$ are generated by

$$\eta_1 = \frac{dx}{y} \quad \text{and} \quad \eta_2 = \frac{dx}{y^*}, \eta_3 = \frac{xdx}{y^*}, \dots, \eta_g = \frac{x^{g-2}dx}{y^*}.$$

with

$$y^* = \frac{x(1-x)\prod_{i=2}^{d+1}(x-\lambda_i)}{y}.$$

Suppose that $b \in C$ is fixed under the action of μ_n. Let $\widetilde{C}$ be the universal covering of C and $\widetilde{b}$ be a lifting of b in $\widetilde{C}$. Then the action of μ_n on $\widetilde{C}$ can be extended to that of $\widetilde{C}$ fixing $\widetilde{b}$. We define a vector valued function I^* on $\widetilde{C}$ by

$$I^*(\widetilde{z}) = (\int_{\widetilde{b}}^{\widetilde{z}} \eta_1, \int_{\widetilde{b}}^{\widetilde{z}} \eta_2, \dots, \int_{\widetilde{b}}^{\widetilde{z}} \eta_g)$$

and we set $I = I^*\Omega_B^{-1}$.

Proposition 4.1. (1) *The map $I : \widetilde{C} \to H^{10}(C)^* = \mathbf{C}^g$ induces a map $C \to J(C)$*

(2) *The action of μ_d on the vector valued function ξ is given by*

$$I(\zeta\widetilde{z}) = I(\widetilde{z})\chi(\zeta).$$

(3) *Under the notations and assumption in Theorem 3.8, the function*

$$\left(\frac{\vartheta_{\epsilon',\epsilon''}(I(\widetilde{z}),\tau)}{\vartheta_{\delta',\delta''}(I(\widetilde{z}),\tau)}\right)^{2n}$$

becomes a rational function on C if the denominator is not identically zero.

Now consider the branch locus $\overline{w}$ of the covering $C \to \mathbf{P}^1$. Let $w \in C$ be a lifting of $\overline{w}$. Let $\mu_m \subset \mu_n$ be stabilizer of the point w. Let $\widetilde{w}$ be a point in the universal covering $\widetilde{C}$ over w.

$$\widetilde{C} \to C \to \mathbf{P}^1 : \widetilde{w} \mapsto w \mapsto \overline{w}$$

Proposition 4.2. *Let ζ be an element in μ_m, We have*

$$I(\zeta\widetilde{w}) - I(\widetilde{w}) \in \tau\mathbf{Z}^g \oplus \mathbf{Z}^g$$

In particular $I(\widetilde{w}) \in \dfrac{1}{m}(\tau\mathbf{Z}^g \oplus \mathbf{Z}^g)$.

Proof. We use the relation $I(\zeta\widetilde{z}) = I(\widetilde{z})\chi(\zeta)$ and

$$m = \frac{d}{dx}(x^m - 1)\mid_{x=1} = \prod_{i=1}^{m-1}(x - \mathbf{e}(i/m))$$

□

Let U be a small neighborhood of w stable under the action of ζ. Then there exists a small neighborhood $\widetilde{U}$ of $\widetilde{w}$ which maps isomorphically to U under the universal covering map. The action of ζ induces an action on $\widetilde{U}$ which is denoted by $\widehat{\zeta}$. For a point $u \in \widetilde{U}$, we define

$$J^*(u) = (\int_{\widetilde{w}}^{u} \eta_1, \int_{\widetilde{w}}^{u} \eta_2, \dots, \int_{\widetilde{w}}^{u} \eta_g)$$

and we set $J = J^* \Omega_B^{-1}$.

Proposition 4.3. *Suppose that $\zeta^{\#}(\epsilon) = \epsilon \mod \mathbf{Z}^{2g}$. For $u \in \widetilde{U}$, we have*

$$\vartheta_\epsilon(I(\widetilde{w}) + J(\widehat{\zeta}u), \tau) = \varphi(u) \cdot \vartheta_\epsilon(I(\widetilde{w}) + J(u), \tau)$$

and $u_1 = \lim_{u \to \widetilde{w}} \varphi(u)$ is a root of unity. Moreover, u_1 is computable using transformation formula.

Proof. By Proposition 4.2, we have $I(\widetilde{w}) = \delta'\tau + \delta''$ with $\delta', \delta'' \in \frac{1}{m}\mathbf{Z}^g$. Since

$$J(\widehat{\zeta}u) = (\int_{\widetilde{w}}^{\widehat{\zeta}u} \eta_i) = (\int_{\zeta\widetilde{w}}^{\zeta u} \eta_i) = (\int_{\widetilde{w}}^{u} \eta_i)\chi(\zeta) = J(u)\chi(u)$$

and

$$\vartheta_\epsilon(I(\widetilde{w}) + J(u), \tau) = E(J(u))\vartheta_{\epsilon+\delta}(J(u), \tau),$$

with

$$E(z) = \mathbf{e}(-\frac{1}{2}\delta'\tau\ {}^t\delta' - \delta'\ {}^t(z + \delta'') - \delta'\ {}^t\epsilon''),$$

we have

$$\begin{aligned}\vartheta_\epsilon(I(\widetilde{w}) + J(\zeta^* u), \tau) =& E(J(u)\chi(\zeta))\vartheta_{\epsilon+\delta}(J(u)\chi(\zeta), \tau)\\ =& E(J(u)\chi(\zeta))v(\epsilon+\delta, u)\vartheta_{\epsilon+\delta}(J(u), \tau)\\ =& E(J(u)\chi(\zeta))E(J(u))^{-1}v(\epsilon+\delta, u)\vartheta_\epsilon(I(\widetilde{w}) + J(u), \tau)\end{aligned}$$

and $\lim_{u \to \widetilde{w}} v(\epsilon + \delta, u)$ is a root of unity. Since $\lim_{u \to \widetilde{w}} J(u) = 0$, we have

$$\lim_{u \to \widetilde{w}} E(J(u)\chi(\zeta))E(J(u))^{-1} = 1$$

Thus, we have the proposition. □

By the following proposition, we can compute the order of the analytic function $\vartheta_\epsilon(I(\widetilde{w}) + J(u), \tau)$ on U at $u = \widetilde{w}$.

Proposition 4.4. (1) *Let t_w be a uniformaizer of C at w such that $\zeta^* t_w = t_w \mathbf{e}(\theta_w)$, $(\theta_w \in \mathbf{Q})$. We set $\lim_{u \to \widetilde{w}} v(\epsilon+\delta, w) = \mathbf{e}(\gamma(\epsilon+\delta, w))$. Then we have*

$$\theta_w \operatorname{ord}_{u=\widetilde{w}} \vartheta_\epsilon(I(\widetilde{u}), \tau) = \gamma(\epsilon + \delta, w) \ mod\ \mathbf{Z}.$$

(2) *Suppose that*

$$f(\widetilde{z}) = \left(\frac{\vartheta_\epsilon(I(\widetilde{z}), \tau)}{\vartheta_\gamma(I(\widetilde{z}), \tau)}\right)^n$$

is a rational function of C. Then we have

$$\theta_w \operatorname{ord}_w f(z) \equiv n(\gamma(\epsilon + \delta, w) - \gamma(\gamma + \delta, w))\ mod\ n. \tag{4.1}$$

Proposition 4.5. *The zero divisor of $\varphi_\epsilon(I\widetilde{z},\tau)$ is the pull back of the divisor D of C. The degree of the divisor D is equal to the genus g. If $\epsilon = 0$, it is called the theta divisor associated to the symplectic basis and the base point.*

5. Inverse period map for some cases

Using the relation (4.1), one can compute the order of zero of $\vartheta_\epsilon(I(\widetilde{z}),\tau)$ mod n. We already know the order of theta divisor is equal to the genus g. Using these conditions, if we can detect the zero locus and order of theta function with character, we can specify the rational function on the curve defined by theta functions. By this computation, inverse period map can be computed for several cases.

5.1. $\mu = (2/3, 1/3, 1/3, 1/3, 1/3)$**(**[S2]**,**[S3]**).** This case is related to a K3 surface. Inverse period map is a spacialization of §5.2. In the continuation of this paper, 9-families of K3 surfaces are mentioned.

(1) $(1/6, 1/3, 1/3, 1/3, 5/6)$(2,specialization of §5.3)
(2) $(1/6, 1/6, 1/6, 2/3, 5/6)$(3,Mostow-list)
(3) $(1/6, 1/6, 1/3, 1/2, 5/6)$(4,Mostow-list)
(4) $(1/3, 1/3, 1/3, 1/2, 1/2)$(5)
(5) $(1/6, 1/6, 1/3, 2/3, 2/3)$(6,Mostow-list,specialization of §5.3)
(6) $(1/6, 1/3, 1/3, 1/2, 2/3)$(7,specialization of §5.3)
(7) $(1/6, 1/6, 1/2, 1/2, 2/3)$(8,Mostow-list)
(8) $(1/6, 1/3, 1/2, 1/2, 1/2)$(9)

5.2. $\mu = (1/3, 1/3, 1/3, 1/3, 1/3, 1/3)$**(**[M2]**).** This is a specialization of §5.3.

Here we give how the story goes in this case. The other case is more or less similar. We consider distinct complex numbers $\lambda_1, \lambda_2, \lambda_3$ and triple covering

$$C : y^3 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$$

of $\mathbf{P}^1$ branching at 6-points. We set $H_1 = H_1(C)$. Then

$$H^{10}(\chi) = \langle \frac{dx}{y} \rangle, H^{10}(\overline{\chi}) = \langle \frac{dx}{y^2}, \frac{xdx}{y^2}, \frac{x^2dx}{y^2} \rangle$$

We choose a reference curve C_0 to be $\lambda_i \in \mathbf{R}$ and $0 < 1 < \lambda_1 < \lambda_2 < \lambda_3$. We choose a symplectic basis $A_1, \dots, A_4, B_1, \dots, B_4$ of $H^1(C_0)$. (See the paper [M2].) Under this base the action of σ on $H_1(C)$ is given by

$$\sigma = \begin{pmatrix} O & H \\ -H & -I_4 \end{pmatrix}$$

where $H = \mathrm{diag}(1,1,1,-1)$. We concider a theta characteristic $\epsilon \in (\frac{1}{6}\mathbf{Z}^4)^2$ such that $\epsilon\sigma - \epsilon \in (\mathbf{Z}^4)^2$. This condition is equivalent to $\epsilon'' = -\epsilon' H$. The parameter ϵ'' is determined by ϵ'. From now on, the index ϵ is denoted by $6 \times \epsilon'$. By considering the order of zeros of theta functions, we have the following proposition.

Proposition 5.1. *We choose a base point as a lifting $\widetilde{1}$ of* 1*. The function*

$$f(z) = \frac{\vartheta_{(-1,-3,-3,-3)}(I(\widetilde{z}),\tau)^3}{\vartheta_{(1,3,3,3)}(I(\widetilde{z}),\tau)^3}$$

is a rational function of C. The divisor (f) of f is equal to $3(0) - 3(\infty)$*. Therefore f is a constant multiple of u.*

Using the above proposition, we have

$$\lambda_1 = \frac{f(\lambda_1)}{f(1)} = \frac{\vartheta_\epsilon(I(\widetilde{\lambda_1}),\tau)^3 \vartheta_\delta(I(\widetilde{1}),\tau)^3}{\vartheta_\delta(I(\widetilde{\lambda_1}),\tau)^3 \vartheta_\epsilon(I(\widetilde{1}),\tau)^3} = \frac{\vartheta_\epsilon(I(\widetilde{\lambda_1}),\tau)^3}{\vartheta_\delta(I(\widetilde{\lambda_1}),\tau)^3} = \frac{\vartheta_{(1,1,3,-3)}(\tau)^3}{\vartheta_{(1,-1,3,-3)}(\tau)^3}$$

Here, we use the equality $I(\widetilde{1}) = 0$

$$\lambda_1 = \frac{(\lambda_1 - 0)(1-\infty)}{(\lambda_1 - \infty)(1-0)}$$

The cross ratios of 4 points out of 6-points $\infty, 0, 1, \lambda_1, \lambda_2, \lambda_3$ is computed.

Cross ratio of 4 points out of 6-points is expressed as determinant of submatrix as follows. Let A be the following matrix consisting of homogeneous coordinate of 6-points $(\nu_{0i} : \nu_{1i})$.

$$A = \begin{pmatrix} \nu_{01} & \nu_{02} & \nu_{03} & \nu_{04} & \nu_{05} & \nu_{06} \\ \nu_{11} & \nu_{12} & \nu_{13} & \nu_{14} & \nu_{15} & \nu_{16} \end{pmatrix}$$

The determinant of 2×2-matrix consisting of i, j columns $(0 \le i < j \le 5)$ is denoted by D_{ij}. Then we have

$$\frac{\vartheta_{(1,1,3,-3)}(\tau)^3}{\vartheta_{(1,-1,3,-3)}(\tau)^3} = \frac{D_{24}D_{13}}{D_{14}D_{23}} = \frac{D_{24}D_{13}D_{56}}{D_{14}D_{23}D_{56}}$$

Proposition 5.2 ([M2])**.** *There exists a set $\{\vartheta_\epsilon(\tau)\}$ consisting of* 15 *theta characteristics and a labeling*

$$\theta_\epsilon(\tau) = \vartheta(ij : kl : mn, \tau)$$

such that the point

$$(\vartheta(ij : kl : mn))_{(ij:kl:mn)\in S_{15}} = (D_{ij}D_{kl}D_{mn})_{(ij:kl:mn)\in S_{15}}$$

in $\mathbf{P}^{14}$*.*

5.3. $\mu = (1/6, 1/6, 1/3, 1/3, 1/3, 1/3, 1/3)$**(**[ACT]**,** [MT1]**).** There are unexpected relation between the action of Weyl group $W(E_6)$ of type E_6. Degree three Del-Pezzo surface S is obtained by blowing up of 6 points in $\mathbf{P}^2$. The linear system of cubic curves passing through the 6-points defines a embedding into $\mathbf{P}^3$ whose image is a cubic surface. By Griffith-Clemens algebraic correspondences, the period map for triple coverings of $\mathbf{P}^3$ branching at this cubic surface is nothing but that of 6-ple covering of $\mathbf{P}^1$. Using this geometry, the inverse period map is completely understood by using $W(E_6)$-invariant set of polynomials.

5.4. $\mu = (1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4)$**([MT2]).** This case belongs to the list of Mostow. We use Prym variety of 4-ple covering of $\mathbf{P}^1$. This abelian variety is not principal. Therefore the treatment is subtle.

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