

# Exercises for the Hypergeometric School 2023

Saiei-Jaeyeong Matsubara-Heo, Anna-Laura Sattelberger, and Simon Telen

## 1 Positive Toric Geometry

**Exercise 1.1 (Toric ideals).** Prove the inclusion  $I_A \subset I(Y_A)$ .

**Exercise 1.2 (Affine lattices).** Prove that  $\mathbb{Z}(A - a_i) = \mathbb{Z}(A - a_j)$  for all  $1 \leq i, j \leq s$ .

**Exercise 1.3 (Dimension, degree, and smoothness).** Verify Theorems 1.6, 1.9, and 1.13 for the matrices  $A_1, A_2$ , and  $A_3$  in Example 1.3.

**Exercise 1.4 (Double pillow).** Consider the integer matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

- (a) Determine the vertices of  $\text{Conv}(A)$ . For one of these vertices  $v$ , write down a monomial parametrization of the affine toric surface  $Y_{A-v}$ , and compute its toric ideal. Is this a smooth surface?
- (b) Compute the toric ideal of the projective toric surface  $X_A \subset \mathbb{P}^4$ .
- (c) Compute the defining equation of the projection of  $X_A$  under

$$\pi = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3.$$

- (d) Plot the surface  $\pi(X_A)$  in the affine chart of  $\mathbb{R}\mathbb{P}^3$  with nonzero first coordinate. Do you recognize the real part of  $\pi(Y_{A-v})$  from part (a) in the picture?
- (e) Identify  $\pi((X_A)_{\geq 0})$  in the plot. Verify that  $(X_A)_{\geq 0}$  is homeomorphic to  $\text{Conv}(A)$ .

**Exercise 1.5 (Real toric varieties).** Show that  $\overline{\phi_A((\mathbb{R}^*)^n)} \subset Y_A(\mathbb{R})$ , where  $\overline{(\cdot)}$  denotes the Euclidean closure, might be a strict inclusion.

**1.6 The boundary of an affine toric variety.** Prove the proposition below.

**Proposition 1.** Let  $x \in Y_A \subset \mathbb{C}^s$  and let  $\text{supp}(x) = \{a_i \in A : x_i \neq 0\}$ . We have  $\text{supp}(x) = Q \cap A$  for some face  $Q$  of the cone  $\text{pos}(A) = \{r_1 a_1 + \cdots + r_s a_s : r_i \in \mathbb{R}_{\geq 0}\}$ .

**Exercise 1.7 (Stratification of affine toric varieties).**

**Proposition 2.** For a face  $Q \subset \text{pos}(A)$ , let  $A \cap Q = \{a_{i_1}, \dots, a_{i_r}\}$  and define the projection  $\pi_Q : \mathbb{C}^s \rightarrow \mathbb{C}^r$ ,  $x \mapsto (x_{i_1}, \dots, x_{i_r})$ . Let  $Y_{A, A \cap Q} = \{x \in Y_A, \text{supp}(x) \subset A \cap Q\}$ . We have

$$\pi_Q(Y_A) = \pi_Q(Y_{A, A \cap Q}) = Y_{A \cap Q}.$$

In particular, if  $\text{supp}(x) = A \cap Q$ , then there is  $t \in (\mathbb{C}^*)^n$  such that

$$x_i = \begin{cases} t^{a_i} & a_i \in A \cap Q, \\ 0 & \text{otherwise.} \end{cases}$$

**Exercise 1.8 (Nonnegative projective toric varieties).** Prove the second part of Proposition 2.6.

**Exercise 1.9 (Positive toric models).** Show that the positive part  $(X_A)_{>0}$  of a toric model consists of all probability distributions  $(x_1, \dots, x_s) \in \text{relint}(\Delta_{s-1})$  whose coordinate-wise logarithm  $(\log x_1, \dots, \log x_s)$  belongs to the row span of  $\hat{A}$ .

**Exercise 1.10 (Maximum likelihood estimation).** Compute the MLE for Example 3.5. Based on these fictional data, does your result confirm the ansatz that being vegetarian or not is independent of someone's subject preference? Can you estimate  $p_Y$ ,  $p_N$ ,  $p_A$ ,  $p_G$ , and  $p_H$ ?

**Exercise 1.11 (Toric patches).** Plot the Bézier plane curve of degree 6, i.e.,  $A = (0 \ 1 \ \dots \ 6)$ , for any matrix of control points  $P \in \mathbb{R}^{2 \times 6}$  and weights  $w \in \mathbb{R}_{>0}^6$ . Investigate the influence of the control points and the weights by experimenting. Solve the same exercise for  $k = 3$  and

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix}.$$

## 2 Algebraic and Holonomic Statistics

**Exercise 2.1 (Another coin model).**

Consider the following discrete statistical model  $\mathcal{M}$  with 3 states. A gambler has a biased coin which shows *head* with probability  $x$ , and *tail* with probability  $1 - x$ . She flips the coin thrice and records three possible outcomes:  $\diamond$  only heads  $\diamond$  mixed outcome  $\diamond$  only tails

- Compute an implicitization of the model. More precisely, determine a homogeneous polynomial  $f \in \mathbb{C}[p_0, p_1, p_2]$  of degree 3, such that  $\mathcal{M} = V(f)$ . Interpret the two images in Figure 1. Can you reproduce them in **Mathematica**?
- Determine the very affine variety  $X$  of the model and compute its ML degree  $d_{\text{ML}}(X)$ .
- For data  $s = (s_0, s_1, s_2) \in \mathbb{N}^3$ , compute the likelihood function  $L_s$ , and the the maximum likelihood estimator of the model.
- Compute the Bernstein–Sato ideal of the model  $\mathcal{M}$ . You can compute it using the library **Dmod\_lib** in the computer algebra software **Singular**<sup>1</sup> by running the following code.

<sup>1</sup>An online version of **Singular** is available at the following link: <https://www.singular.uni-kl.de:8003/>

```
LIB "dmod.lib";
ring r = 0,x,dp; setring r;
ideal F = (x^2,x*(1-x),1-x);
def A = annfsBMI(F); setring A;
BS;
```

In order to interpret the output of the code above, have a look at the documentation of the command `annfsBMI`: [https://www.singular.uni-kl.de/Manual/4-0-3/sing\\_598.htm](https://www.singular.uni-kl.de/Manual/4-0-3/sing_598.htm)

- (e) Compute the tropical variety  $\text{trop}(X)$ .

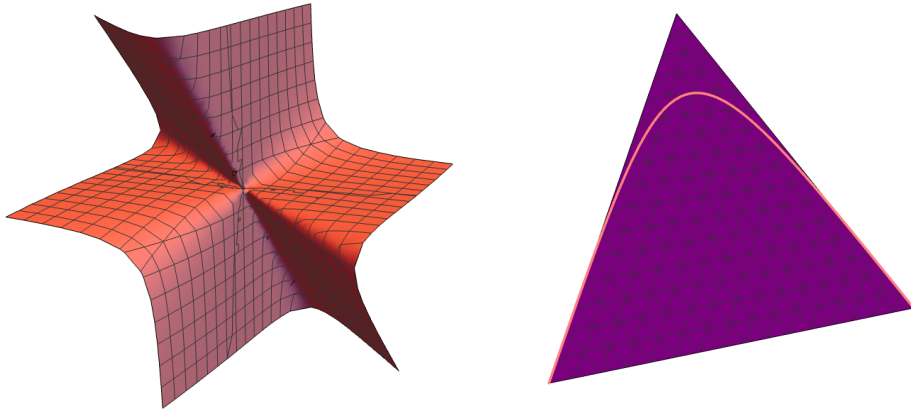


Figure 1: Images accompanying Exercise 2.1 (a)

**Exercise 2.2 (Mellin transform).**

The *Mellin transform* of a complex-valued function  $f$  in  $n$  variables  $x = (x_1, \dots, x_n)$  is

$$\mathfrak{M}\{f\}(\nu_1, \dots, \nu_n) = \int_{\Gamma} f(x_1, \dots, x_n) x_1^{\nu_1} \cdots x_n^{\nu_n} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n},$$

where the integration contour  $\Gamma$  is such that the boundary term in IBP vanishes and the integral converges. The ( $n$ -th) *shift algebra* is the non-commutative  $\mathbb{C}$ -algebra obtained from the free  $\mathbb{C}$ -algebra generated by variables  $\nu_1, \dots, \nu_n$  and shift-operators  $\sigma_1^{\pm 1}, \dots, \sigma_n^{\pm 1}$  by imposing the following relations: all generators commute, except  $\nu_i$  and  $\sigma_i^{\pm 1}$ . They obey the rule

$$\sigma_i \nu_i^{\pm 1} = (\nu_i \pm 1) \sigma_i^{\pm 1}.$$

- Compute  $\mathfrak{M}\{x_i \cdot f\}$  and  $\mathfrak{M}\{x_i \partial_i \bullet f\}$ .
- Formulate the action of  $S_n$  on  $\mathfrak{M}\{f\}$ .
- Building on (a), formulate the Mellin transform as an isomorphism of the Weyl algebra on the torus  $D_{\mathbb{G}_m^n} = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \langle \partial_1, \dots, \partial_n \rangle$  and the shift algebra  $S_n$ . This isomorphism is referred to as “algebraic Mellin transform”.
- Let  $f = (x_1 + 1)(x_2 + 1) \in \mathbb{C}[x_1, x_2]$ . Compute its  $s$ -parametric annihilator  $\text{Ann}_{D_2[s]}(f^s)$ . In order to compute it, you can use the command `annfsBMI` in `Singular`. The  $s$ -parametric annihilator is encoded as LD in the output ring. Deduce shift-relations for  $\mathfrak{M}\{f^s\}$ .

### 3 Twisted Cohomology and Likelihood Ideals

**Exercise 3.1 (Convergence).** Describe the convergence condition on  $(s, \nu) \in (0, \infty) \times \mathbb{R}^3$  of an integral

$$\int_{\mathbb{R}_{>0}^3} f(x)^{-s} x_1^{\nu_1} x_2^{\nu_2} x_3^{\nu_3} \frac{dx_1 dx_2 dx_3}{x_1 x_2 x_3}$$

for the following polynomials  $f(x)$ .

(a)  $f(x) = 1 + x_1 + x_2 + x_3 + x_1 x_3 + x_2 x_3$ .

(b)  $f(x) = 1 + x_1 + x_2 + x_3 + x_1 x_3$ .

(c)  $f(x) = 1 + \sum_{\sigma \in \mathfrak{S}_3} x_1^{\sigma(1)} x_2^{\sigma(2)} x_3^{\sigma(3)}$ .

**Exercise 3.2 (ML degree and Euler characteristic).**

(a) Compute the Euler characteristics of  $\mathcal{M}_{0,4}$  and  $\mathcal{M}_{0,5}$ .

We aim to compute the Euler characteristic of  $\mathcal{M}_{0,n}$  for  $n \geq 4$ . To do this, we use the following lemma from algebraic topology (see, e.g., [1, Exercise 14.37]):

**Lemma 3.** *Let  $E, B$  be reasonably good manifolds (such as complex affine algebraic varieties). If  $\pi : E \rightarrow B$  is a fiber bundle with fiber  $F$ , the following product relation hold true:*

$$\chi(E) = \chi(B)\chi(F).$$

(b) Construct a fiber bundle  $\pi : \mathcal{M}_{0,n} \rightarrow \mathcal{M}_{0,n-1}$ .

(c) Compute the Euler characteristic of  $\mathcal{M}_{0,n}$  for all  $n \geq 4$ .

**Exercise 3.3 (Twisted Cohomology).** Consider  $f_1(x), \dots, f_\ell(x) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ . We claimed that the following elements of  $R = \mathbb{C}(s, \nu) \langle \sigma_{s_1}^{\pm 1}, \dots, \sigma_{s_\ell}^{\pm 1}, \sigma_{\nu_1}^{\pm 1}, \dots, \sigma_{\nu_n}^{\pm 1} \rangle$  generates a left ideal of  $R$  which corresponds to the twisted cohomology:

$$1 - \sigma_{s_i} f_i(\sigma_\nu) \quad (i = 1, \dots, \ell), \quad \nu_j - \sum_{i=1}^{\ell} s_i \sigma_{s_i} \sigma_{\nu_j} \frac{\partial f_i}{\partial x_j}(\sigma_\nu) \quad (j = 1, \dots, n).$$

Show that these operators annihilate the cohomology class  $\left[ \frac{dx_1 \wedge \dots \wedge dx_n}{x_1 \dots x_n} \right] \in H^n(X_K, \omega)$ .

**Exercise 3.4 (Variation of coefficients).** Consider an integral of the form

$$I(z; s, \nu) := \int_0^1 (1-x)^{-s_1} (1-z_2 x)^{-s_2} \dots (1-z_\ell x)^{-s_\ell} x^\nu \frac{dx}{x},$$

where  $s_i, \nu$  are complex parameters and  $z_2, \dots, z_\ell \in \mathbb{C}^*$ . Describe the condition on  $z$  that the ML degree is one. Combined with a theorem explained in the lecture, this problem recovers the result of [2].

## References

- [1] Bott, Raoul, and Loring W. Tu. Differential forms in algebraic topology. Vol. 82. New York: Springer, 1982.
- [2] Zeilberger, Doron. Gauss's  ${}_2F_1(1)$  cannot be generalized to  ${}_2F_1(x)$ . Journal of Computational and Applied Mathematics 39.3 (1992): 379–382.