# Structured root finding via eigenvalues

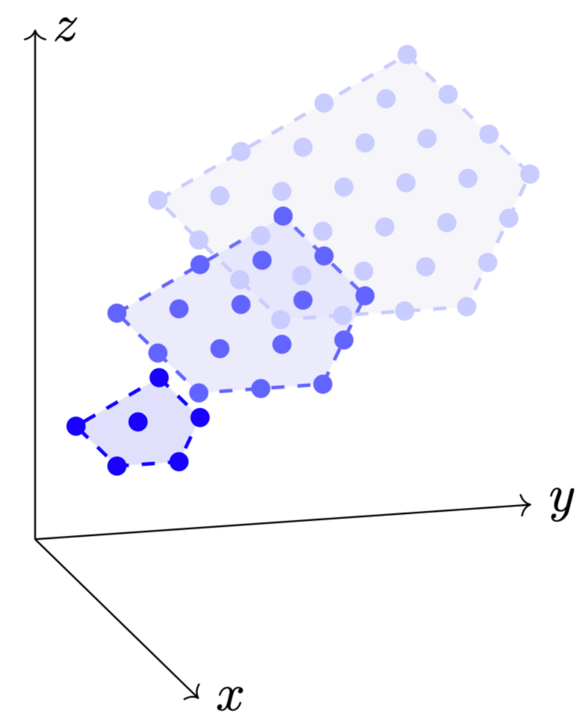
*Beyond toric varieties*

Simon Telen

Hypergeometric Workshop, August 18, Kobe University

Barbara Betti

Marta Panizzut




Max-Planck-Institut für **Mathematik** in den **Naturwissenschaften**

arXiv > math > arXiv:2306.14871

Mathematics > Algebraic Geometry

*[Submitted on 26 Jun 2023]*

**Solving Equations Using Khovanskii Bases**

Barbara Betti, Marta Panizzut, Simon Telen

# Solving polynomial equations

**find** $(t_1, t_2) \in K^2$ **such that**

$$0 = a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2$$

$$0 = b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2$$

# Solving polynomial equations

**find** $(t_1, t_2) \in K^2$ **such that**

$$0 = a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2$$
$$0 = b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2$$

$\swarrow$

**find** $u = (u_0 : u_1 : u_2) \in \mathbb{P}_K^2$ **such that**

$$0 = a_0 u_0^3 + a_1 u_0 u_1 u_2 + a_2 u_1^2 u_2 + a_3 u_1 u_2^2$$
$$0 = b_0 u_0^3 + b_1 u_0 u_1 u_2 + b_2 u_1^2 u_2 + b_3 u_1 u_2^2$$

# Solving polynomial equations

**find** $(t_1, t_2) \in K^2$ **such that**

$$0 \;=\; a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2$$

$$0 \;=\; b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2$$

**find** $x = (x_0 : \cdots : x_3) \in X \subset \mathbb{P}^3_K$ **such that**

$$0 \;=\; a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3$$

$$0 \;=\; b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$

$$X \;=\; \overline{\mathrm{im}(t \mapsto (1 : t_1 t_2 : t_1^2 t_2 : t_1 t_2^2))}$$

**find** $u = (u_0 : u_1 : u_2) \in \mathbb{P}^2_K$ **such that**

$$0 \;=\; a_0 u_0^3 + a_1 u_0 u_1 u_2 + a_2 u_1^2 u_2 + a_3 u_1 u_2^2$$

$$0 \;=\; b_0 u_0^3 + b_1 u_0 u_1 u_2 + b_2 u_1^2 u_2 + b_3 u_1 u_2^2$$

# Solving polynomial equations

**find** $(t_1, t_2) \in K^2$ **such that**

$$0 = a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2$$
$$0 = b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2$$

$\longrightarrow$

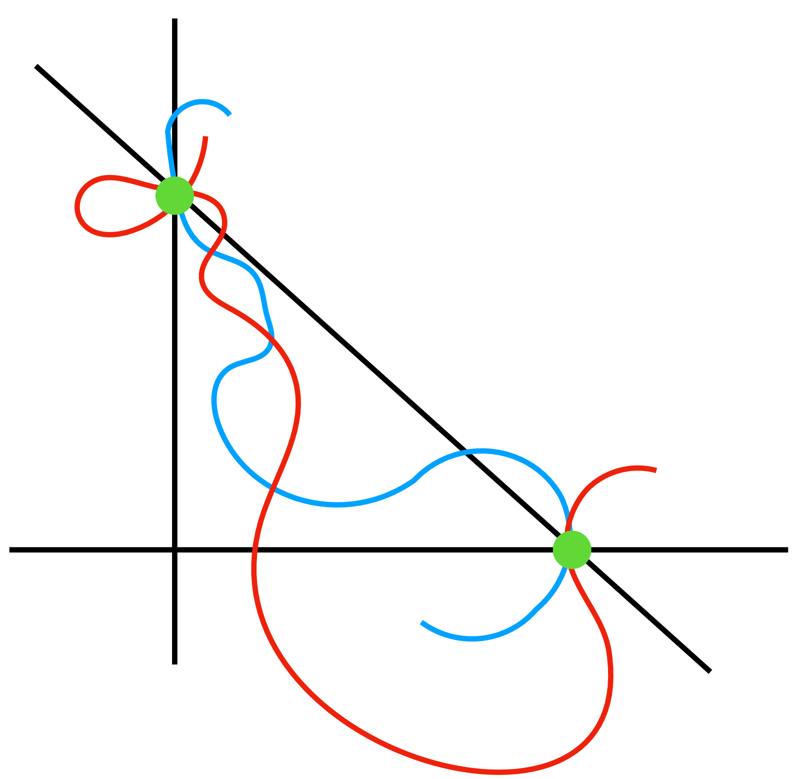
**find** $x = (x_0 : \cdots : x_3) \in X \subset \mathbb{P}^3_K$ **such that**

$$0 = a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3$$
$$0 = b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$
$$X = \overline{\mathrm{im}(t \mapsto (1 : t_1 t_2 : t_1^2 t_2 : t_1 t_2^2))}$$

$\swarrow$

**find** $u = (u_0 : u_1 : u_2) \in \mathbb{P}^2_K$ **such that**

$$0 = a_0 u_0^3 + a_1 u_0 u_1 u_2 + a_2 u_1^2 u_2 + a_3 u_1 u_2^2$$
$$0 = b_0 u_0^3 + b_1 u_0 u_1 u_2 + b_2 u_1^2 u_2 + b_3 u_1 u_2^2$$



# Solving polynomial equations

**find** $(t_1, t_2) \in K^2$ **such that**

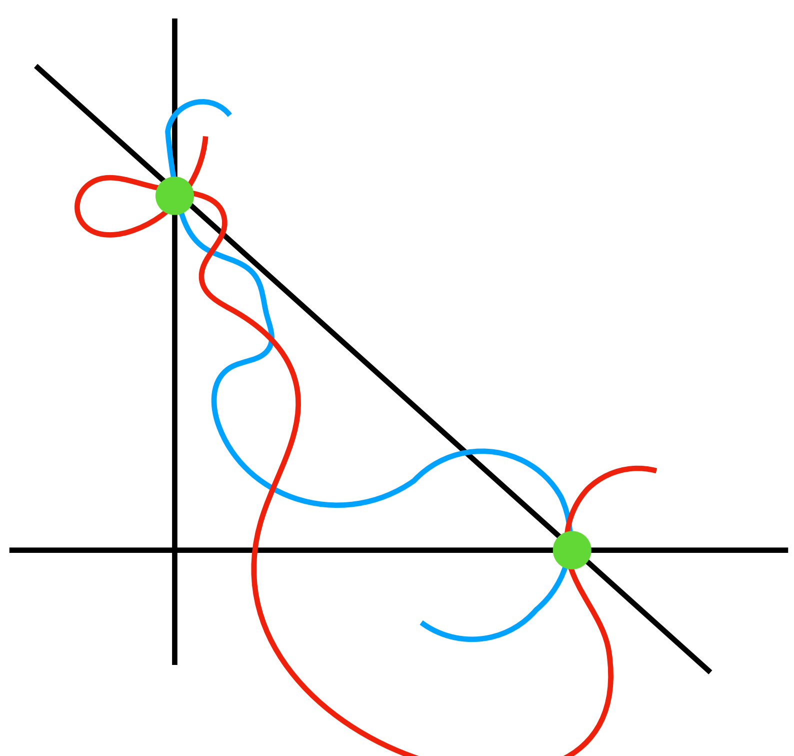
$$\begin{aligned} 0 &= a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2 \\ 0 &= b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2 \end{aligned}$$

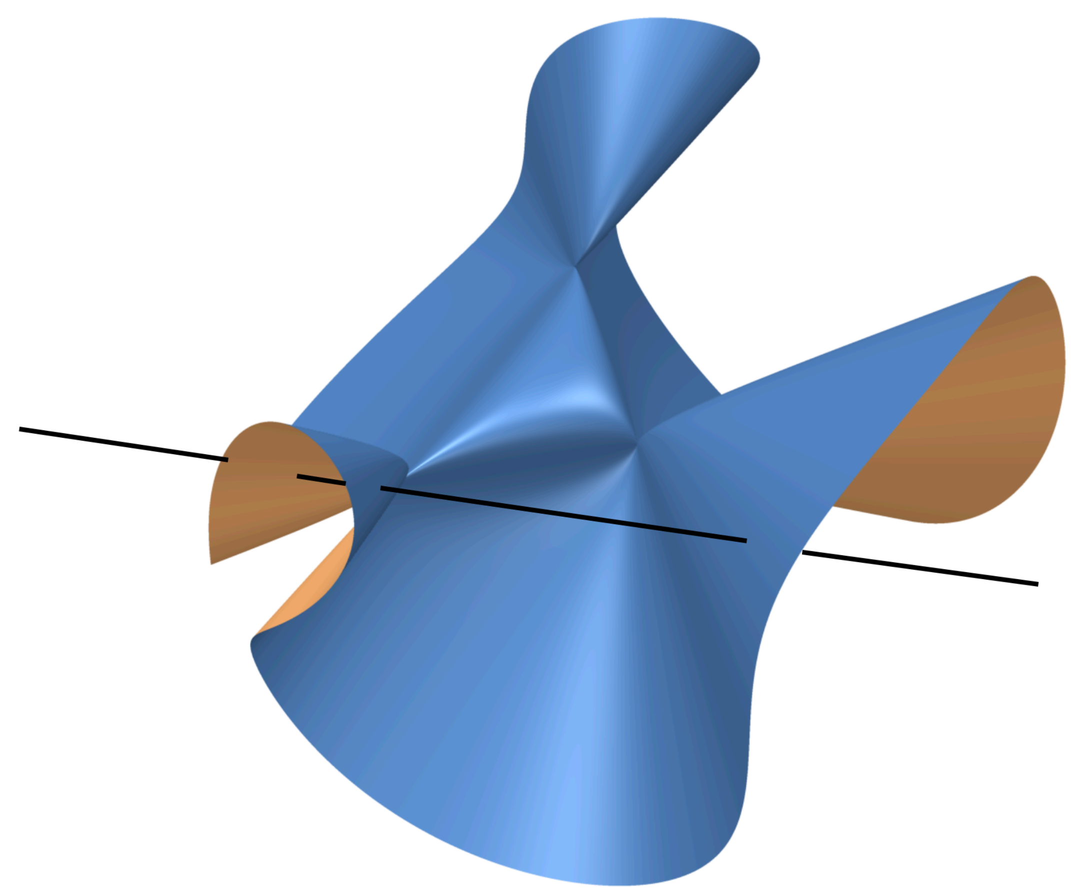
**find** $x = (x_0 : \cdots : x_3) \in X \subset \mathbb{P}_K^3$ **such that**

$$\begin{aligned} 0 &= a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 \\ 0 &= b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 \\ X &= \overline{\mathrm{im}(t \mapsto (1 : t_1 t_2 : t_1^2 t_2 : t_1 t_2^2))} \end{aligned}$$

**find** $u = (u_0 : u_1 : u_2) \in \mathbb{P}_K^2$ **such that**

$$\begin{aligned} 0 &= a_0 u_0^3 + a_1 u_0 u_1 u_2 + a_2 u_1^2 u_2 + a_3 u_1 u_2^2 \\ 0 &= b_0 u_0^3 + b_1 u_0 u_1 u_2 + b_2 u_1^2 u_2 + b_3 u_1 u_2^2 \end{aligned}$$





$x_1^3 - x_0 x_2 x_3 = 0$

# Solving polynomial equations

**General strategy:**

0. formulation

**find** $(t_1, t_2) \in K^2$ **such that**

$$0 \;=\; a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2$$
$$0 \;=\; b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2$$

**find** $x \in X \subset \mathbb{P}^3$ **such that**

$$0 \;=\; a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3$$
$$0 \;=\; b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$
$$X \;=\; \overline{\mathrm{im}(t \mapsto (1 : t_1 t_2 : t_1^2 t_2 : t_1 t_2^2))}$$

# Solving polynomial equations

**General strategy:**

0. formulation

1. build a matrix

**find** $(t_1, t_2) \in K^2$ **such that**

$$0 = a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2$$
$$0 = b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2$$

**find** $x \in X \subset \mathbb{P}^3$ **such that**

$$0 = a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3$$
$$0 = b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$
$$X = \overline{\mathrm{im}(t \mapsto (1 : t_1 t_2 : t_1^2 t_2 : t_1 t_2^2))}$$

$$M(d)$$

# Solving polynomial equations

**General strategy:**

0. formulation

1. build a matrix

2. compute kernel

**find** $(t_1, t_2) \in K^2$ **such that**

$$\begin{aligned} 0 &= a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2 \\ 0 &= b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2 \end{aligned}$$

**find** $x \in X \subset \mathbb{P}^3$ **such that**

$$\begin{aligned} 0 &= a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 \\ 0 &= b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 \\ X &= \overline{\mathrm{im}(t \mapsto (1 : t_1 t_2 : t_1^2 t_2 : t_1 t_2^2))} \end{aligned}$$

$M(d)$ $\qquad$ $N(d)$

# Solving polynomial equations

**General strategy:**

0. formulation

1. build a matrix

2. compute kernel

3. solve EVP

**find** $(t_1, t_2) \in K^2$ **such that**

$$\begin{aligned} 0 &= a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2 \\ 0 &= b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2 \end{aligned}$$

**find** $x \in X \subset \mathbb{P}^3$ **such that**

$$\begin{aligned} 0 &= a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 \\ 0 &= b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 \\ X &= \overline{\mathrm{im}(t \mapsto (1 : t_1 t_2 : t_1^2 t_2 : t_1 t_2^2))} \end{aligned}$$

$M(d)$

$N(d)$

$Av = \lambda v$

# Solving polynomial equations

**General strategy:**

| 0. formulation | 1. build a matrix | 2. compute kernel | 3. solve EVP |
|---|---|---|---|
| | | EXPENSIVE | CHEAP |

**find** $(t_1, t_2) \in K^2$ **such that**

$$0 = a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2$$
$$0 = b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2$$

**find** $x \in X \subset \mathbb{P}^3$ **such that**

$$0 = a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3$$
$$0 = b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$
$$X = \overline{\mathrm{im}(t \mapsto (1 : t_1 t_2 : t_1^2 t_2 : t_1 t_2^2))}$$

$$M(d) \qquad N(d) \qquad Av = \lambda v$$

# Solving polynomial equations

**General strategy:**

EXPENSIVE

CHEAP

0. formulation → 1. build a matrix → 2. compute kernel → 3. solve EVP

**find** $(t_1, t_2) \in K^2$ **such that**

$$0 = a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2$$
$$0 = b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2$$

**find** $x \in X \subset \mathbb{P}^3$ **such that**

$$0 = a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3$$
$$0 = b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$
$$X = \overline{\mathrm{im}(t \mapsto (1 : t_1 t_2 : t_1^2 t_2 : t_1 t_2^2))}$$

$$M(d) \qquad N(d) \qquad Av = \lambda v$$

0. formulation

equations generate an ideal $I = \langle f_1, f_2 \rangle \subset K[X]$

$$\langle a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3, b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 \rangle \subset K[X] = K[x_0, x_1, x_2, x_3]/\langle x_1^3 - x_0 x_2 x_3 \rangle$$

# Solving polynomial equations

**General strategy:**

EXPENSIVE CHEAP

0. formulation | 1. build a matrix | 2. compute kernel | 3. solve EVP

**find** $(t_1, t_2) \in K^2$ **such that**

$$\begin{aligned} 0 &= a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2 \\ 0 &= b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2 \end{aligned}$$

**find** $x \in X \subset \mathbb{P}^3$ **such that**

$$\begin{aligned} 0 &= a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 \\ 0 &= b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 \\ X &= \overline{\mathrm{im}(t \mapsto (1 : t_1 t_2 : t_1^2 t_2 : t_1 t_2^2))} \end{aligned}$$

$$M(d) \qquad N(d) \qquad Av = \lambda v$$

**0. formulation** equations generate an ideal $I = \langle f_1, f_2 \rangle \subset K[X]$

$$\langle a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3, b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 \rangle \subset K[X] = K[x_0, x_1, x_2, x_3] / \langle x_1^3 - x_0 x_2 x_3 \rangle$$

**1. build a matrix** $M(d)$ columns: basis for $K[X]_d$ rows: generate $I_d$

$$M(1) = \begin{array}{c c} & \begin{array}{cccc} x_0 & x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \end{array} & \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix} \end{array}$$

# Solving polynomial equations

**General strategy:**

| 0. formulation | 1. build a matrix | 2. compute kernel | 3. solve EVP |
|---|---|---|---|
| | | EXPENSIVE | CHEAP |

**find** $(t_1, t_2) \in K^2$ **such that**

$$\begin{aligned} 0 &= a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2 \\ 0 &= b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2 \end{aligned}$$

**find** $x \in X \subset \mathbb{P}^3$ **such that**

$$\begin{aligned} 0 &= a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 \\ 0 &= b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 \\ X &= \overline{\mathrm{im}(t \mapsto (1 : t_1 t_2 : t_1^2 t_2 : t_1 t_2^2))} \end{aligned}$$

$$M(d) \qquad N(d) \qquad Av = \lambda v$$

**0. formulation** equations generate an ideal $I = \langle f_1, f_2 \rangle \subset K[X]$

$$\langle a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3, b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 \rangle \subset K[X] = K[x_0, x_1, x_2, x_3] / \langle x_1^3 - x_0 x_2 x_3 \rangle$$

**1. build a matrix** $M(d)$ columns: basis for $K[X]_d$ rows: generate $I_d$

$$M(1) = \begin{array}{cc} & \begin{array}{cccc} x_0 & x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \end{array} & \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix} \end{array}$$

$$M(2) = \begin{array}{cc} & \begin{array}{cccccccccc} x_0^2 & x_0x_1 & x_0x_2 & x_0x_3 & x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 \end{array} \\ \begin{array}{c} x_0f_1 \\ x_1f_1 \\ x_2f_1 \\ x_3f_1 \\ x_0f_2 \\ x_1f_2 \\ x_2f_2 \\ x_3f_2 \end{array} & \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & & & & & & \\ & a_0 & & & a_1 & a_2 & & a_3 & & \\ & & a_0 & & & a_1 & a_2 & & a_3 & \\ & & & a_0 & & & & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & & & & & & \\ & b_0 & & & b_1 & b_2 & & b_3 & & \\ & & b_0 & & & b_1 & b_2 & & b_3 & \\ & & & b_0 & & & & b_1 & b_2 & b_3 \end{pmatrix} \end{array}$$

size: Hilbert function of $X$

$d$ : regularity of $K[X]/I$

# Macaulay matrices

**find** $(t_1, t_2) \in K^2$ **such that**

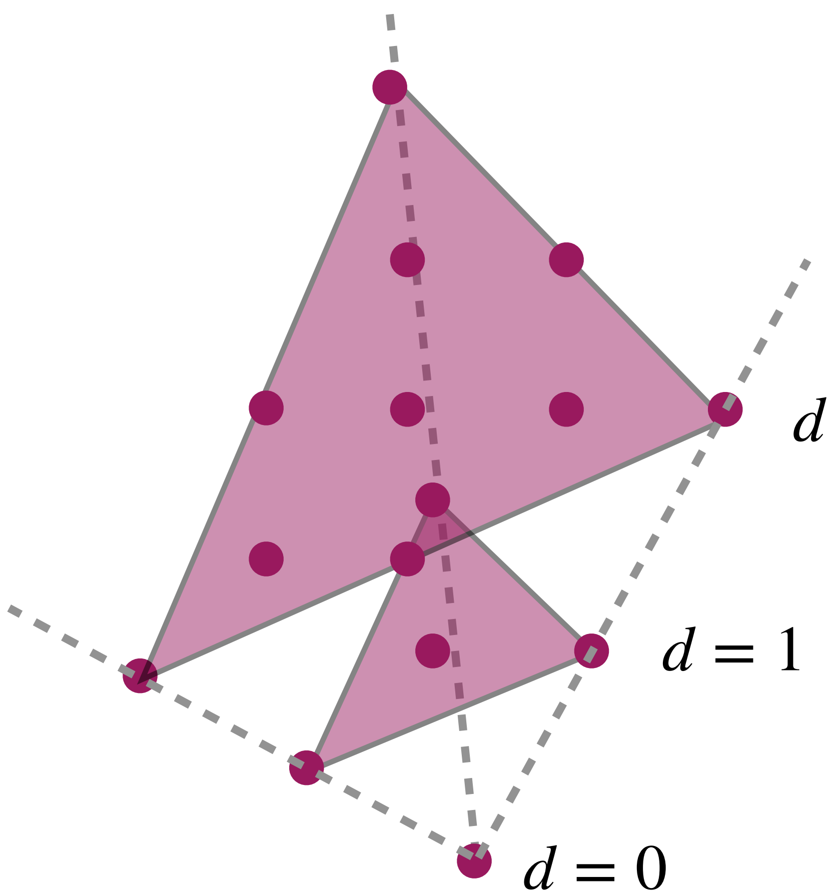
$$0 \;=\; a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2$$
$$0 \;=\; b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2$$

$$\begin{aligned} \langle f_1, f_2 \rangle \subset K[X] \;&=\; K[x_0, x_1, x_2, x_3]/\langle x_1^3 - x_0 x_2 x_3 \rangle \\ &\simeq\; K[t_0, t_0 t_1 t_2, t_0 t_1^2 t_2, t_0 t_1 t_2^2] \;=:\; R \end{aligned}$$

# Macaulay matrices

**find** $(t_1, t_2) \in K^2$ **such that**

$$0 \;=\; a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2$$
$$0 \;=\; b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2$$



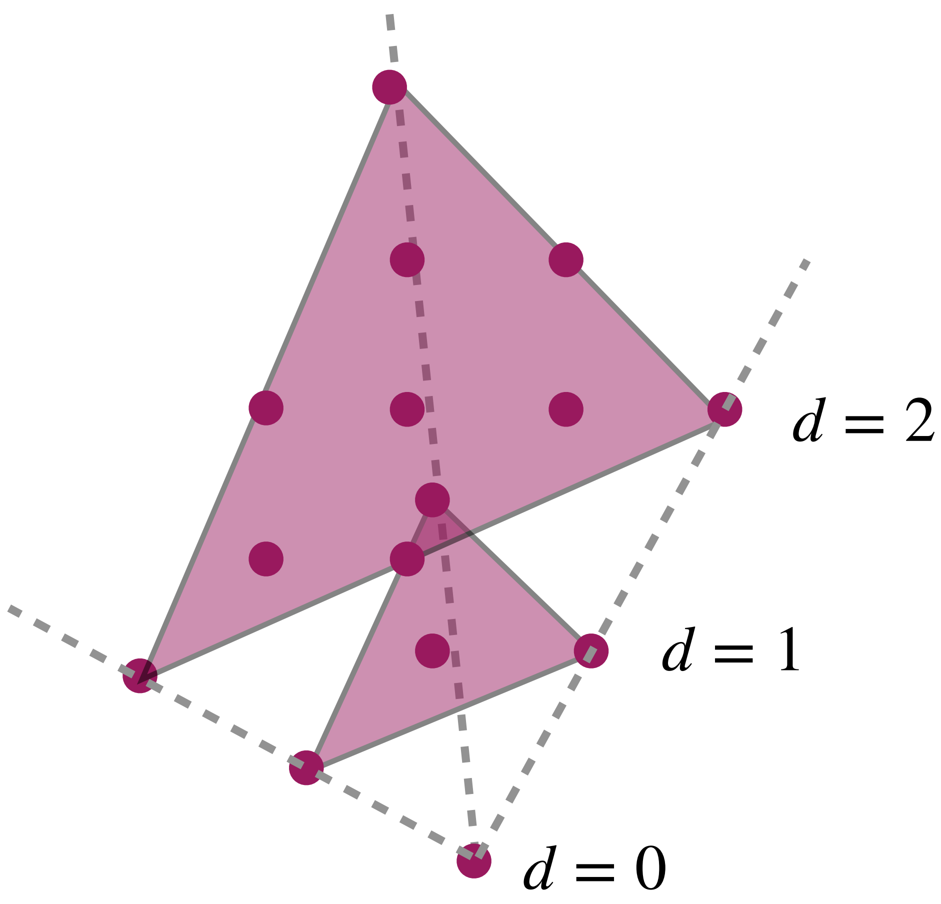

$$\langle f_1, f_2 \rangle \subset K[X] \;=\; K[x_0, x_1, x_2, x_3]/\langle x_1^3 - x_0 x_2 x_3 \rangle$$
$$\simeq\; K[t_0, t_0 t_1 t_2, t_0 t_1^2 t_2, t_0 t_1 t_2^2] \;=:\; R$$

$$R = \bigoplus_{d=0}^{\infty} R_d, \qquad \deg(t_0^d \cdot f(t_1, t_2)) \;=\; d$$

# Macaulay matrices

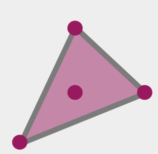
**find** $(t_1, t_2) \in K^2$ **such that**

$$0 = a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2$$
$$0 = b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2$$




$$\langle f_1, f_2 \rangle \subset K[X] = K[x_0, x_1, x_2, x_3]/\langle x_1^3 - x_0 x_2 x_3 \rangle$$
$$\simeq K[t_0, t_0 t_1 t_2, t_0 t_1^2 t_2, t_0 t_1 t_2^2] =: R$$

$$R = \bigoplus_{d=0}^{\infty} R_d, \quad \deg(t_0^d \cdot f(t_1, t_2)) = d$$

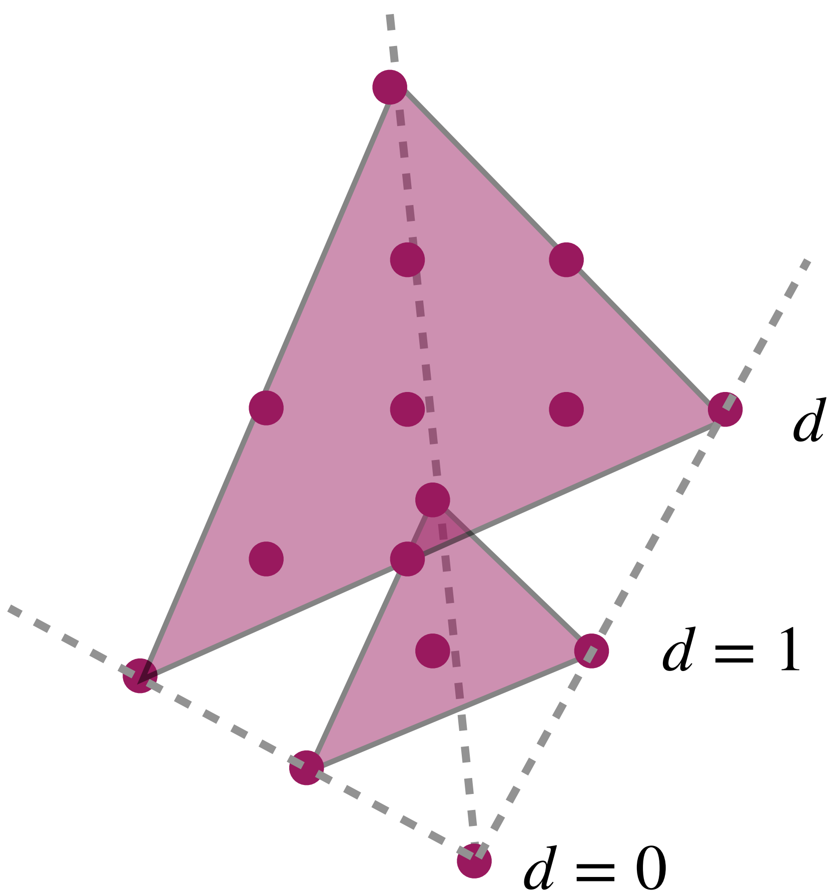


$$M(1) = \begin{array}{c} \\ f_1 \\ f_2 \end{array} \begin{array}{c} \begin{array}{cccc} x_0 & x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix} \end{array}$$

# Macaulay matrices

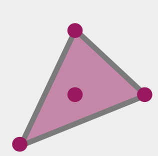
**find** $(t_1, t_2) \in K^2$ **such that**

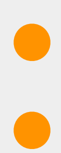
$$0 = a_0 + a_1 t_1 t_2 + a_2 t_1^2 t_2 + a_3 t_1 t_2^2$$
$$0 = b_0 + b_1 t_1 t_2 + b_2 t_1^2 t_2 + b_3 t_1 t_2^2$$



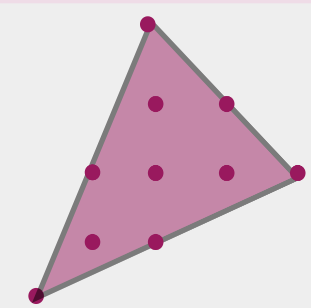

$$\langle f_1, f_2 \rangle \subset K[X] = K[x_0, x_1, x_2, x_3]/\langle x_1^3 - x_0 x_2 x_3 \rangle$$
$$\simeq K[t_0, t_0 t_1 t_2, t_0 t_1^2 t_2, t_0 t_1 t_2^2] =: R$$

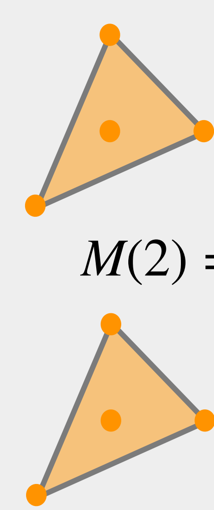
$$R = \bigoplus_{d=0}^{\infty} R_d, \quad \deg(t_0^d \cdot f(t_1, t_2)) = d$$





$$M(1) = \begin{array}{cc} & \begin{matrix} x_0 & x_1 & x_2 & x_3 \end{matrix} \\ \begin{matrix} f_1 \\ f_2 \end{matrix} & \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix} \end{array}$$





$$M(2) = \begin{array}{cc} & \begin{matrix} x_0^2 & x_0x_1 & x_0x_2 & x_0x_3 & x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 \end{matrix} \\ \begin{matrix} x_0f_1 \\ x_1f_1 \\ x_2f_1 \\ x_3f_1 \\ x_0f_2 \\ x_1f_2 \\ x_2f_2 \\ x_3f_2 \end{matrix} & \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & & & & & & \\ & a_0 & & & a_1 & a_2 & & a_3 & & \\ & & a_0 & & & a_1 & a_2 & & a_3 & \\ & & & a_0 & & & & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & & & & & & \\ & b_0 & & & b_1 & b_2 & & b_3 & & \\ & & b_0 & & & b_1 & b_2 & & b_3 & \\ & & & b_0 & & & & b_1 & b_2 & b_3 \end{pmatrix} \end{array}$$

# Beyond toric varieties

Sometimes **0. formulation** is for free:

Compute all lines in 3-space touching four given lines

# Beyond toric varieties

Sometimes **0. formulation** is for free:

Compute all lines in 3-space touching four given lines

**find** $p = (p_{12} : \cdots : p_{34}) \in \mathrm{Gr}(2,4) \subset \mathbb{P}^5$ **such that**

$$\begin{aligned}
0 &= a_{12}p_{12} + a_{13}p_{13} + a_{14}p_{14} + a_{23}p_{23} + a_{24}p_{24} + a_{34}p_{34} \\
0 &= b_{12}p_{12} + b_{13}p_{13} + b_{14}p_{14} + b_{23}p_{23} + b_{24}p_{24} + b_{34}p_{34} \\
0 &= c_{12}p_{12} + c_{13}p_{13} + c_{14}p_{14} + c_{23}p_{23} + c_{24}p_{24} + c_{34}p_{34} \\
0 &= d_{12}p_{12} + d_{13}p_{13} + d_{14}p_{14} + d_{23}p_{23} + d_{24}p_{24} + d_{34}p_{34}
\end{aligned}$$

# Beyond toric varieties

Sometimes **0. formulation** is for free:

Compute all lines in 3-space touching four given lines

**find** $p = (p_{12} : \cdots : p_{34}) \in \mathrm{Gr}(2,4) \subset \mathbb{P}^5$ **such that**

$$\begin{aligned}
0 &= a_{12}p_{12} + a_{13}p_{13} + a_{14}p_{14} + a_{23}p_{23} + a_{24}p_{24} + a_{34}p_{34} \\
0 &= b_{12}p_{12} + b_{13}p_{13} + b_{14}p_{14} + b_{23}p_{23} + b_{24}p_{24} + b_{34}p_{34} \\
0 &= c_{12}p_{12} + c_{13}p_{13} + c_{14}p_{14} + c_{23}p_{23} + c_{24}p_{24} + c_{34}p_{34} \\
0 &= d_{12}p_{12} + d_{13}p_{13} + d_{14}p_{14} + d_{23}p_{23} + d_{24}p_{24} + d_{34}p_{34}
\end{aligned}$$

expand in monomials and solve on TV ✗ solve directly on the Grassmannian ✓

# Beyond toric varieties

Sometimes **0. formulation** is for free:

Compute all lines in 3-space touching four given lines

**find** $p = (p_{12} : \cdots : p_{34}) \in \mathrm{Gr}(2,4) \subset \mathbb{P}^5$ **such that**

$$
\begin{aligned}
0 &= a_{12}p_{12} + a_{13}p_{13} + a_{14}p_{14} + a_{23}p_{23} + a_{24}p_{24} + a_{34}p_{34} \\
0 &= b_{12}p_{12} + b_{13}p_{13} + b_{14}p_{14} + b_{23}p_{23} + b_{24}p_{24} + b_{34}p_{34} \\
0 &= c_{12}p_{12} + c_{13}p_{13} + c_{14}p_{14} + c_{23}p_{23} + c_{24}p_{24} + c_{34}p_{34} \\
0 &= d_{12}p_{12} + d_{13}p_{13} + d_{14}p_{14} + d_{23}p_{23} + d_{24}p_{24} + d_{34}p_{34}
\end{aligned}
$$

expand in monomials and solve on TV ✗ solve directly on the Grassmannian ✓

$$\phi_0 = t_1 - t_2, \quad \phi_1 = t_2^2 - t_2, \quad \phi_2 = t_1t_2 - t_2, \quad \phi_3 = t_1^2 - t_2, \quad \phi_4 = t_1t_2^2 - t_2, \quad \phi_5 = t_1^2t_2 - t_2$$

$$f_1 = -7\phi_0 + 10\phi_1 + 17\phi_2 - 10\phi_3 - 17\phi_4 + 16\phi_5, \qquad f_2 = 2\phi_0 + 5\phi_1 + 5\phi_2 + 5\phi_3 - 6\phi_4 - 6\phi_5$$

# Beyond toric varieties

Sometimes **0. formulation** is for free:

Compute all lines in 3-space touching four given lines

**find** $p = (p_{12} : \cdots : p_{34}) \in \mathrm{Gr}(2{,}4) \subset \mathbb{P}^5$ **such that**

$$\begin{aligned}
0 &= a_{12}p_{12} + a_{13}p_{13} + a_{14}p_{14} + a_{23}p_{23} + a_{24}p_{24} + a_{34}p_{34}\\
0 &= b_{12}p_{12} + b_{13}p_{13} + b_{14}p_{14} + b_{23}p_{23} + b_{24}p_{24} + b_{34}p_{34}\\
0 &= c_{12}p_{12} + c_{13}p_{13} + c_{14}p_{14} + c_{23}p_{23} + c_{24}p_{24} + c_{34}p_{34}\\
0 &= d_{12}p_{12} + d_{13}p_{13} + d_{14}p_{14} + d_{23}p_{23} + d_{24}p_{24} + d_{34}p_{34}
\end{aligned}$$

expand in monomials and solve on TV ✗ solve directly on the Grassmannian ✓

$$\phi_0 = t_1 - t_2, \quad \phi_1 = t_2^2 - t_2, \quad \phi_2 = t_1 t_2 - t_2, \quad \phi_3 = t_1^2 - t_2, \quad \phi_4 = t_1 t_2^2 - t_2, \quad \phi_5 = t_1^2 t_2 - t_2$$

$$f_1 = -7\phi_0 + 10\phi_1 + 17\phi_2 - 10\phi_3 - 17\phi_4 + 16\phi_5, \qquad f_2 = 2\phi_0 + 5\phi_1 + 5\phi_2 + 5\phi_3 - 6\phi_4 - 6\phi_5$$

**find** $x = (x_0 : \cdots : x_5) \in X \subset \mathbb{P}^5$ **such that**

$$\begin{aligned}
0 &= -7x_0 + 10x_1 + 17x_2 - 10x_3 - 17x_4 + 16x_5\\
0 &= 2x_0 + 5x_1 + 5x_2 + 5x_3 - 6x_4 - 6x_5
\end{aligned}$$

$$X = \overline{\mathrm{im}(t \mapsto (t_1 - t_2 : t_2^2 - t_2 : \cdots : t_1^2 t_2 - t_2))}$$

A del Pezzo surface of degree 5

# Generalized Macaulay matrices

$$\phi_0 = t_1 - t_2, \quad \phi_1 = t_2^2 - t_2, \quad \phi_2 = t_1 t_2 - t_2, \quad \phi_3 = t_1^2 - t_2, \quad \phi_4 = t_1 t_2^2 - t_2, \quad \phi_5 = t_1^2 t_2 - t_2$$

$$f_1 = -7\phi_0 + 10\phi_1 + 17\phi_2 - 10\phi_3 - 17\phi_4 + 16\phi_5, \qquad f_2 = 2\phi_0 + 5\phi_1 + 5\phi_2 + 5\phi_3 - 6\phi_4 - 6\phi_5$$

$$K[X] = K[x_0, \ldots, x_5]/\langle 5 \textbf{ quadrics}, 1 \textbf{ cubic}\rangle$$

# Generalized Macaulay matrices

$$\phi_0 = t_1 - t_2, \quad \phi_1 = t_2^2 - t_2, \quad \phi_2 = t_1 t_2 - t_2, \quad \phi_3 = t_1^2 - t_2, \quad \phi_4 = t_1 t_2^2 - t_2, \quad \phi_5 = t_1^2 t_2 - t_2$$

$$f_1 = -7\phi_0 + 10\phi_1 + 17\phi_2 - 10\phi_3 - 17\phi_4 + 16\phi_5, \qquad f_2 = 2\phi_0 + 5\phi_1 + 5\phi_2 + 5\phi_3 - 6\phi_4 - 6\phi_5$$

$$K[X] = K[x_0, \ldots, x_5]/\langle 5 \textbf{ quadrics}, 1 \textbf{ cubic} \rangle$$

$$M(1) = \begin{array}{c c} & \begin{array}{cccccc} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \end{array} & \begin{bmatrix} -7 & 10 & 17 & -10 & -17 & 16 \\ 2 & 5 & 5 & 5 & -6 & -6 \end{bmatrix} \end{array} \quad \textbf{2 x 6}$$

# Generalized Macaulay matrices

$$\phi_0 = t_1 - t_2, \quad \phi_1 = t_2^2 - t_2, \quad \phi_2 = t_1 t_2 - t_2, \quad \phi_3 = t_1^2 - t_2, \quad \phi_4 = t_1 t_2^2 - t_2, \quad \phi_5 = t_1^2 t_2 - t_2$$

$$f_1 = -7\phi_0 + 10\phi_1 + 17\phi_2 - 10\phi_3 - 17\phi_4 + 16\phi_5, \qquad f_2 = 2\phi_0 + 5\phi_1 + 5\phi_2 + 5\phi_3 - 6\phi_4 - 6\phi_5$$

$$K[X] = K[x_0, \ldots, x_5]/\langle 5 \textbf{ quadrics}, 1 \textbf{ cubic} \rangle$$

$$M(1) = \begin{array}{cc} & \begin{matrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{matrix} f_1 \\ f_2 \end{matrix} & \begin{bmatrix} -7 & 10 & 17 & -10 & -17 & 16 \\ 2 & 5 & 5 & 5 & -6 & -6 \end{bmatrix} \end{array}$$

**2 x 6**

**12 x 16**

$$M(2) = \begin{array}{cc} & \begin{matrix} x_1^2 & x_0x_1 & x_1x_2 & \cdots \end{matrix} \\ \begin{matrix} x_0 \cdot f_1 \\ x_1 \cdot f_1 \\ x_2 \cdot f_1 \\ \vdots \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix} & \begin{bmatrix} 0 & 10 & 0 & 0 & -7 & 17 & -17 & 0 & 0 & -10 & 16 & 0 & 0 & 0 & 0 & 0 \\ 10 & -7 & 7 & -17 & 0 & 10 & -10 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 17 & 0 & 0 & 3 & -10 & -17 & 0 & 0 & -10 & 16 & 0 & 0 & 0 & 0 \\ 0 & -17 & 27 & 0 & 0 & -10 & 10 & 0 & 0 & -7 & 17 & -17 & 0 & -10 & 16 & 0 \\ 0 & -17 & 0 & 10 & 0 & 10 & 0 & 17 & -17 & 0 & 0 & -10 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -7 & 17 & -17 & 0 & -10 & 16 \\ 0 & 5 & 0 & 0 & 2 & 5 & -6 & 0 & 0 & 5 & -6 & 0 & 0 & 0 & 0 & 0 \\ 5 & 2 & 10 & -6 & 0 & -5 & 5 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 15 & 0 & 0 & -3 & 4 & -6 & 0 & 0 & 5 & -6 & 0 & 0 & 0 & 0 \\ 0 & -5 & 10 & 0 & 0 & -4 & 4 & 0 & 0 & 2 & 5 & -6 & 0 & 5 & -6 & 0 \\ 0 & -5 & 0 & 5 & 0 & -5 & 12 & 5 & -6 & 0 & 0 & 5 & -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 2 & 5 & -6 & 0 & 5 & -6 \end{bmatrix} \end{array}$$

# Generalized Macaulay matrices

$$\phi_0 = t_1 - t_2, \quad \phi_1 = t_2^2 - t_2, \quad \phi_2 = t_1 t_2 - t_2, \quad \phi_3 = t_1^2 - t_2, \quad \phi_4 = t_1 t_2^2 - t_2, \quad \phi_5 = t_1^2 t_2 - t_2$$

$$f_1 = -7\phi_0 + 10\phi_1 + 17\phi_2 - 10\phi_3 - 17\phi_4 + 16\phi_5, \qquad f_2 = 2\phi_0 + 5\phi_1 + 5\phi_2 + 5\phi_3 - 6\phi_4 - 6\phi_5$$

$$K[X] = K[x_0, \ldots, x_5]/\langle 5 \textbf{ quadrics}, 1 \textbf{ cubic} \rangle$$

$$M(1) = \begin{array}{c} \\ f_1 \\ f_2 \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} -7 & 10 & 17 & -10 & -17 & 16 \\ 2 & 5 & 5 & 5 & -6 & -6 \end{bmatrix} \end{array}$$

**2 x 6**

**12 x 16**

$$M(2) = \begin{array}{c} \\ x_0 \cdot f_1 \\ x_1 \cdot f_1 \\ x_2 \cdot f_1 \\ \vdots \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \!\!\begin{array}{c} \begin{array}{cccc} x_1^2 & x_0 x_1 & x_1 x_2 & \cdots \end{array} \\ \begin{bmatrix}
0 & 10 & 0 & 0 & -7 & 17 & -17 & 0 & 0 & -10 & 16 & 0 & 0 & 0 & 0 & 0 \\
10 & -7 & 7 & -17 & 0 & 10 & -10 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 17 & 0 & 0 & 3 & -10 & -17 & 0 & 0 & -10 & 16 & 0 & 0 & 0 & 0 \\
0 & -17 & 27 & 0 & 0 & -10 & 10 & 0 & 0 & -7 & 17 & -17 & 0 & -10 & 16 & 0 \\
0 & -17 & 0 & 10 & 0 & 10 & 0 & 17 & -17 & 0 & 0 & -10 & 16 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -7 & 17 & -17 & 0 & -10 & 16 \\
0 & 5 & 0 & 0 & 2 & 5 & -6 & 0 & 0 & 5 & -6 & 0 & 0 & 0 & 0 & 0 \\
5 & 2 & 10 & -6 & 0 & -5 & 5 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & 15 & 0 & 0 & -3 & 4 & -6 & 0 & 0 & 5 & -6 & 0 & 0 & 0 & 0 \\
0 & -5 & 10 & 0 & 0 & -4 & 4 & 0 & 0 & 2 & 5 & -6 & 0 & 5 & -6 & 0 \\
0 & -5 & 0 & 5 & 0 & -5 & 12 & 5 & -6 & 0 & 0 & 5 & -6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 2 & 5 & -6 & 0 & 5 & -6
\end{bmatrix} \end{array}$$

need a basis for $K[X]_d$

# Khovanskii bases

Gröbner Bases
and Convex Polytopes

Ch 11

Bernd Sturmfels

# Khovanskii bases

Gröbner Bases and Convex Polytopes

Ch 11

Bernd Sturmfels

$R = K[t_0\phi_0, \ldots, t_0\phi_\ell] \subset K[t_0, \ldots, t_n]$

$R$ is the coordinate ring $K[X]$

# Khovanskii bases

Gröbner Bases
and Convex Polytopes
**Ch 11**
Bernd Sturmfels

$R = K[t_0\phi_0, \ldots, t_0\phi_\ell] \subset K[t_0, \ldots, t_n]$  $R$ is the coordinate ring $K[X]$

Definition: $t_0\phi_0, \ldots, t_0\phi_\ell$ is a Khovanskii basis for $R$ with respect to $\prec$ if

$$R_{\text{mon}} := K[\text{in}_\prec(t_0\phi_0), \ldots, \text{in}_\prec(t_0\phi_\ell)] = \text{in}_\prec(R) \implies \text{HF}_{K[X]}(d) = \text{HF}_{R_{\text{mon}}}(d)$$

*or SAGBI basis, canonical basis, …*

# Khovanskii bases

Gröbner Bases and Convex Polytopes

Ch 11

Bernd Sturmfels

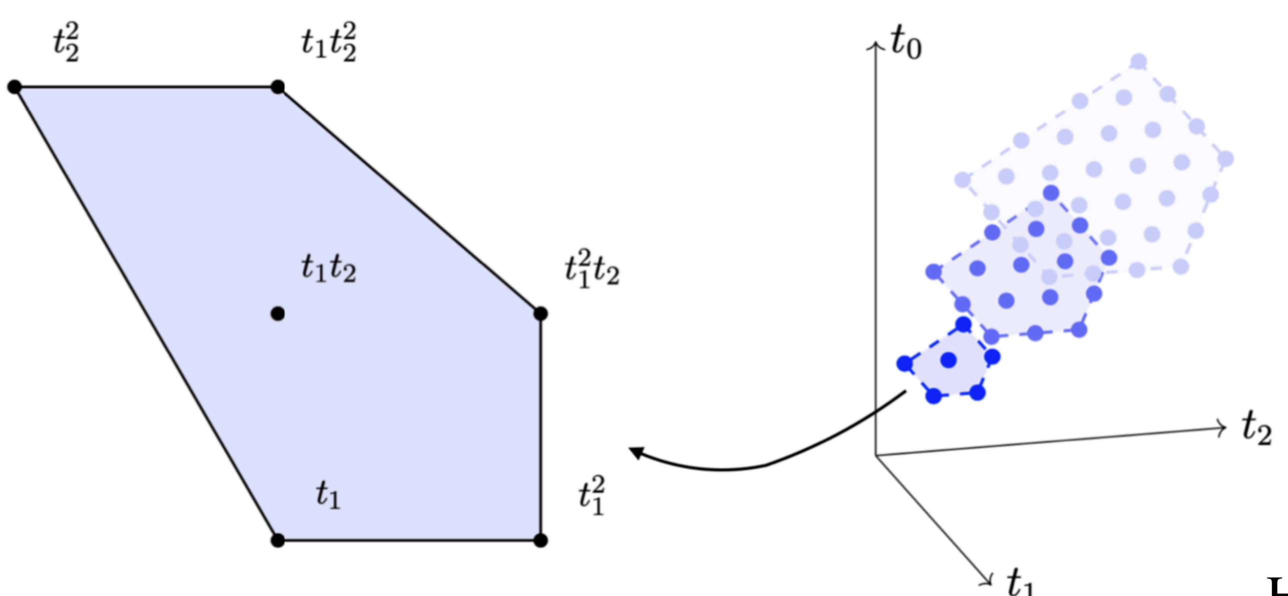
$R \;=\; K[t_0\phi_0, \ldots, t_0\phi_\ell] \subset K[t_0, \ldots, t_n]$ $\qquad$ $R$ is the coordinate ring $K[X]$

**Definition:** $t_0\phi_0, \ldots, t_0\phi_\ell$ is a Khovanskii basis for $R$ with respect to $\prec$ if

$$R_{\text{mon}} := K[\text{in}_\prec(t_0\phi_0), \ldots, \text{in}_\prec(t_0\phi_\ell)] = \text{in}_\prec(R) \implies \text{HF}_{K[X]}(d) = \text{HF}_{R_{\text{mon}}}(d)$$

*or SAGBI basis, canonical basis, …*

$\phi_0 = \underline{t_1} - t_2, \quad \phi_1 = \underline{t_2^2} - t_2, \quad \phi_2 = \underline{t_1 t_2} - t_2, \quad \phi_3 = \underline{t_1^2} - t_2, \quad \phi_4 = \underline{t_1 t_2^2} - t_2, \quad \phi_5 = \underline{t_1^2 t_2} - t_2$




| | | | | | |
|---|---|---|---|---|---|
| $d$ : | 0 | 1 | 2 | 3 | $\cdots$ |
| $\text{HF}_{K[X]}(d)$ : | 1 | 6 | 16 | 31 | $\cdots$ |

# Khovanskii bases

Gröbner Bases
and Convex Polytopes
**Ch 11**
Bernd Sturmfels

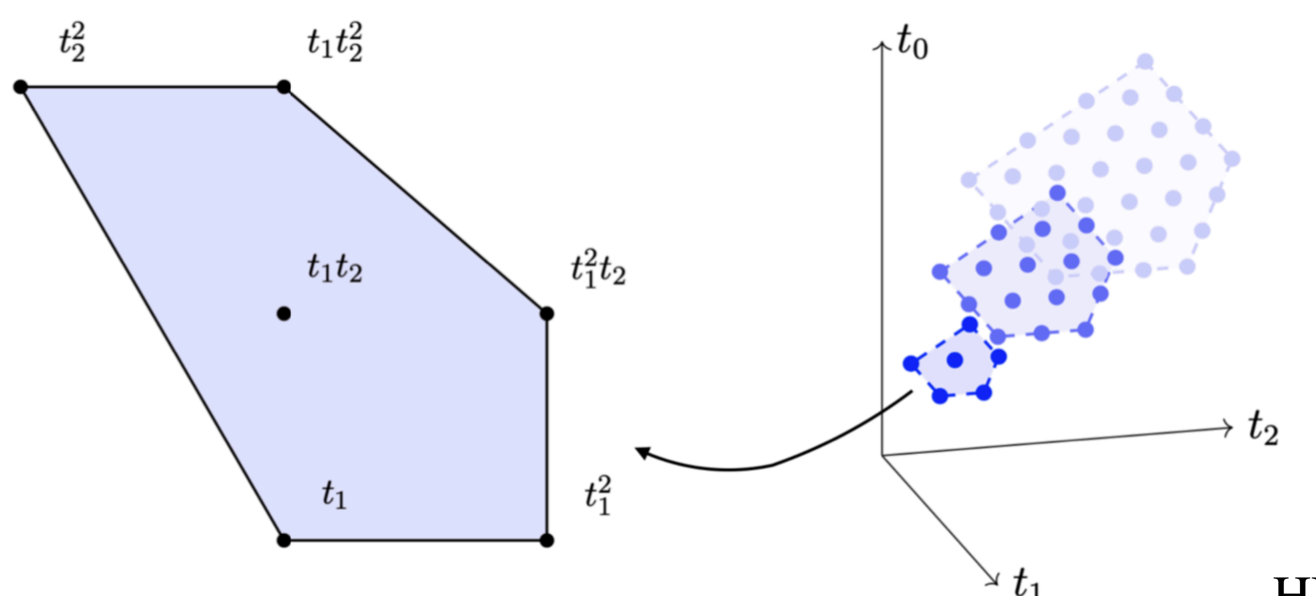
$R \ = \ K[t_0\phi_0, \ldots, t_0\phi_\ell] \subset K[t_0, \ldots, t_n]$ $\quad$ $R$ is the coordinate ring $K[X]$

Definition: $t_0\phi_0, \ldots, t_0\phi_\ell$ is a Khovanskii basis for $R$ with respect to $\prec$ if

$$R_{\text{mon}} := K[\text{in}_\prec(t_0\phi_0), \ldots, \text{in}_\prec(t_0\phi_\ell)] \ = \ \text{in}_\prec(R) \implies \text{HF}_{K[X]}(d) \ = \ \text{HF}_{R_{\text{mon}}}(d)$$

*or SAGBI basis, canonical basis, …*

$$\phi_0 = \underline{t_1} - t_2, \quad \phi_1 = \underline{t_2^2} - t_2, \quad \phi_2 = \underline{t_1t_2} - t_2, \quad \phi_3 = \underline{t_1^2} - t_2, \quad \phi_4 = \underline{t_1t_2^2} - t_2, \quad \phi_5 = \underline{t_1^2t_2} - t_2$$




| | | | | | |
|---|---|---|---|---|---|
| $d:$ | 0 | 1 | 2 | 3 | $\cdots$ |
| $\text{HF}_{K[X]}(d):$ | 1 | 6 | 16 | 31 | $\cdots$ |

For each dot at level $d$, it is easy to construct a basis monomial of $K[X]_d$

# Equations on a del Pezzo surface

$$\phi_0 = t_1 - t_2, \quad \phi_1 = t_2^2 - t_2, \quad \phi_2 = t_1 t_2 - t_2, \quad \phi_3 = t_1^2 - t_2, \quad \phi_4 = t_1 t_2^2 - t_2, \quad \phi_5 = t_1^2 t_2 - t_2$$

$$f_1 = -7\phi_0 + 10\phi_1 + 17\phi_2 - 10\phi_3 - 17\phi_4 + 16\phi_5, \qquad f_2 = 2\phi_0 + 5\phi_1 + 5\phi_2 + 5\phi_3 - 6\phi_4 - 6\phi_5$$

```
d = 2; p = 9716633; K = GF(p)
R, (t1,t2) = PolynomialRing(K, ["t1";"t2"])
φ = [t1-t2; t2^2-t2; t1*t2-t2; t1^2-t2; t1*t2^2-t2; t1^2*t2-t2]
S, x = PolynomialRing(K, ["x$i" for i = 1:length(φ)])
f = [-7 10 17 -10 -17 16;2 5 5 5 -6 -6]*φ;
degs_f = [1;1]; dreg = sum(degs_f)+1;
leadexps = [leadexp(h,[-2;-1]) for h in φ]
Mul = get_commuting_matrices(f,dreg,degs_f,φ,K,(t1,t2),leadexps)
```

# Equations on a Bott-Samelson threefold

$$f_1 = 1 + t_1 + t_2 + t_3 + t_1t_3 + t_2t_3 + t_1(t_1t_3 + t_2) + t_2(t_1t_3 + t_2)$$
$$f_2 = 1 - 2t_1 + 3t_2 - 4t_3 + 5t_1t_3 - 6t_2t_3 + 7t_1(t_1t_3 + t_2) - 8t_2(t_1t_3 + t_2)$$
$$f_3 = 2 + 3t_1 + 5t_2 + 7t_3 + 11t_1t_3 + 13t_2t_3 + 17t_1(t_1t_3 + t_2) + 19t_2(t_1t_3 + t_2)$$

Okounkov bodies and toric degenerations

**Authors:** Dave Anderson

Numerical homotopies from Khovanskii bases

**Authors:** Michael Burr, Frank Sottile, Elise Walker

# Equations on a Bott-Samelson threefold

0. formulation

$$X \subset \mathbb{P}^7$$

$$f_1 = 1 + t_1 + t_2 + t_3 + t_1t_3 + t_2t_3 + t_1(t_1t_3 + t_2) + t_2(t_1t_3 + t_2)$$
$$f_2 = 1 - 2t_1 + 3t_2 - 4t_3 + 5t_1t_3 - 6t_2t_3 + 7t_1(t_1t_3 + t_2) - 8t_2(t_1t_3 + t_2)$$
$$f_3 = 2 + 3t_1 + 5t_2 + 7t_3 + 11t_1t_3 + 13t_2t_3 + 17t_1(t_1t_3 + t_2) + 19t_2(t_1t_3 + t_2)$$

Okounkov bodies and toric degenerations

**Authors:** Dave Anderson

Numerical homotopies from Khovanskii bases

**Authors:** Michael Burr, Frank Sottile, Elise Walker

# Equations on a Bott-Samelson threefold

0. formulation

$$X \subset \mathbb{P}^7$$

$$\begin{aligned} f_1 &= 1 + t_1 + t_2 + t_3 + t_1t_3 + t_2t_3 + t_1(t_1t_3 + t_2) + t_2(t_1t_3 + t_2) \\ f_2 &= 1 - 2t_1 + 3t_2 - 4t_3 + 5t_1t_3 - 6t_2t_3 + 7t_1(t_1t_3 + t_2) - 8t_2(t_1t_3 + t_2) \\ f_3 &= 2 + 3t_1 + 5t_2 + 7t_3 + 11t_1t_3 + 13t_2t_3 + 17t_1(t_1t_3 + t_2) + 19t_2(t_1t_3 + t_2) \end{aligned}$$

```
K = QQ; R, (t1,t2,t3) = PolynomialRing(K, ["t1";"t2";"t3"])
φ = [t1^0; t1; t2; t3; t1*t3; t2*t3; t1*(t1*t3+t2); t2*(t1*t3+t2)]
```

Okounkov bodies and toric degenerations

Authors: Dave Anderson

Numerical homotopies from Khovanskii bases

Authors: Michael Burr, Frank Sottile, Elise Walker

# Equations on a Bott-Samelson threefold

0. formulation

$$\begin{aligned} f_1 &= 1 + t_1 + t_2 + t_3 + t_1t_3 + t_2t_3 + t_1(t_1t_3 + t_2) + t_2(t_1t_3 + t_2) \\ f_2 &= 1 - 2t_1 + 3t_2 - 4t_3 + 5t_1t_3 - 6t_2t_3 + 7t_1(t_1t_3 + t_2) - 8t_2(t_1t_3 + t_2) \\ f_3 &= 2 + 3t_1 + 5t_2 + 7t_3 + 11t_1t_3 + 13t_2t_3 + 17t_1(t_1t_3 + t_2) + 19t_2(t_1t_3 + t_2) \end{aligned}$$

$$X \subset \mathbb{P}^7$$

```
K = QQ; R, (t1,t2,t3) = PolynomialRing(K, ["t1";"t2";"t3"])
φ = [t1^0; t1; t2; t3; t1*t3; t2*t3; t1*(t1*t3+t2); t2*(t1*t3+t2)]
```

```
f = [1 1 1 1 1 1 1 1; 1 -2 3 -4 5 -6 7 -8; 2 3 5 7 11 13 17 19]*φ
```

Okounkov bodies and toric degenerations

**Authors:** Dave Anderson

Numerical homotopies from Khovanskii bases

**Authors:** Michael Burr, Frank Sottile, Elise Walker

# Equations on a Bott-Samelson threefold

0. formulation

$$X \subset \mathbb{P}^7$$

$$\begin{aligned}
f_1 &= 1 + t_1 + t_2 + t_3 + t_1 t_3 + t_2 t_3 + t_1(t_1 t_3 + t_2) + t_2(t_1 t_3 + t_2)\\
f_2 &= 1 - 2t_1 + 3t_2 - 4t_3 + 5t_1 t_3 - 6t_2 t_3 + 7t_1(t_1 t_3 + t_2) - 8t_2(t_1 t_3 + t_2)\\
f_3 &= 2 + 3t_1 + 5t_2 + 7t_3 + 11t_1 t_3 + 13t_2 t_3 + 17t_1(t_1 t_3 + t_2) + 19t_2(t_1 t_3 + t_2)
\end{aligned}$$

```
K = QQ; R, (t1,t2,t3) = PolynomialRing(K, ["t1";"t2";"t3"])
φ = [t1^0; t1; t2; t3; t1*t3; t2*t3; t1*(t1*t3+t2); t2*(t1*t3+t2)]
```

```
f = [1 1 1 1 1 1 1 1; 1 -2 3 -4 5 -6 7 -8; 2 3 5 7 11 13 17 19]*φ
```

```
degs_f = [1;1;1]; dreg = 3; leadexps = [leadexp(h,[0;-1;0]) for h in φ]
sols = solve_Khovanskii(f,dreg,degs_f,φ,(t1,t2,t3),leadexps)
```

Okounkov bodies and toric degenerations
**Authors:** Dave Anderson

Numerical homotopies from Khovanskii bases
**Authors:** Michael Burr, Frank Sottile, Elise Walker

# Equations on a Bott-Samelson threefold

0. formulation

$X \subset \mathbb{P}^7$

OSCAR SYMBOLIC TOOLS

$$
\begin{aligned}
f_1 &= 1 + t_1 + t_2 + t_3 + t_1t_3 + t_2t_3 + t_1(t_1t_3 + t_2) + t_2(t_1t_3 + t_2)\\
f_2 &= 1 - 2t_1 + 3t_2 - 4t_3 + 5t_1t_3 - 6t_2t_3 + 7t_1(t_1t_3 + t_2) - 8t_2(t_1t_3 + t_2)\\
f_3 &= 2 + 3t_1 + 5t_2 + 7t_3 + 11t_1t_3 + 13t_2t_3 + 17t_1(t_1t_3 + t_2) + 19t_2(t_1t_3 + t_2)
\end{aligned}
$$

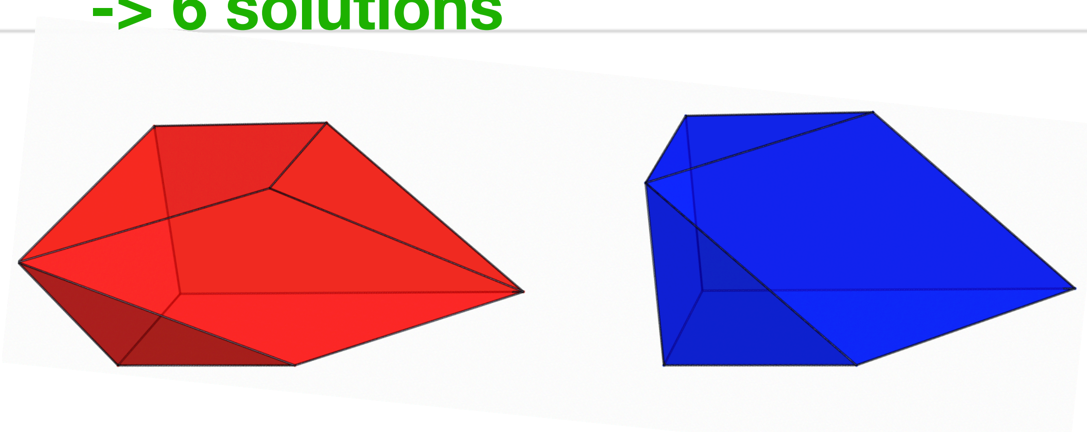
```
K = QQ; R, (t1,t2,t3) = PolynomialRing(K, ["t1";"t2";"t3"])
φ = [t1^0; t1; t2; t3; t1*t3; t2*t3; t1*(t1*t3+t2); t2*(t1*t3+t2)]
```

```
f = [1 1 1 1 1 1 1 1; 1 -2 3 -4 5 -6 7 -8; 2 3 5 7 11 13 17 19]*φ
```

```
degs_f = [1;1;1]; dreg = 3; leadexps = [leadexp(h,[0;-1;0]) for h in φ]
sols = solve_Khovanskii(f,dreg,degs_f,φ,(t1,t2,t3),leadexps)
```

-> 6 solutions

Vol = 10



Vol = 6

Okounkov bodies and toric degenerations

Authors: Dave Anderson

Numerical homotopies from Khovanskii bases

Authors: Michael Burr, Frank Sottile, Elise Walker

# Complete intersections

$$I = \langle f_1, \ldots, f_n \rangle \subset K[X], \quad V_X(I) = Z = \{z_1, \ldots, z_r\}$$

$d_{\mathrm{reg}} =$ 2nd smallest $d$ such that $I_d = I(Z)_d$ **and** $\mathrm{HF}_{K[X]/I}(d) = r$ ?

# Complete intersections

$$I = \langle f_1, \ldots, f_n \rangle \subset K[X], \quad V_X(I) = Z = \{z_1, \ldots, z_r\}$$

$d_{\mathrm{reg}} =$ 2nd smallest $d$ such that $I_d = I(Z)_d$ **and** $\mathrm{HF}_{K[X]/I}(d) = r$ ?

**Theorem (Hilbert-Serre)**

There exists an integer $L$ such that, for all $d \geq L$ the Hilbert function is given by

$$\mathrm{HF}_{K[X]}(d) = \mathrm{HP}_{K[X]}(d) = \frac{\deg X}{(\dim X)!} d^{\dim X} + \text{ lower order terms.}$$

# Complete intersections

$$I = \langle f_1, \ldots, f_n \rangle \subset K[X], \quad V_X(I) = Z = \{z_1, \ldots, z_r\}$$

$d_{\mathrm{reg}} =$ 2nd smallest $d$ such that $I_d = I(Z)_d$ **and** $\mathrm{HF}_{K[X]/I}(d) = r$ ?

**Theorem (Hilbert-Serre)**

There exists an integer $L$ such that, for all $d \geq L$ the Hilbert function is given by

$$\mathrm{HF}_{K[X]}(d) = \mathrm{HP}_{K[X]}(d) = \frac{\deg X}{(\dim X)!} d^{\dim X} + \text{ lower order terms.}$$

**Theorem.** If $X$ is arithmetically Cohen-Macaulay, then

$$d_{\mathrm{reg}} \leq \sum_{i=1}^{n} \deg(f_i) + L + 1$$

# Intersecting Chow forms

**Good news:** the Plücker embedding provides a Khovanskii basis!

# Intersecting Chow forms

**Good news:** the Plücker embedding provides a Khovanskii basis!

Which lines in 3-space touch $q$ given curves of degree 4, and $4$-$q$ curves of degree 2?

*intersect two generic quadrics* *intersect quadric and hyperplane*

# Intersecting Chow forms

**Good news:** the Plücker embedding provides a Khovanskii basis!

Which lines in 3-space touch *q* given curves of degree 4, and *4-q* curves of degree 2?

*intersect two generic quadrics* *intersect quadric and hyperplane*

*q* degree 4 equations in Plücker coordinates + *(4-q)* degree 2 equations

| $q$ | # solutions | $d_{\text{reg}}$ | $\text{HF}_{K[X]}(d_{\text{reg}})$ | # equations | $t_{\mathbb{Q}}$ | $t_{\mathbb{F}_{9716633}}$ |
|---|---|---|---|---|---|---|
| 0 | 32 | 6 | 336 | 4 | $39s$ | $4s$ |
| 1 | 64 | 8 | 825 | 4 | $711s$ | $36s$ |
| 2 | 128 | 10 | 1716 | 4 | $6833s$ | $247s$ |
| 3 | 256 | 12 | 3185 | 4 | $55319s$ | $1336s$ |
| 4 | 512 | 14 | 5440 | 4 | $\times$ | $5572s$ |

Table 2: Computational results for intersecting Chow forms on $\text{Gr}(2, 4)$.

# Intersecting Chow forms

**Good news:** the Plücker embedding provides a Khovanskii basis!

Which lines in 3-space touch *q* given curves of degree 4, and *4-q* curves of degree 2?

*intersect two generic quadrics*

*intersect quadric and hyperplane*

*q* degree 4 equations in Plücker coordinates + *(4-q)* degree 2 equations

| $q$ | # solutions | $d_{\text{reg}}$ | $\text{HF}_{K[X]}(d_{\text{reg}})$ | # equations | $t_{\mathbb{Q}}$ | $t_{\mathbb{F}_{9716633}}$ |
|---|---|---|---|---|---|---|
| 0 | 32 | 6 | 336 | 4 | $39s$ | $4s$ |
| 1 | 64 | 8 | 825 | 4 | $711s$ | $36s$ |
| 2 | 128 | 10 | 1716 | 4 | $6833s$ | $247s$ |
| 3 | 256 | 12 | 3185 | 4 | $55319s$ | $1336s$ |
| 4 | 512 | 14 | 5440 | 4 | $\times$ | $5572s$ |

Table 2: Computational results for intersecting Chow forms on $\text{Gr}(2, 4)$.

*size of the matrix*

# Intersecting Chow forms

**Good news:** the Plücker embedding provides a Khovanskii basis!

Which lines in 3-space touch *q* given curves of degree 4, and *4-q* curves of degree 2?

*intersect two generic quadrics* *intersect quadric and hyperplane*

*q* degree 4 equations in Plücker coordinates + *(4-q)* degree 2 equations

| $q$ | # solutions | $d_{\mathrm{reg}}$ | $\mathrm{HF}_{K[X]}(d_{\mathrm{reg}})$ | # equations | $t_{\mathbb{Q}}$ | $t_{\mathbb{F}_{9716633}}$ |
|---|---|---|---|---|---|---|
| 0 | 32 | 6 | 336 | 4 | $39s$ | $4s$ |
| 1 | 64 | 8 | 825 | 4 | $711s$ | $36s$ |
| 2 | 128 | 10 | 1716 | 4 | $6833s$ | $247s$ |
| 3 | 256 | 12 | 3185 | 4 | $55319s$ | $1336s$ |
| 4 | 512 | 14 | 5440 | 4 | $\times$ | $5572s$ |

Table 2: Computational results for intersecting Chow forms on $\mathrm{Gr}(2, 4)$.

*size of the matrix*

**Theorem (Betti, Panizzut, T)**

For a complete intersection on $\mathrm{Gr}(k, m)$, $d_{\mathrm{reg}} = \sum_i d_i - m + 2.$

# Khovanskii bases are hot!

3. arXiv:2306.07897 [pdf, ps, other] math.AG cond-mat.soft math-ph nlin.PS physics.class-ph

   Khovanskii bases for semimixed systems of polynomial equations -- a case of approximating stationary nonlinear Newtonian dynamics

   **Authors:** Viktoriia Borovik, Paul Breiding, Javier del Pino, Mateusz Michałek, Oded Zilberberg

4. arXiv:2302.12473 [pdf, ps, other] math.AC cs.MS

   SubalgebraBases in Macaulay2

   **Authors:** Michael Burr, Oliver Clarke, Timothy Duff, Jackson Leaman, Nathan Nichols, Elise Walker

10. arXiv:2208.08179 [pdf, other] math.AG

    The Algebraic Degree of Coupled Oscillators

    **Authors:** Paul Breiding, Mateusz Michałek, Leonid Monin, Simon Telen

11. arXiv:2208.05258 [pdf, ps, other] math.AG math.CO

    Discrete geometry of Cox rings of blow-ups of $\mathbb{P}^3$

    **Authors:** Mara Belotti, Marta Panizzut

15. arXiv:2206.07838 [pdf, ps, other] math.AG

    Generic root counts and flatness in tropical geometry

    **Authors:** Paul Alexander Helminck, Yue Ren

29. arXiv:2008.13055 [pdf, ps, other] math.AG

    Numerical homotopies from Khovanskii bases ⟶ MathComp '23

    **Authors:** Michael Burr, Frank Sottile, Elise Walker

**Thank you!**

# From kernel to EVP

**(1)** Construct $M(d_{\mathrm{reg}})$ and let $N(d_{\mathrm{reg}})$ be its kernel

**(2)** View $N(d_{\mathrm{reg}})$ as a map $N : K[X]_{d_{\mathrm{reg}}} \to K^{\delta}$ $\delta =$ number of solutions

**(3)** Set $d := d_{\mathrm{reg}} - 1.$ For any degree one form $p \in K[X]_1$ , define

$$N_p : K[X]_d \longrightarrow K^{\delta}, \quad f \longmapsto N(pf)$$

**(4)** Pick a random $h \in K[X]_1$

**Theorem 4.5.** *With the above assumptions and notation, there exists a subspace $B \subset K[X]_d$ such that the restriction $(N_h)_{|B}$ is invertible. Moreover, for any $p \in K[X]_1$, the eigenvalues of $(N_h)^{-1}_{|B} \circ (N_p)_{|B} : B \longrightarrow B$ are $\frac{p}{h}(z)$, for $z \in V_X(I)$.*