Positive Toric Geometry

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August 4, 2023

Abstract

These are lecture notes for a minicourse taught by the author at the *Hypergeometric school*, taking place at Kobe University on August 16-17, 2023. We study the positive and nonnegative points of affine and projective toric varieties, embedded via a monomial parametrization. We prove the well-known fact that a nonnegative toric variety is homeomorphic to a cone, in the affine case, or a polytope, in the projective case. We establish this via the algebraic moment map. We discuss applications of positive toric varieties in algebraic statistics, linear programming and geometric modelling.

1 Toric varieties

We start with a lightning introduction to complex, embedded toric varieties. We take the approach of references like [7, 9, 11]: our toric varieties are not necessarily normal, and they come with an embedding in affine or projective space. For further reading, we point the reader to the lecture notes [10, 14], and the standard text books [4, 6].

Most toric varieties in applications arise via a parametrization map which uses only monomials. The data defining such a map is an integer matrix

$$A = (a_1 \; a_2 \; \cdots \; a_s) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ns} \end{pmatrix} \in \mathbb{Z}^{n \times s},$$

recording the exponent vectors $a_1, \ldots, a_s \in \mathbb{Z}^n$ of these monomials in its columns. That is, the *monomial map* ϕ_A associated to A is

$$\phi_A : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}^s, \quad \text{where} \quad \phi_A(t) = (t^{a_1}, \ldots, t^{a_s}). \tag{1}$$

Here $t = (t_1, \ldots, t_n)$ and t^{a_i} is short for $t_1^{a_{1i}} \cdots t_n^{a_{ni}}$.

We associate an affine variety $Y_A \subset \mathbb{C}^s$ to A by taking the Zariski closure of $\operatorname{im} \phi_A$:

$$Y_A = \overline{\operatorname{im} \phi_A} = \overline{\{\phi_A(t) \, : \, t \in (\mathbb{C}^*)^n\}} \subset \mathbb{C}^s.$$

This is what is called an *affine toric variety*, and all affine toric varieties arise in this way. Many familiar varieties are of the form Y_A for some $A \in \mathbb{Z}^{n \times s}$. Here are some examples.

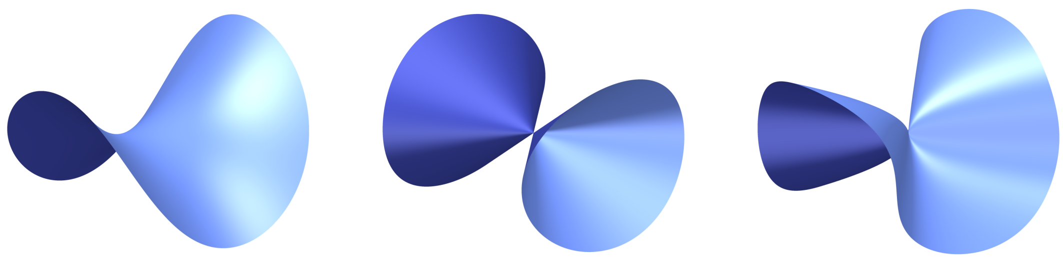

Figure 1: Three toric surfaces in three-space.

Example 1.1 (affine spaces and tori)**.** The variety Y_A for the identity matrix $A = \mathrm{id}_n$ is $\mathbb{C}^n$. appending a column with entries -1, we obtain the $n \times (n+1)$-matrix

$$A = \begin{pmatrix} 1 & & & -1 \\ & \ddots & & \vdots \\ & & 1 & -1 \end{pmatrix}.$$

Now Y_A is parametrized by $(t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_n, (t_1 \cdots t_n)^{-1}) \in \mathbb{C}^{n+1}$. Its equation is $Y_A = \{x_1 \cdots x_n x_{n+1} = 1\} \subset \mathbb{C}^{n+1}$. In fact, $Y_A \simeq (\mathbb{C}^*)^n$, hence ϕ_A is a closed embedding. $\diamond$

Example 1.2 (moment curves)**.** The *moment curve* of degree d arises from

$$A = \begin{pmatrix} 1 & 2 & \cdots & d \end{pmatrix} \quad \in \mathbb{Z}^{1 \times d}.$$

It is embedded in $\mathbb{C}^d$ via the parametrization $\phi_A(t) = (t, t^2, \ldots, t^d)$. For $d = 2$, the moment curve is the parabola $\{x^2 - y = 0\}$ in the affine plane $\mathbb{C}^2$. For $d = 3$, it is the twisted cubic from [3, p. 8], defined by $Y_{(1\ 2\ 3)} = \{x^2 - y = x^3 - z = 0\} \subset \mathbb{C}^3$. $\diamond$

Example 1.3 (toric surfaces)**.** Consider the 2×3-matrices

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

These correspond to toric surfaces Y_{A_1}, Y_{A_2} and Y_{A_3} in $\mathbb{C}^3$, with defining equation $x - yz = 0, x^2 - yz = 0$ and $x^3 - yz = 0$ respectively. The parametrization of A_1 is $\phi_{A_1}(t_1, t_2) = (t_1 t_2, t_1, t_2)$, and $x - yz$ vanishes on the image. The degree of Y_{A_1} and Y_{A_2} is 2, while Y_{A_3} has degree 3. Some real points of these surfaces are shown in Figure 1. Note that Y_{A_1} is smooth, while Y_{A_2} and Y_{A_3} have a singular point at the origin. $\diamond$

Example 1.4 (rank-one matrices)**.** Consider the matrix $A \in \mathbb{Z}^{5 \times 6}$ given by

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

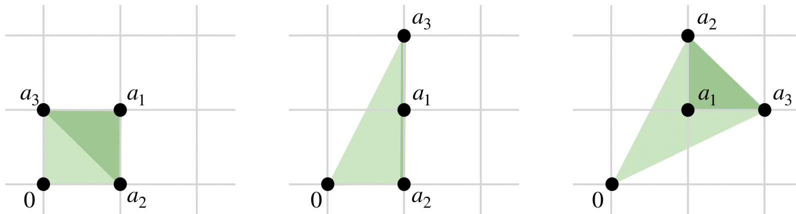

Figure 2: The polytopes $\mathrm{conv}(0 \cup A_1)$, $\mathrm{conv}(0 \cup A_2)$ and $\mathrm{conv}(0 \cup A_3)$, with A_i as in Example 1.3, are the regions shaded in light green. The dark green regions are $\mathrm{conv}(A_1), \mathrm{conv}(A_2)$ and $\mathrm{conv}(A_3)$ (left to right).

The corresponding monomial map ϕ_A parametrizes rank-one 2×3-matrices:

$$\phi_A(t_1, \ldots, t_5) \;=\; \begin{pmatrix} t_1 t_3 & t_1 t_4 & t_1 t_5 \\ t_2 t_3 & t_2 t_4 & t_2 t_5 \end{pmatrix} \;=\; \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \cdot \begin{pmatrix} t_3 & t_4 & t_5 \end{pmatrix} \;\in \mathbb{C}^{2\times 3} = \mathbb{C}^6.$$

Analogously, one easily finds the matrix $A \in \mathbb{Z}^{(d_1+d_2)\times d_1 d_2}$ for rank-one $d_1 \times d_2$-matrices. $\diamond$

The two most important invariants of an algebraic variety are its dimension and degree. For the toric variety Y_A, these invariants are expressed nicely in terms of the matrix A. It is our first encounter with a *convex polytope*. We define

$$\mathrm{conv}(0 \cup A) \;=\; \mathrm{conv}(0, a_1, \ldots, a_s) \;\subset\; \mathbb{R}^n.$$

Here $\mathrm{conv}(\cdot)$ takes the convex hull of a list of points in a real vector space, $0 \in \mathbb{R}^n$ is the origin and $a_i \in \mathbb{Z}^n \subset \mathbb{R}^n$ is the i-th column of A. In general, a convex polytope $P \subset \mathbb{R}^n$ is the convex hull of finitely many points. Here is a warm up example.

Example 1.5 (convex polytopes)**.** In dimension $n = 1$, polytopes are line segments. For instance, the polytope $\mathrm{conv}(0 \cup A)$ for the matrix $A = (1\ 2\ 3)$ for the twisted cubic from Example 1.2 is the line segment $[0, 3]$. Two dimensional polytopes are called polygons. Three examples, coming from Example 1.3, are shown in Figure 2. $\diamond$

The dimension of a convex polytope $P \subset \mathbb{R}^n$ is that of the smallest affine-linear subspace containing it, and the volume is $\mathrm{Vol}(P) = \int_P 1\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$. We write $\mathbb{Z}A$ for the lattice

$$\mathbb{Z}A \;=\; \{n_1 a_1 + \cdots + n_s a_s \;:\; n_j \in \mathbb{Z}\} \;\subset\; \mathbb{Z}^n. \tag{3}$$

Theorem 1.6. *The dimension and degree of the affine toric variety Y_A are given by*

$$\dim Y_A \;=\; \dim \mathrm{conv}(0 \cup A) \;=\; \mathrm{rank}(A), \qquad \deg Y_A \;=\; n! \cdot \mathrm{Vol}(\mathrm{conv}(0 \cup A)). \tag{4}$$

Here the formula for $\deg Y_A$ *assumes that* $\mathrm{rank}(A) = n$ *and* $\mathbb{Z}A = \mathbb{Z}^n$.

Proof. For the statement about dimension, see for instance [14, Corollary 2.14]. The statement about the degree of Y_A will follow from Theorem 1.13 and the observation that Y_A is an affine open subset of the projective toric variety $X_{0\cup A}$. $\square$

Having determined the dimension and degree of Y_A, we now ask for its defining equations.

Definition 1.7. The *toric ideal* of an integer matrix $A \in \mathbb{Z}^{n\times s}$ is the binomial ideal

$$I_A \;=\; \langle x^u - x^v \;:\; u, v \in \mathbb{N}^s,\; A(u-v) = 0\rangle \;\subset\; \mathbb{C}[x_1, \dots, x_s]. \tag{5}$$

Theorem 1.8. *The toric ideal I_A is the prime ideal $I(Y_A)$ of the affine toric variety Y_A.*

Proof. See [4, Theorem 1.1.9]. The reader should check $I_A \subset I(Y_A)$ as an exercise. □

We close the discussion on affine toric varieties with a criterion for smoothness.

Theorem 1.9. *The affine toric variety Y_A is smooth if and only if the semigroup $\mathbb{N}A \subset \mathbb{Z}^n$ generated by the columns of A can be generated by only* $\operatorname{rank}(A)$ *elements.*

We now switch to the projective toric variety defined by A. We replace (1) by

$$\Phi_A : (\mathbb{C}^*)^n \longrightarrow \mathbb{P}^{s-1}, \quad \text{where} \quad \Phi_A(t) = (t^{a_1} : \dots : t^{a_s}). \tag{6}$$

We associate a projective variety $X_A \subset \mathbb{P}^{s-1}$ to A by taking the Zariski closure of $\operatorname{im}\Phi_A$:

$$X_A \;=\; \overline{\operatorname{im}\Phi_A} \;=\; \overline{\{\Phi_A(t) \;:\; t \in (\mathbb{C}^*)^n\}} \;\subset \mathbb{P}^{s-1}.$$

We revisit some matrices we have seen before from this projective point of view.

Example 1.10 (projective space is toric)**.** The variety X_A for $A = \mathrm{id}_n$ is $\mathbb{P}^{n-1}$. ◇

Example 1.11 (Example 1.3 in $\mathbb{P}^2$)**.** The matrices A_1, A_3 from Example 1.3 give the projective toric variety $X_{A_1} = X_{A_3} = \mathbb{P}^2$. Indeed, the maps Φ_{A_1} and Φ_{A_3} are dominant. The matrix A_2 leads to a smooth toric curve $X_{A_2} = \{x_1^2 - x_2x_3 = 0\} \subset \mathbb{P}^2$. ◇

Example 1.12 (Segre embedding)**.** The threefold $X_A \subset \mathbb{P}^5$ corresponding to the matrix A from (2) is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ into $\mathbb{P}^5$. It is described by

$$\operatorname{rank}\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_5 \end{pmatrix} \;\le\; 2.$$

That is, the defining equations are the 2×2 minors of this matrix. ◇

Projective toric varieties are closely related to polytopes. The right polytope to associate with X_A is the convex hull in $\mathbb{R}^n$ of the columns of A, denoted $\operatorname{conv}(A)$. These polytopes for A_1, A_2, A_3 in Example 1.3 are the areas shaded in dark green in Figure 2. Note that $\operatorname{conv}(A_2)$ is the line segment connecting $(1,0)$ and $(1,2)$.

For $1 \le i \le s$, let $A - a_i = \{a_1 - a_i, \dots, a_{i-1} - a_i, a_{i+1} - a_i, \dots, a_s - a_i\}$ and let $\mathbb{Z}(A - a_i)$ be the lattice generated by these shifted exponent vectors, as in (3). You will prove in the exercises that $\mathbb{Z}(A - a_i) = \mathbb{Z}(A - a_j)$, for all $1 \le i, j \le s$. The projective analog of (4) is:

Theorem 1.13. *The dimension and degree of the projective toric variety X_A are given by*

$$\dim X_A \;=\; \dim \operatorname{conv}(A) \;=\; \operatorname{rank}(A - a_1), \qquad \deg X_A \;=\; n! \cdot \operatorname{Vol}(\operatorname{conv}(A)). \tag{7}$$

Here the formula for $\deg X_A$ *assumes that* $\operatorname{rank}(A - a_1) = n$*, and* $\mathbb{Z}(A - a_1) = \mathbb{Z}^n$*.*

For a proof of this theorem, see for instance [14, Corollary 3.11 and Theorem 3.16]. To obtain the defining equations of X_A, we must add a row of ones to the matrix A:

$$\hat{A} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_s \end{pmatrix} \in \mathbb{Z}^{(n+1)\times s}. \tag{8}$$

This trick forces the binomial generators of $I_{\hat{A}}$ to be homogeneous [14, Proposition 3.7].

Theorem 1.14. *The toric ideal $I_{\hat{A}}$, with $\hat{A}$ as in (8), is the prime homogeneous ideal $I(X_A)$ of the projective toric variety X_A.*

A projective variety $X \subset \mathbb{P}^{s-1}$ is covered by affine varieties $Y_i = X \cap U_i$, where $U_i = \{x_i \neq 0\}$ and $x_1, \ldots, x_s$ are homogeneous coordinates on $\mathbb{P}^{s-1}$. For the toric variety X_A, the vertices of $\operatorname{conv}(A)$ provide a more efficient affine covering [14, Proposition 3.33]:

Theorem 1.15. *Suppose $\{a_1, \ldots, a_m\} \subset A$ are the vertices of* $\operatorname{conv}(A)$ *($m \leq s$). We have*

$$X_A = (X_A \cap U_1) \cup \cdots \cup (X_A \cap U_s) = (X_A \cap U_1) \cup \cdots \cup (X_A \cap U_m).$$

Moreover, the affine variety $(X_A \cap U_i)$ is the affine toric variety $Y_{A-a_i} \subset \mathbb{C}^{s-1}$.

2 Positive points and the algebraic moment map

We now switch from complex to real numbers. The real points of the affine (resp. projective) toric variety Y_A (resp. X_A) are the points

$$Y_A(\mathbb{R}) = Y_A \cap \mathbb{R}^s, \quad X_A(\mathbb{R}) = X_A \cap \mathbb{RP}^{s-1}.$$

By Theorem 1.8, points in $Y_A(\mathbb{R})$ are represented by real coordinate vectors $(x_1, \ldots, x_s) \in \mathbb{R}^s$ satisfying $x^u - x^v = 0$, for all $u - v \in \ker A$. Similarly, $X_A(\mathbb{R}) = \{(x_1 : \cdots : x_s) \in \mathbb{RP}^{s-1} : x^u - x^v = 0, \text{ for all } u - v \in \ker \hat{A}\}$. Among those are the nonnegative and positive points.

Definition 2.1. The *positive affine toric variety* $(Y_A)_{>0} \subset Y_A(\mathbb{R})$ is given by $\phi_A(\mathbb{R}^n_{>0})$, i.e., the image of the positive orthant $\mathbb{R}^n_{>0}$ under the monomial map $\phi_A : (\mathbb{C}^*)^n \to \mathbb{C}^s$. We say that $(Y_A)_{>0}$ is the *positive part* of Y_A. Similarly, the *positive projective toric variety* $(X_A)_{>0} \subset X_A(\mathbb{R})$, also called the *positive part* of X_A, is $\Phi_A(\mathbb{R}^n_{>0})$.

Example 2.2. By Example 1.1, the positive part of affine space $\mathbb{C}^n$ is $\mathbb{R}^n_{>0}$. Similarly, we have $((\mathbb{C}^*)^n)_{>0} = \mathbb{R}^n_{>0}$. Example 1.10 constructs $\mathbb{P}^{n-1}$ as the image of Φ_{id_n}, hence

$$(\mathbb{P}^{n-1})_{>0} = \{(x_1 : \cdots : x_n) \in \mathbb{P}^{n-1} : x_i > 0, \, i = 1, \ldots, n\}. \qquad \diamond$$

Proposition 2.3. *The following equalities hold:*

$$(Y_A)_{>0} = Y_A \cap \mathbb{R}^s_{>0}, \quad (X_A)_{>0} = X_A \cap (\mathbb{P}^{s-1})_{>0}.$$

Proof. The inclusion $(Y_A)_{>0} \subset Y_A \cap \mathbb{R}^s_{>0}$ is easy. To show the opposite inclusion, notice that $Y_A \cap \mathbb{R}^s_{>0} \subset Y_A \cap (\mathbb{C}^*)^s = \operatorname{im} \phi_A$. For this last equality, see for instance the proof of Proposition 2.10 in [14]. Hence, a point $x \in Y_A \cap \mathbb{R}^s_{>0}$ can be written

$$x \;=\; (t^{a_1}, \dots, t^{a_s}) \;=\; (|t^{a_1}|, \dots, |t^{a_s}|) \;=\; (|t|^{a_1}, \dots, |t|^{a_s}) \quad \text{for some } t \in (\mathbb{C}^*)^n.$$

Here $|\cdot|$ takes the absolute value, and $|t| = (|t_1|, \dots, |t_n|) \in \mathbb{R}^n_{>0}$. This shows that $x \in \phi_A(\mathbb{R}^n_{>0})$, and concludes the proof for Y_A. The proof for X_A is analogous. □

Definition 2.4. The *nonnegative affine toric variety* $(Y_A)_{\geq 0} \subset Y_A(\mathbb{R})$ is the closure $\overline{(Y_A)_{>0}}$ of $(Y_A)_{>0}$ in $\mathbb{R}^s$. Similarly, the *nonnegative projective toric variety* $(X_A)_{\geq 0}$ is $\overline{(X_A)_{>0}} \subset \mathbb{RP}^{s-1}$.

Example 2.5. The nonnegative part of affine space $\mathbb{C}^n$ is the nonnegative orthant $(\mathbb{C}^n)_{\geq 0} = \mathbb{R}^n_{\geq 0}$. The nonnegative part of $(\mathbb{C}^*)^n$ is the same as its positive part: $((\mathbb{C}^*)^n)_{\geq 0} = \mathbb{R}^n_{>0}$. The nonnegative part of projective space is

$$(\mathbb{P}^{n-1})_{\geq 0} = \{(x_1 : \cdots : x_n) \in \mathbb{P}^{n-1} \,:\, x_i \geq 0,\, i = 1, \dots, n\}. \qquad \diamond$$

Proposition 2.6. *The following equalities hold:*

$$(Y_A)_{\geq 0} \;=\; Y_A \cap \mathbb{R}^s_{\geq 0}, \quad (X_A)_{\geq 0} \;=\; X_A \cap (\mathbb{P}^{s-1})_{\geq 0}.$$

Proof. By Proposition 2.3, we have $\overline{(Y_A)_{>0}} = \overline{Y_A \cap \mathbb{R}^s_{>0}} \subset \overline{Y_A} \cap \overline{\mathbb{R}^s_{>0}} = Y_A \cap \mathbb{R}^s_{\geq 0}$, which proves one inclusion for $(Y_A)_{\geq 0}$. To show the other inclusion, let $x \in Y_A \cap \mathbb{R}^s_{\geq 0}$. Let $\operatorname{supp}(x) = \{a_i \in A \,:\, x_i \neq 0\}$. We write $\operatorname{pos}(A) \subset \mathbb{R}^n$ for the polyhedral cone generated by the integer vectors in A: $\operatorname{pos}(A) = \{r_1 a_1 + \cdots + r_s a_s \,:\, r_i \in \mathbb{R}_{\geq 0}\}$. You will show in the exercises that

1. $\operatorname{supp}(x) = Q \cap A$, for some face $Q \subset \operatorname{pos}(A)$,
2. there is $t \in (\mathbb{C}^*)^n$ such that

$$x_i = \begin{cases} t^{a_i} & a_i \in \operatorname{supp}(x), \\ 0 & \text{otherwise} \end{cases}.$$

Since $x_i > 0$ for $a_i \in \operatorname{supp}(x)$, we may replace t by $|t|$, and assume $t \in \mathbb{R}^n_{>0}$. Let $w \in (\mathbb{R}^n)^\vee$ be such that $\langle w, a \rangle = 0$ for all $a \in Q$, and $\langle w, a \rangle > 0$ for all $a \in A \setminus Q$. We have that

$$\lim_{u \to 0} \phi_A(u^{w_1} t_1, \dots, u^{w_n} t_n) \;=\; \lim_{u \to 0} (t^{a_1} u^{\langle w, a_1 \rangle}, \dots, t^{a_s} u^{\langle w, a_s \rangle}) \;=\; x.$$

Here u is a positive parameter. This shows that $x \in \overline{(Y_A)_{>0}}$. The proof for the nonnegative projective toric variety $(X_A)_{\geq 0}$ is analogous, and left as an exercise. □

Example 2.7. Let A_3 be the 2×3 matrix from Example 1.3. The intersection of the toric surface $Y_{A_3} = \{x^3 - yz = 0\} \subset \mathbb{C}^3$ with $\mathbb{R}^3_{\geq 0}$ is its nonnegative part. This is the blue surface in Figure 3, obtained by intersecting Y_{A_3} with $\mathbb{R}^3_{\geq 0}$. $\diamond$

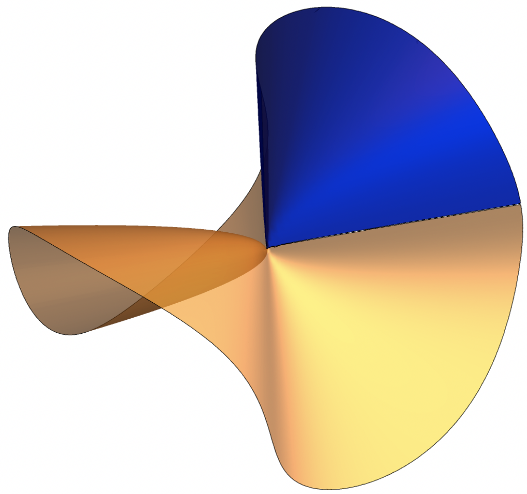

Figure 3: The nonnegative part $(Y_{A_3})_{\geq 0}$ of the toric surface $Y_{A_3} = \{x^3 - yz = 0\}$.

We now identify the nonnegative part $(Y_A)_{\geq 0}$ of the affine toric variety Y_A with the cone $\mathrm{pos}(A)$. The map that realizes this identification is the *algebraic moment map*.

Definition 2.8. The *affine algebraic moment map* $\mu_{A,w} : Y_A \to \mathbb{R}^n$ with weights $w \in \mathbb{R}^s_{>0}$ is

$$\mu_{A,w}(x) \;=\; \sum_{i=1}^{s} w_i \cdot |x_i| \cdot a_i.$$

Note that restricted to the nonnegative part $(Y_A)_{\geq 0}$, $\mu_{A,w}$ is the linear map $\mathbb{R}^s \to \mathbb{R}^n$ given by the matrix $A \cdot \mathrm{diag}(w)$. Here is the theorem that justifies our claim that $(Y_A)_{\geq 0} \simeq \mathrm{pos}(A)$.

Theorem 2.9. *For any positive weights w, the restriction of the affine algebraic moment map $\mu_{A,w}$ to the nonnegative affine toric variety $(Y_A)_{\geq 0}$ is a homeomorphism onto* $\mathrm{pos}(A)$.

Example 2.10. The image of the blue surface in the left part of Figure 3 under the linear projection $A_3 : \mathbb{R}^3 \to \mathbb{R}^2$ is the cone $\mathrm{pos}(A_3)$ shown in blue in Figure 4. ⋄

To prove Theorem 2.9, we follow the approach in [6, Chapter 4]. It is a consequence of Theorem 2.11 below. For each face $Q \subset \mathrm{pos}(A)$, let $(Y_{A,A\cap Q})_{>0} = \{x \in (Y_A)_{\geq 0} : \mathrm{supp}(x) = A \cap Q\}$. By Proposition 4.1, we have $(Y_A)_{\geq 0} = \bigsqcup_{Q \subset \mathrm{pos}(A)} (Y_{A,A\cap Q})_{>0}$. This is a disjoint union over all faces of $\mathrm{pos}(A)$, including $\mathrm{pos}(A)$ itself, for which $(Y_{A,A})_{>0} = (Y_A)_{>0}$.

Theorem 2.11. *For each face Q of* $\mathrm{pos}(A)$, *the restriction of the affine algebraic moment map $\mu_{A,w}$ to $(Y_{A,A\cap Q})_{>0}$ is a real analytic isomorphism onto the relative interior of Q.*

Proof. By Proposition 4.2, $(Y_{A,A\cap Q})_{>0}$ is the positive part of the affine toric variety $Y_{A\cap Q} \subset \mathbb{C}^{|A\cap Q|} = \mathbb{C}^r$, parametrized by the monomial map $\phi_{A\cap Q}$. By Theorem 1.6, the dimension of $Y_{A\cap Q}$ is the rank of the submatrix $A \cap Q = (a_{i_1} \;\cdots\; a_{i_r})$, which is the dimension $q = \dim Q$ of Q. We will show that, as a manifold, $\mathrm{relint}(Q)$ is isomorphic to $\mathbb{R}^q$, and this isomorphism is realized by precomposing μ_A with an isomorphism $\mathbb{R}^q \to (Y_{A\cap Q})_{>0}$.

By Theorem 1.8, $Y_{A\cap Q}$ only depends on the row space of $A \cap Q$, so that we may replace the $n \times r$ matrix $A \cap Q$ by a $q \times r$ integer matrix $A_Q = (u_1 \ \cdots \ u_r)$ whose row span over $\mathbb{Q}$ equals that of $A \cap Q$. We have $Y_{A\cap Q} = Y_{A_Q}$ and $(Y_{A\cap Q})_{>0} = \phi_{A_Q}(\mathbb{R}^q_{>0})$. The restriction of ϕ_{A_Q} to $\mathbb{R}^q_{>0}$ is a real analytic isomorphism onto $(Y_{A\cap Q})_{>0}$. Consider the diagram

$$\begin{array}{ccccc} \mathbb{R}^q_{>0} & \xrightarrow{\phi_{A_Q}} & (Y_{A\cap Q})_{>0} & & \\ \uparrow{\scriptstyle \exp} & & & \searrow^{\mu_A} & \\ \mathbb{R}^q & \xrightarrow{F} & \mathrm{int}(\mathrm{pos}(A_Q)) & \xrightarrow{\simeq} & \mathrm{relint}(Q) \end{array}$$

where $\exp(y_1, \ldots, y_q) = (e^{y_1}, \ldots, e^{y_q})$, and the identification between $\mathrm{pos}(A_Q)$ and Q is induced by the identification of the column spans of $A \cap Q$ and A_Q. We define

$$F : \mathbb{R}^q \to \mathrm{relint}(\mathrm{pos}(A_Q)), \quad y \mapsto \sum_{j=1}^{r} w_{i_j} \cdot e^{\langle y, u_j\rangle} \cdot u_j,$$

where u_j is the j-th column of A_Q. This map makes the diagram commute. The blue arrows in the diagram are analytic isomorphisms. To prove the proposition, it suffices to show that F is a real analytic isomorphism as well. This is part of Theorem 2.13 below. □

Example 2.12. We consider again the matrix A_3 from Example 1.3. The cone $C = \mathrm{pos}(A_3)$ is the image of $(Y_{A_3})_{\geq 0}$ under the moment map. The proof of Theorem 2.11 parametrizes this cone in two more ways. First, the map $\mathbb{R}^2_{>0} \to \mathrm{int}(C)$, using weights 1, is given by

$$(t_1, t_2) \longmapsto (t_1t_2 + t_1t_2^2 + 2t_1^2t_2,\ t_1t_2 + 2t_1t_2^2 + t_1^2t_2).$$

This is obtained by composing ϕ_{A_3} with the moment map. The meshed orange area in the left part of Figure 4 is the image of $(0, 1)^2$ under this map. In lighter shades of orange, the images of $(0, 2)^2$ and $(0, 3)^2$ are shown. When $\alpha \to \infty$, the image of $(0, \alpha)^2$ fills the interior. The other parametrization of $\mathrm{int}(C)$ precomposes this map with the exponential:

$$(y_1, y_2) \longmapsto (e^{y_1+y_2} + e^{y_1+2y_2} + 2e^{2y_1+y_2},\ e^{y_1+y_2} + 2e^{y_1+2y_2} + e^{2y_1+y_2}).$$

This is the map $\mathbb{R}^2 \to \mathrm{int}(C)$ from the proof of Theorem 2.11. The images of $(-\alpha, \alpha)^2$ for $\alpha = 1, 2, 3$ are shown in the right part of Figure 4. ◇

Theorem 2.13. *Let $F : \mathbb{R}^q \to \mathbb{R}^q$ be a map of the form $F(y) = \sum_{j=1}^r w_j \cdot e^{\langle y,u_j\rangle} \cdot u_j$, where $u_1, \ldots, u_r \in \mathbb{R}^q$ span $\mathbb{R}^q$ and $w_k > 0$. We have*

- (A_q) *F is a real analytic isomorphism onto $C = \mathrm{int}(\mathrm{pos}(u_1, \ldots, u_r))$,*
- $(B_{q,m})$ *For any linear surjection $\pi : \mathbb{R}^q \to \mathbb{R}^m$, $\pi \circ F$ is onto the interior of $\pi(C)$, and the fibre of π over each point of $\pi(C)$ is a connected manifold isomorphic to $\mathbb{R}^{q-m}$.*

Proof. First, we show (A_1). The derivative of $F(y) = \sum_{j=1}^r w_j e^{yu_j} u_j$ is $\sum_{j=1}^r w_j u_j^2 e^{yu_j} > 0$. This shows that F is injective. To show surjectivity onto $\mathrm{int}(C)$, we distinguish three cases:

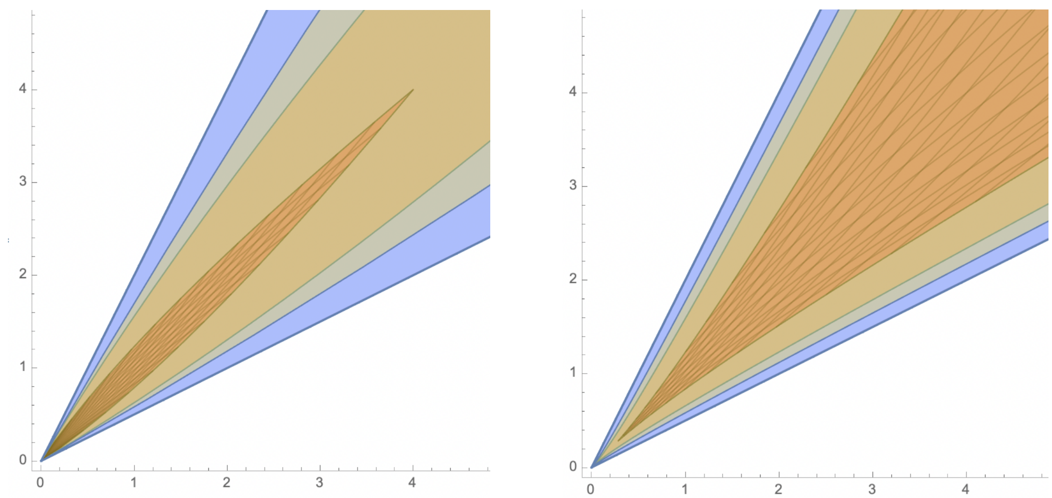

Figure 4: The cone $\mathrm{pos}(A_3)$ is parametrized by $\mathbb{R}^2_{>0}$ and by $\mathbb{R}^2$.

1. all u_j are negative, $C = \mathbb{R}_{\leq 0}$, $\lim_{y\to-\infty} F(y) = -\infty$, $\lim_{y\to\infty} F(y) = 0$,
2. all u_j are positive, $C = \mathbb{R}_{\geq 0}$, $\lim_{y\to-\infty} F(y) = 0$, $\lim_{y\to\infty} F(y) = \infty$,
3. some u_j are negative, some positive, $C = \mathbb{R}$, $\lim_{y\to-\infty} F(y) = -\infty$, $\lim_{y\to\infty} F(y) = \infty$.

This proves (A_1). Next, we prove the following two helpful claims:

(i) F is one-to-one and

(ii) the Jacobian matrix $\left(\frac{\partial F_i}{\partial y_k}\right)_{1\leq i,k\leq n}$ is positive definite for all $y \in \mathbb{R}^q$.

For (i), it suffices to show that the restriction of F to any line is one-to-one. This is clearly necessary, and it is sufficient because if $F(y) = F(y')$, then F is not injective on the line connecting y and y'. Fix any line $L \subset \mathbb{R}^q$. After a change of coordinates, we may assume that L is given by fixing the last $q-1$ coordinates: $L = \{y_2 = y_2^*, \ldots, y_q = y_q^*\}$. Let $\tilde{w}_j = w_j e^{y_2^* u_{j,2} + \cdots + y_q^* u_{j,q}}$, where $u_{j,k}$ is the k-th coordinate of u_j. The restriction of F to L is

$$y_1 \longmapsto \sum_{j=1}^{r} \tilde{w}_j \cdot e^{y_1 u_{j,1}} \cdot u_j.$$

The first coordinate of this function is one-to-one by (A_1), so $F_{|L}$ is one-to-one.

To show (ii), we compute $\frac{\partial F_i}{\partial y_k} = \sum_{j=1}^{r} w_j \cdot u_{j,k} \cdot e^{\langle y, u_j \rangle} u_{j,i}$. Note that this matrix is symmetric. It represents a positive definite quadratic form given by

$$v = (v_1, \ldots, v_q)^t \longmapsto v^t \cdot \left(\frac{\partial F_i}{\partial y_k}\right)_{1\leq i,k\leq n} \cdot v = \sum_j w_j \cdot e^{\langle y, u_j \rangle} \langle v, u_j \rangle^2.$$

The next step is to show $(A_m) \Rightarrow (B_{q,m})$ for all $m \leq q$. After changing coordinates $\pi : \mathbb{R}^q \to \mathbb{R}^m$ is the projection $(y_1, \ldots, y_q) \mapsto (y_1, \ldots, y_m)$ onto the first m coordinates. We write

$\bar{\pi} : (y_1, \ldots, y_q) \mapsto (y_{m+1}, \ldots, y_q)$ for the complementary projection. To simplify notation, let us write $\underline{y} = \pi(y) \in \mathbb{R}^m$, and $\bar{y} = \bar{\pi}(y) \in \mathbb{R}^{q-m}$. Notice that

$$F(y) \;=\; F(\underline{y}, \bar{y}) \;=\; \sum_{j=1}^{r} w_j \cdot e^{\langle \underline{y}, \underline{u}_j \rangle} \cdot e^{\langle \bar{y}, \bar{u}_j \rangle} \cdot u_j.$$

When we fix the last $q - m$ coordinates of y, that is, we fix $\bar{y} = \bar{\pi}(y)$, we see that the map $F_{\bar{y}} : \underline{y} \mapsto \pi(F(\underline{y}, \bar{y}))$ is a real analytic isomorphism $\mathbb{R}^m \to \mathrm{int}(\pi(C))$ by (A_m). The positive weights are $\tilde{w}_j = w_j \cdot e^{\langle \bar{y}, \bar{u}_j \rangle}$. This proves, in particular, that $\pi \circ F$ is onto $\mathrm{int}(\pi(C))$. To show that fibres are isomorphic to $\mathbb{R}^{q-m}$, for each $p \in \mathrm{int}(\pi(C))$, consider the map

$$G_p : \mathbb{R}^{q-m} \longrightarrow (\pi \circ F)^{-1}(p), \quad \bar{y} \longmapsto (F_{\bar{y}}^{-1}(p), \bar{y}).$$

By the above discussion, this is one-to-one and onto. To show that it is an isomorphism of manifolds, we use the implicit function theorem. Using coordinates $(\underline{z}, \bar{z})$ on the image, we see that the graph of our map is given by $\mathcal{G}(\bar{y}, \underline{z}, \bar{z}) = 0$, where $\mathcal{G} : \mathbb{R}^{(q-m)+q} \to \mathbb{R}^q$ is

$$\mathcal{G}(\bar{y}, \underline{z}, \bar{z}) = ((\pi \circ F)(\underline{z}, \bar{z}) - p, \; \bar{z} - \bar{y}).$$

Indeed, we have $\mathcal{G}(\bar{y}, G_p(\bar{y})) = 0$. The derivatives with respect to the variables $\underline{z}, \bar{z}$ give a $q \times q$ Jacobian matrix of $\mathcal{G}$, whose first m rows are the first m rows of the Jacobian matrix from (ii). The last $q-m$ rows consist of a $(q-m) \times m$ block of zeros, and a $(q-m) \times (q-m)$ identity matrix. By our computations in the proof of (ii), the Jacobian is invertible for all values of $(\bar{y}, \underline{z}, \bar{z})$. Hence G_p is analytic, and establishes $\mathbb{R}^{q-m} \simeq (\pi \circ F)^{-1}(p)$ as manifolds.

The last step is to show $(B_{q,q-1}) \Rightarrow (A_q)$. By (i)-(ii), F is one-to-one, and it is a local isomorphism. We need to show that $(B_{q,q-1})$ implies that F is onto. We first prove that

$$\mathrm{im}\, F \text{ contains a point arbitrarily close to any point on each ray of } (C). \tag{9}$$

Suppose that u_1 spans ray. Let $J \subset [r] = \{1, \ldots, r\}$ be defined as $J = \{j \in [r] : u_j = s_j u_1 \text{ for some } s_j \in \mathbb{R}\}$. I.e., J indexes the vectors u_j which lie on the same line through 0 as u_1. There is $v \in (\mathbb{R}^q)^\vee$ such that $\langle v, u_j \rangle = 0$ for $j \in J$ and $\langle v, u_j \rangle < 0$ for $j \notin J$. We have

$$\lim_{\lambda \to \infty} F(\lambda v + v') \;=\; \lim_{\lambda \to \infty} \sum_{j=1}^{r} w_j \cdot e^{\langle \lambda v + v', u_j \rangle} \cdot u_j \;=\; \left(\sum_{j \in J} w_j e^{\langle v', u_1 \rangle s_j} s_j \right) \cdot u_1.$$

Here the s_j are the scaling factors appearing in the definition of J, and $v' \in (\mathbb{R}^q)^\vee$ is arbitrary. We now apply (A_1) to the expression between parentheses on the right hand side. If all s_j are positive, we can choose v' to approach any point on the ray $\mathbb{R}_{>0} \cdot u_1$. If at least one of the s_j is negative, than $\mathbb{R} \cdot u_1$ belongs to the lineality space of C, and v' can be chosen to approach any point on this line. This establishes the claim (9).

Knowing (9), to show that F is onto, it suffices to show that $\mathrm{im}\, F \subset \mathbb{R}^q$ is convex. Equivalently, the intersection of $\mathrm{im}\, F$ with any line $L \subset \mathbb{R}^q$ is either connected or empty. Any such line L is the fibre $\pi^{-1}(p)$ of a linear projection $\pi : \mathbb{R}^q \to \mathbb{R}^{q-1}$. We have

$$F(\mathbb{R}^q) \cap \pi^{-1}(p) \;=\; F((\pi \circ F)^{-1}(p)).$$

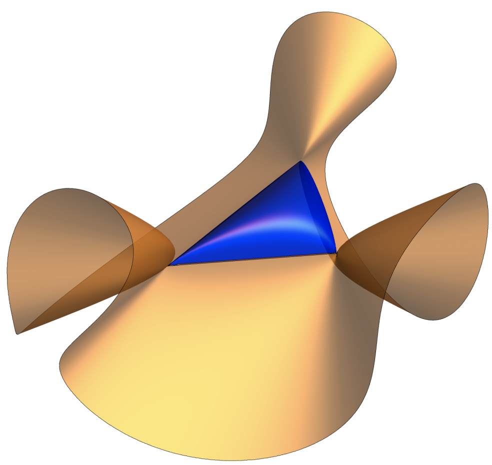
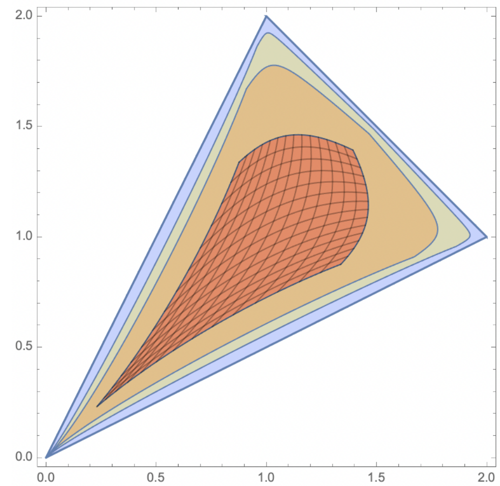

Figure 5: The nonnegative toric variety $(X_A)_{\geq 0}$ from Example 2.16 is a triangle.

By $(B_{q,q-1})$, $(\pi \circ F)^{-1}(p)$ is connected or empty. Since F is continuous, so is $F((\pi \circ F)^{-1}(p))$.

We have shown (A_1), $(A_m) \Rightarrow (B_{q,m})$ and $(B_{q,q-1}) \Rightarrow (A_q)$. The theorem now follows by induction: $(A_1) \Rightarrow (B_{2,1}) \Rightarrow (A_2) \Rightarrow (B_{3,2}) \Rightarrow (A_3) \Rightarrow \dots$. □

We now turn to projective toric varieties. We start with a projective moment map.

Definition 2.14. The *algebraic moment map* $\bar{\mu}_{A,w} : X_A \to \mathbb{R}^n$ with weights $w \in \mathbb{R}^s_{>0}$ is

$$\bar{\mu}_{A,w}(x) \;=\; \frac{1}{|w_1 \cdot x_1| + \cdots + |w_s \cdot x_s|} \sum_{i=1}^{s} w_i \cdot |x_i| \cdot a_i.$$

Notice that $\bar{\mu}_{A,w}$ is well defined on $\mathbb{P}^{s-1}$. Ignoring absolute values, the map $\bar{\mu}_{A,w}$ is given by the matrix $\hat{A} \cdot \mathrm{diag}(w)$, viewed as a map between projective spaces: $\hat{A} \cdot \mathrm{diag}(w) : \mathbb{P}^{s-1} \dashrightarrow \mathbb{P}^n$, followed by the dehomogenization $\mathbb{P}^n \dashrightarrow \mathbb{R}^n$ which divides by the first coordinate.

Theorem 2.15. *For any positive weights w, the restriction of the algebraic moment map $\bar{\mu}_{A,w}$ to the nonnegative projective toric variety $(X_A)_{\geq 0}$ is a homeomorphism onto* $\mathrm{Conv}(A)$.

Example 2.16. The surface $Y_{A_3} = \{x^3 - yz = 0\}$ is an affine open subset of $X_A = \{x^3 - yzw = 0\} \subset \mathbb{P}^3$. The matrix A is given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

The corresponding polygon $\mathrm{Conv}(A_3)$ is the triangle in the right part of Figure 2. The surface X_A (orange) and its nonnegative part $(X_A)_{\geq 0}$ (blue) are plotted in Figure 5, using $x + y + z + w = 1$. The surface has three singular points, one for each vertex of the triangle. The nonnegative part is homeomorphic to our triangle, as predicted by Theorem 2.13. ⋄

Like in the affine case, Theorem 2.15 follows from the following statement:

Theorem 2.17. *For each face Q of* $\mathrm{Conv}(A)$*, the restriction of the algebraic moment map* $\bar{\mu}_{A,w}$ *to* $(X_{A,A\cap Q})_{>0}$ *is a real analytic isomorphism onto the relative interior of* Q*.*

Here $X_{A,A\cap Q} = \{x \in X_A : \mathrm{supp}(x) \subset Q\}$. After suitable coordinate changes, we are left with the following analog of Theorem 2.13:

Theorem 2.18. *For positive weights w_j and $u_j \in \mathbb{R}^q$, let $\overline{F} : \mathbb{R}^q \to \mathbb{R}^q$ be a map of the form*

$$\overline{F}(y) = \frac{1}{\sum_{j=1}^r w_j \cdot e^{\langle y, u_j \rangle}} \cdot \sum_{j=1}^r w_j \cdot e^{\langle y, u_j \rangle} \cdot u_j.$$

If $P = \mathrm{Conv}(u_1, \ldots, u_r)$ has dimension q, then $\overline{F}$ is a real analytic isomorphism onto $\mathrm{int}(P)$*.*

Example 2.19. In Example 2.16, the two-dimensional face $Q = \mathrm{conv}(A)$ leads to the following map $\mathbb{R}^2 \to \mathbb{R}^2$ (we take all weights w_j to be 1):

$$(y_1, y_2) \longmapsto \left(\frac{e^{y_1+y_2} + e^{y_1+2y_2} + 2e^{2y_1+y_2}}{1 + e^{y_1+y_2} + e^{y_1+2y_2} + e^{2y_1+y_2}}, \frac{e^{y_1+y_2} + 2e^{y_1+2y_2} + e^{2y_1+y_2}}{1 + e^{y_1+y_2} + e^{y_1+2y_2} + e^{2y_1+y_2}} \right).$$

Theorem 2.18 claims this is a real analytic isomorphism of the plane onto the interior of the triangle $\mathrm{Conv}(A)$. The image of $[-\alpha, \alpha]^2$ for $\alpha = 1, 2, 3$ is shown in the right part of Figure 5 in different shades of orange. When $\alpha \to \infty$, the image fills the blue triangle. ⋄

Proof of Theorem 2.18. Let $U = (u_1 \ \cdots \ u_r) \in \mathbb{Z}^{q \times r}$ be the matrix whose columns are u_j. The assumption $\dim P = q$ implies $\mathrm{rank}(\hat{U}) = q + 1$. Theorem 2.13 says that $F : \mathbb{R}^{q+1} \to \mathrm{int}(\mathrm{pos}(\hat{U}))$, with

$$F(y_0, y) = \sum_{j=1}^r w_j \cdot e^{\langle y, u_j \rangle} \cdot e^{y_0} \cdot \hat{u}_j,$$

is a real analytic isomorphism. Here $(y_0, y) = (y_0, y_1, \ldots, y_q)$ are coordinates on $\mathbb{R}^{q+1}$ and $\hat{u}_j = (1, u_j) \in \mathbb{R}^{q+1}$. We identify $\mathrm{int}(P) \simeq \mathrm{int}(\mathrm{pos}(\hat{U})) \cap \{\text{first coordinate equal to } 1\}$. Its preimage under F is $y_0 = -\log(\sum_{j=1}^r w_j \cdot e^{\langle y, u_j \rangle})$. The restriction of F to this preimage is precisely $\bar{F}$. More precisely, we have $F(-\log(\sum_{j=1}^r w_j \cdot e^{\langle y, u_j \rangle}), y) = (1, \overline{F}(y))$. □

3 Selected applications

This section presents some applications of nonnegative toric varieties. We start with an interior point method for linear programming [12]. Next, we switch to statistics [1]. Finally, we discuss toric patches in geometric modelling [9].

3.1 Entropic regularization for linear programming

A linear program seeks to minimize a linear function on a polyhedron. It is given by

$$\text{minimize } c^t \cdot x, \quad \text{such that } A\,x = b \text{ and } x \geq 0. \tag{10}$$

Here $c = (c_1, \ldots, c_s)^t \in \mathbb{R}^s$ is a real vector of length x, $A \in \mathbb{N}^{n \times s}$ is an $n \times s$ matrix with nonnegative integer entries and $b \in \mathbb{R}^n$. The requirement that A has nonnegative entries ensures that the *feasible region* $P_{A,b} = \{x \in \mathbb{R}^s \,:\, Ax = b,\, x \geq 0\}$ is compact. That is, $P_{A,b}$ is a convex polytope. The restriction that A has integer entries is required for the connection with toric varieties. The problem is feasible, i.e., $P_{A,b}$ is nonempty, if and only if $b \in \mathrm{pos}(A)$. We will assume that $b \in \mathrm{relint}(\mathrm{pos}(A))$, so that $P_{A,b}$ is full-dimensional. Otherwise, the problem can be reformulated for smaller s. We also assume that the minimizer is unique. This happens for most vectors c, and the unique minimizer is a vertex of $P_{A,b}$.

Interior point methods solve (10) by first replacing the objective function $c^t \cdot x$ by a strictly convex function on $P_{A,b}$. For instance, one adds a strictly convex function $h : \mathbb{R}^s_{>0} \to \mathbb{R}$ to $c^t \cdot x$. This is called *regularization*. Our choice of h corresponds to *entropic regularization*:

$$\text{minimize } c^t \cdot x + \varepsilon \cdot \sum_{i=1}^{s} (x_i \log x_i - x_i), \qquad \text{such that } \; A\,x = b \text{ and } \; x \geq 0. \tag{11}$$

Here ε is a positive parameter. One checks that $t \log t - t$ is strictly convex on $\mathbb{R}_{>0}$, and its derivative diverges for $t \to 0^+$. This ensures that for each ε, there is a unique minimizer $x^*(\varepsilon) \in \mathrm{int}(P_{A,b})$. When $\varepsilon \to 0^+$, the optimizer $x^*(\varepsilon)$ approaches a minimizer of (10).

Since $x^*(\varepsilon) \in \mathrm{int}(P_{A,b})$, it has to satisfy first order optimality conditions. The Lagrangian

$$\mathcal{L} \;=\; c^t \cdot x + \varepsilon \cdot \sum_{i=1}^{s} (x_i \log x_i - x_i) - \lambda^t \cdot (Ax - b)$$

has partial derivatives $\nabla_\lambda \mathcal{L} = Ax - b$, $\nabla_x \mathcal{L} = c - \varepsilon \log(x) - \lambda^t \cdot A$. Here ∇_λ takes derivatives with respect to the n Lagrange multipliers $\lambda_1, \ldots, \lambda_n$, ∇_x takes derivatives with respect to $x_1, \ldots, x_s$, and $\log(x) = (\log(x_1), \ldots, \log(x_s))$. From this, we see that the first order optimality conditions $\nabla_\lambda \mathcal{L} = \nabla_x \mathcal{L} = 0$ are equivalent to

$$Ax = b \quad \text{and} \quad x = \left(e^{\frac{c_1}{\varepsilon}} e^{\langle \lambda, a_1 \rangle}, \ldots, e^{\frac{c_s}{\varepsilon}} e^{\langle \lambda, a_s \rangle}\right) \qquad \text{for some } \lambda \in \mathbb{R}^n_{\geq 0}.$$

Here $a_i \in \mathbb{N}^n$ are the columns of A. Changing coordinates $t_i = e^{\lambda_i}$, we obtain

$$Ax = b \quad \text{and} \quad x = \left(e^{\frac{c_1}{\varepsilon}} t^{a_1}, \ldots, e^{\frac{c_s}{\varepsilon}} t^{a_s}\right) \qquad \text{for some } t \in \mathbb{R}^n_{>0}. \tag{12}$$

We let $w = w(\varepsilon) = (e^{\frac{c_1}{\varepsilon}}, \ldots, e^{\frac{c_s}{\varepsilon}}) \in \mathbb{R}^s_{>0}$ and define the *scaled affine toric variety*

$$w \star Y_A \;=\; \{(w_1 x_1, \ldots, w_s x_s) \in \mathbb{C}^s \;:\; x \in Y_A\}.$$

Its *positive part* is $(w \star Y_A)_{>0} = (w \star Y_A) \cap \mathbb{R}^s_{>0}$.

Theorem 3.1. *The minimizer $x^*(\varepsilon)$ of (11) is the unique intersection point of the positive scaled toric variety $(w(\varepsilon) \star Y_A)_{>0}$ with the feasible polytope $P_{A,b}$.*

Proof. By (12), we have that $x^*(\varepsilon) = w(\varepsilon) \star \mu^{-1}_{A,w(\varepsilon)}(b)$, where $\star$ multiplies vectors entry-wise. Here we use that the moment map $\mu_{A,w}$ is a homeomorphism (Theorem 2.9). By the assumption that $b \in \mathrm{relint}(\mathrm{pos}(A))$, Theorem 2.11 ensures that $\mu^{-1}_{A,w(\varepsilon)}(b) \in (Y_A)_{>0}$. Clearly, this implies $x^*(\varepsilon) \in (w(\varepsilon) \star Y_A)_{>0} \cap P_{A,b}$, and this intersection consists of only one point. ☐

In the limit $\varepsilon \to \infty$, the optimizer $x^*(\varepsilon)$ converges to the *Birch point* $(Y_A)_{>0} \cap P_{A,b}$ of Y_A associated to b. We will motivate this name in the next section.

Example 3.2 (Optimal transport)**.** We describe a transportation problem that leads to a linear program which is often solved via entropic regularization. Suppose μ_i units of a product are stored at stations $i \in \{1, 2\}$, and ν_j units are desired at stations $j \in \{1, 2, 3\}$. The cost of transporting one unit from station i to station j is c_{ij}. Let x_{ij} be the 2×3 matrix representing the number of units transported from unit i to unit j. This is called the *transportation plan*. Optimal transport asks for the transportation plan that minimizes the total cost. This is formulated as a linear program (10) with parameters

$$c = (c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23})^t, \quad A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}.$$

We have seen this matrix in (2). This formulation assumes that all units at stations i are transported to stations j, and precisely ν_j units arrive at each station j. Clearly, this requires $\mu_1 + \mu_2 = \nu_1 + \nu_2 + \nu_3$. If this is not satisfied, the problem is infeasible. The feasible polytope $P_{A,b}$ for optimal transport problems is called the *transportation polytope*. By Theorem 3.1, the Birch point is the unique positive rank-1 2×3 matrix inside the transportation polytope. $\diamond$

3.2 Toric models in algebraic statistics

A probability distribution for a discrete random variable with s states is a point in the $(s-1)$-dimensional probability simplex

$$\Delta_{s-1} = \{(x_1, \ldots, x_s) \in \mathbb{R}^s_{\geq 0} \,:\, x_1 + \cdots + x_s = 1\}.$$

Maximum likelihood estimation means estimating a distribution $\hat{p} \in \Delta_{s-1}$ from empirical observations, using the ansatz that $\hat{p}$ belongs to a *model*. More precisely, suppose that the state i is observed $u_i \in \mathbb{N}$ times in an experiment, for $i = 1, \ldots, s$. The *likelihood* of observing the data $(u_1, \ldots, u_s)$, assuming the probability distribution $(p_1, \ldots, p_s) \in \Delta_{s-1}$, is $p_1^{u_1} \cdots p_s^{u_s}$. Using a priori knowledge on the random variable, we expect that the true distribution $\hat{p}$ belongs to a *statistical model* $X \subset \Delta_{s-1}$. The *maximum likelihood estimate* (MLE) is the distribution $p^* \in X$ that maximizes the likelihood $p_1^{u_1} \cdots p_s^{u_s}$. Equivalently, and often more practically, we maximize the *log-likelihood* $u_1 \log p_1 + \cdots + u_s \log p_s$ on X.

In *algebraic statistics* [5, 13], the model X is an *semialgebraic set*, obtained by intersecting an algebraic variety with the probability simplex. In our setting, we will identify $\Delta_{s-1} \simeq (\mathbb{P}^{s-1})_{\geq 0}$ with the nonnegative part of projective space, and our models are called *log-linear models*, *exponential families* or *toric models*, see for instance [5, §1.1 and §1.2] or [1].

Definition 3.3. The *toric model* associated to an integer matrix $A \in \mathbb{Z}^{n \times s}$ is the nonnegative projective toric variety $(X_A)_{\geq 0} \subset (\mathbb{P}^{s-1})_{\geq 0}$.

In the exercises, you will check that the positive part $(X_A)_{>0}$ of the toric model consists of all probability distributions $(x_1, \ldots, x_s) \in \text{relint}(\Delta_{s-1})$ whose coordinate-wise logarithm $(\log x_1, \ldots, \log x_s)$ belongs to the row span of $\hat{A}$. Since the matrices A and $\hat{A}$ define the same projective toric variety X_A, we will assume from now on that the vector of all ones $(1, \ldots, 1)$ is in the row span of A. The following theorem describes the MLE for toric models.

Theorem 3.4 (Birch's theorem)**.** *The MLE for the toric model $(X_A)_{\geq 0} \subset (\mathbb{P}^{s-1})_{\geq 0} \simeq \Delta_{s-1}$ for a vector $u = (u_1, \ldots, u_s) \in (\mathbb{N} \setminus \{0\})^s$ is the unique point $p^* \in (X_A)_{\geq 0}$ satisfying*

$$A\,p^* \;=\; A\,\bar{u}, \quad \textit{where} \quad \bar{u} = \left(\frac{u_1}{u_1 + \cdots + u_s}, \, \cdots, \, \frac{u_s}{u_1 + \cdots + u_s} \right).$$

Proof. The MLE is obtained from the following optimization problem:

$$\text{minimize } \; -\sum_{i=1}^{s} u_i \, \log x_i \quad \text{ such that } \; \log x \in \text{Row}(A), \; \sum_{i=1}^{s} x_i = 1 \text{ and } x \in \mathbb{R}^s_{>0}.$$

Let $B \in \mathbb{Q}^{s \times \ell}$ be a kernel matrix for A: $A \cdot B = 0$, $\text{rank}(A) = s - \ell$, $\text{rank}(B) = \ell$. We express the condition $\log x \in \text{Row}(A)$ as $B^t \cdot \log x = 0$. The Lagrangian is

$$\mathcal{L} \;=\; -u^t \cdot \log x - \lambda^t \cdot (B^t \cdot \log x) - \mu \cdot (\sum_i x_i - 1),$$

where $\lambda \in \mathbb{R}^\ell$ and $\mu \in \mathbb{R}$ are Lagrange multipliers. Partial derivation with respect to x gives

$$-u - B \cdot \lambda = \mu \cdot x, \quad \text{ and hence } \quad A\,x = \frac{-A\,u}{\mu}.$$

Here we just multiplied from the left with A. Now, by the assumption that $(1, \ldots, 1) \in \text{Row}(A)$ there is $c \in \mathbb{R}^n$ so that $c^t \cdot A = (1, \ldots, 1)$. Applying c^t from the left to our equation, we see that $\sum_i x_i = -\mu^{-1}(\sum_i u_i)$. Together with the condition $\sum_i x_i = 1$, this gives $\mu = -\sum_i u_i$. This gives $Ax = A\bar{u}$ as desired, and there is only one such point $x = p^* \in (X_A)_{>0}$ by Theorem 2.17. For this we use $A\bar{u} \in \text{int}(\text{Conv}(A))$. □

Example 3.5 (Independence models)**.** Suppose we ask a population of 100 people whether they are vegetarian (Y = yes, N = no), and which of the three subjects algebra, geography and history they find most interesting (A = algebra, G = geography, H = history). A purely fictional outcome of the experiment is the following table:

	A	G	H
Y	15	7	16
N	24	14	24

We consider the random variable with six outcomes YA, YG, YH, NA, NG, NH. For instance YH is the state in which a person is vegetarian and likes history the most. The probabilities of being vegetarian or not are denoted by p_{Y} and p_{N}, and the probabilities of being most interested in algebra, geography or history are $p_{\text{A}}, p_{\text{G}}, p_{\text{H}}$. Assuming that being vegetarian or not is independent from someone's subject preference, the probability of liking algebra and being vegetarian is simply the product $x_{\mathrm{YA}} = p_{\mathrm{Y}} p_{\mathrm{A}}$. This parametrizes a model

$$(p_{\mathrm{Y}}, p_{\mathrm{N}}, p_{\mathrm{A}}, p_{\mathrm{G}}, p_{\mathrm{H}}) \longmapsto (p_{\mathrm{Y}}p_{\mathrm{A}} : p_{\mathrm{Y}}p_{\mathrm{G}} : p_{\mathrm{Y}}p_{\mathrm{H}} : p_{\mathrm{N}}p_{\mathrm{A}} : p_{\mathrm{N}}p_{\mathrm{G}} : p_{\mathrm{N}}p_{\mathrm{H}}) \ \in \mathbb{P}^5 \simeq \Delta_5.$$

We recognize the toric model given by the matrix A seen in (2) and Example 3.2. The MLE is the unique positive solution to the system of polynomial equations

$$\operatorname{rank} \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_5 \end{pmatrix} \leq 2 \quad \text{and} \quad A\,x = (0.15,\, 0.07,\, 0.16,\, 0.24,\, 0.14,\, 0.24)^t. \qquad \diamond$$

3.3 Toric patches in geometric modelling

A matrix $\hat{P} = (\hat{p}_1 \ \cdots \ \hat{p}_s) \in \mathbb{C}^{(k+1)\times s}$ represents a rational map $\pi_{\hat{P}} : \mathbb{P}^{s-1} \dashrightarrow \mathbb{P}^k$. This map is called the *central projection* from $\mathbb{P}(\ker \hat{P})$, where $\ker \hat{P} \subset \mathbb{C}^s$ is the kernel of $\hat{P}$, viewed as a linear map $\mathbb{C}^s \to \mathbb{C}^{k+1}$. The image of the projective toric variety $X_A \subset \mathbb{P}^{s-1}$ is parametrized by polynomials whose monomial support is contained in A:

$$(\pi_{\hat{P}} \circ \Phi_A)(t) \ = \ \hat{p}_1 \cdot t^{a_1} + \cdots + \hat{p}_s \cdot t^{a_s} \ \in \mathbb{P}^k. \tag{13}$$

It is clear that any unirational variety arises as the closure of $\pi_{\hat{P}}(X_A)$ for some $\hat{P} \in \mathbb{C}^{(k+1)\times s}$, and some $A \in \mathbb{Z}^{n\times s}$. Here, we consider only real matrices $\hat{P} \in \mathbb{R}^{(k+1)\times s}$, and we are interested in the image $\pi_{\hat{P}}((X_A)_{\geq 0}) \subset \mathbb{RP}^k$ of the nonnegative projective toric variety associated to A. From (13), we see that this image consists of nonnegative combinations of the columns of $\hat{P}$, viewed as points in $\mathbb{P}^k$. After a coordinate transformation, we may assume that their first coordinates are nonzero. Furthermore, below we will introduce positive weights for each of the columns of $\hat{P}$, so that we may scale the first row of the matrix to be the all-ones vector $(1 \ \cdots \ 1)$. In analogy with our notation above, to denote the matrix consisting of all but this first row, we drop the hat and use $P = (p_1 \ \cdots \ p_s) \in \mathbb{R}^{k\times s}$.

Definition 3.6. The *toric patch* associated to the matrices $P \in \mathbb{R}^{k\times s}$, $A \in \mathbb{Z}^{n\times s}$ and the vector of weights $w \in \mathbb{R}^s_{>0}$ is the image of $(X_A)_{\geq 0}$ under the algebraic moment map

$$\bar{\mu}_{P,w}(x) \ = \ \frac{1}{|w_1 \cdot x_1| + \cdots + |w_s \cdot x_s|} \sum_{i=1}^{s} w_i \cdot |x_i| \cdot p_i.$$

Up to taking absolute values, the map $\bar{\mu}_{P,w}$ is constructed by composing $\pi_{\hat{P}}$ with scaling and dehomogenization, like in the discussion following Definition 2.14. In particular, by Definition 2.14 and Theorem 2.15, the toric patch for $A = P$ (in particular, $n = k$) equals the polytope $\operatorname{Conv}(A)$. This is called the *tautological patch* in [2, Example 3.8].

In geometric modelling, the columns of P are called the *control points*, and the coefficients of p_i in the formula for $\bar{\mu}_{P,w}$, restricted to X_A, are the *blending functions*. Toric patches with $k = 2$ and $n = 1$ are *Bézier curves*. They are linear projections of nonnegative rational normal curves, the homogeneous version of the moment curves in Example 1.2. These are used, for instance, in font design [2, Example 3.2]. Toric patches with $k = 3$ and $n = 2$ are

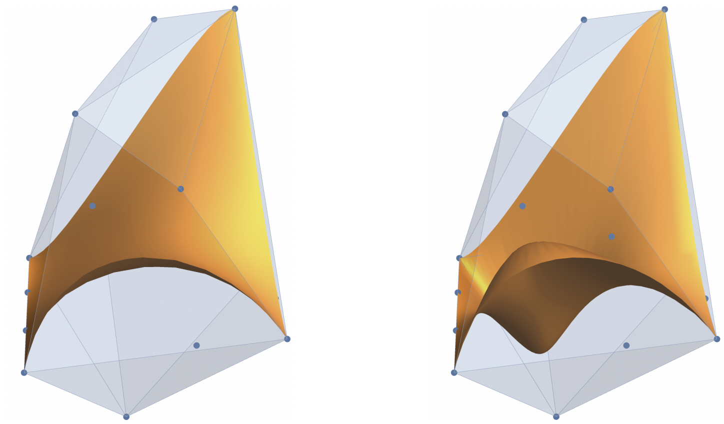

Figure 6: Two toric patches from Example 3.7 with weights w_1 (left) and w_2 (right).

projections of nonnegative toric surfaces which are used to create and approximate all kinds of shapes. For instance, [2, p. 94] mentions the Guggenheim museum in Bilbao. The most commonly used surface patches are the *tensor product surface of degree* (d_1, d_2), which uses $A = \{(a,b) \in \mathbb{Z}^2 : 0 \leq a \leq d_1, 0 \leq b \leq d_2\}$, and the *triangular Bézier patch of degree* d, which uses $A = \{(a,b) \in \mathbb{Z}^2 : a \geq 0, b \geq 0, a+b \leq d\}$. Via the inverse of the moment map, such patches are parametrized by rectangles and triangles respectively. The more general patches from Definition 3.6 were introduced by Krasauskas in [8]. We give one example.

Example 3.7. We draw toric patches corresponding to the following matrices:

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 0 & 3 & 2 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & 0 & -2 & \frac{1}{2} & 0 & 0 & 0 & 3 & 3 \end{pmatrix}.$$

The parameters are $n = 2, s = 16, k = 3$. The projective toric surface X_A corresponds to the polygon $\text{Conv}(A) = [0,3]^2$. It is the Segre-Veronese embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{15}$ of degree $(3,3)$. To illustrate the influence of the weights w, we plotted the toric patch $\bar{\mu}_{P,w}((X_A)_{\geq 0}) \subset \mathbb{R}^3$ for two different sets of weights in Figure 6. The weights are

$$\begin{aligned} w_1 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1), \\ w_2 &= (1,1,1,1,5,100,5,5,5,100,5,5,1,1,1,1). \end{aligned}$$

The figure demonstrates the *convex hull property*: a toric patches is always contained in $\text{Conv}(P)$, the convex hull of the control points. ◇

Acknowledgements. Thanks to Saiei-Jaeyeong Matsubara-Heo and Nobuki Takayama for organizing the summer school in Kobe. Thanks to Leonie Kayser for useful discussions.

4 Exercises

1. Toric ideals. Prove the inclusion $I_A \subset I(Y_A)$.

2. Affine lattices. Prove that $\mathbb{Z}(A - a_i) = \mathbb{Z}(A - a_j)$ for all $1 \leq i, j \leq s$.

3. Dimension, degree and smoothness. Verify Theorems 1.6, 1.9 and 1.13 for the matrices A_1, A_2 and A_3 in Example 1.3.

4. Double pillow [9, Section 7]. Consider the integer matrix $A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$.

1. Determine the vertices of $\text{Conv}(A)$. For one of these vertices v, write down a monomial parametrization of the affine toric surface Y_{A-v}, and compute its toric ideal. Is this a smooth surface?

2. Compute the toric ideal of the projective toric surface $X_A \subset \mathbb{P}^4$.

3. Compute the defining equation of the projection of X_A under

$$\pi = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3.$$

4. Plot the surface $\pi(X_A)$ in the affine chart of $\mathbb{RP}^3$ with nonzero first coordinate. Do you recognize the real part of $\pi(Y_{A-v})$ from part (a) in the picture?

5. Identify $\pi((X_A)_{\geq 0})$ in the plot. Verify that $(X_A)_{\geq 0}$ is homeomorphic to $\text{Conv}(A)$.

5. Real toric varieties. Show that $\overline{\phi_A((\mathbb{R}^*)^n)} \subset Y_A(\mathbb{R})$, where $\bar{\cdot}$ is the Euclidean closure, might be a strict inclusion.

6. The boundary of an affine toric variety.

Proposition 4.1. *Let $x \in Y_A \subset \mathbb{C}^s$ and let $\text{supp}(x) = \{a_i \in A : x_i \neq 0\}$. We have $\text{supp}(x) = Q \cap A$ for some face Q of the cone $\text{pos}(A) = \{r_1 a_1 + \cdots + r_s a_s : r_i \in \mathbb{R}_{\geq 0}\}$.*

7. Stratification of affine toric varieties.

Proposition 4.2. *For a face $Q \subset \text{pos}(A)$, let $A \cap Q = \{a_{i_1}, \ldots, a_{i_r}\}$ and define the projection $\pi_Q : \mathbb{C}^s \to \mathbb{C}^r$, $x \mapsto (x_{i_1}, \ldots, x_{i_r})$. Let $Y_{A, A\cap Q} = \{x \in Y_A, \text{supp}(x) \subset A \cap Q\}$. We have*

$$\pi_Q(Y_A) = \pi_Q(Y_{A, A\cap Q}) = Y_{A\cap Q}.$$

In particular, if $\text{supp}(x) = A \cap Q$, then there is $t \in (\mathbb{C}^*)^n$ such that

$$x_i = \begin{cases} t^{a_i} & a_i \in A \cap Q, \\ 0 & \textit{otherwise} \end{cases}.$$

8. Nonnegative projective toric varieties. Prove the second part of Proposition 2.6.

9. Positive toric models. Show that the positive part $(X_A)_{>0}$ of a toric model consists of all probability distributions $(x_1, \dots, x_s) \in \mathrm{relint}(\Delta_{s-1})$ whose coordinate-wise logarithm $(\log x_1, \dots, \log x_s)$ belongs to the row span of $\hat{A}$.

10. Maximum likelihood estimation. Compute the MLE for Example 3.5. Based on these fictional data, does your result confirm the ansatz that being vegetarian or not is independent of someone's subject preference? Can you estimate $p_{\mathrm{Y}}, p_{\mathrm{N}}, p_{\mathrm{A}}, p_{\mathrm{G}}, p_{\mathrm{H}}$?

11. Toric patches. Plot the Bézier plane curve of degree 6, i.e., $A = (0\ 1\ \cdots\ 6)$, for any matrix of control points $P \in \mathbb{R}^{2\times 6}$ and weights $w \in \mathbb{R}^6_{>0}$. Investigate the influence of the control points and the weights by experimenting. Same exercise for $k = 3$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix}.$$

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