The variant of the q-hypergeometric equation of degree three: a q-analog of Papperitz's differential system

Takahiko Nobukawa
Kobe University (Ph.D student)
mail:tnobukw@math.kobe-u.ac.jp

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Abstract

Two variants of the q-hypergeometric equation $\mathcal{H}_2$ and $\mathcal{H}_3$ are introduced by Hatano-Matsunawa-Sato-Takemura from the viewpoint of some quantum integrable system.
In this talk, we will discuss $\mathcal{H}_3$ mainly.
We will show that this equation can be regarded as a q-analog of the Riemann-Papperitz differential system.
From this aspect, we will give integral solutions and series solutions for the equation $\mathcal{H}_3$.
If time permits, we will show some applications.

Notations

Throughout this talk, we fix $q \in \mathbb{C}$ with $0 < |q| < 1$.

the q-shifted factorial (the q-Pochhammer symbol):

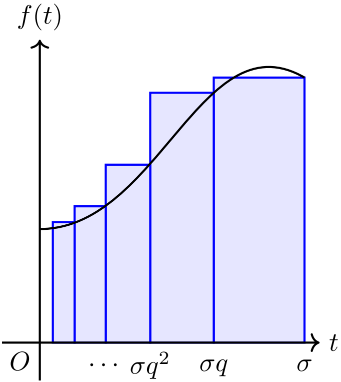
$$(a)_\infty = \prod_{i=0}^{\infty}(1 - aq^i), \ (a)_n = \frac{(a)_\infty}{(aq^n)_\infty}, \ (a_1, \ldots, a_M)_n = (a_1)_n \cdots (a_M)_n.$$

the Jackson integral:

$$\int_0^{\sigma} f(t) d_q t = (1-q)\sigma \sum_{i=0}^{\infty} f(\sigma q^i) q^i,$$

$$\int_0^{\sigma\infty} f(t) d_q t = (1-q)\sigma \sum_{i=-\infty}^{\infty} f(\sigma q^i) q^i,$$

$$\int_{\sigma_1}^{\sigma_2} f(t) d_q t = \int_0^{\sigma_2} f(t) d_q t - \int_0^{\sigma_1} f(t) d_q t.$$



the q-shift operator: $T_x f(x) = f(qx)$.

Main result

The variant of the q-hypergeometric equation of degree three is defined as follows:

$$
\begin{aligned}
&\mathcal{H}_3\, y = 0,\\
&\mathcal{H}_3 = \prod_{i=1}^{3}(x - q^{h_i+1/2}t_i)\cdot T_x^{-1} + q^{2\alpha+1}\prod_{i=1}^{3}(x - q^{l_i-1/2}t_i)\cdot T_x\\
&\quad - q^{\alpha}\Bigg[(q+1)x^3 - q^{1/2}\sum_{i=1}^{3}(q^{h_i}+q^{l_i})t_i x^2\\
&\qquad + q^{(h_1+h_2+h_3+l_1+l_2+l_3+1)/2}t_1t_2t_3\sum_{i=1}^{3}\frac{q^{-h_i}+q^{-l_i}}{t_i}x\\
&\qquad - q^{(h_1+h_2+h_3+l_1+l_2+l_3)/2}(q+1)t_1t_2t_3\Bigg].
\end{aligned}
$$

The functions

$$x^{\nu-\alpha}\int_{\sigma_1}^{\sigma_2}\frac{(q^{\nu}xt,q^{h_1+\frac12}t_1t,q^{h_2+\frac12}t_2t,q^{h_3+\frac12}t_3t)_\infty}{(xt,q^{\nu+l_1-\frac12}t_1t,q^{\nu+l_2-\frac12}t_2t,q^{\nu+l_3-\frac12}t_3t)_\infty}\,d_qt,$$

$$x^{\nu-\alpha}\frac{(q^{\nu-h_3+\frac12}x/t_3)_\infty}{(q^{\frac12-h_3}x/t_3)_\infty}$$

$$\times\,{}_8W_7\left(\frac{t_1q^{h_1-h_3+\nu}}{t_3};\frac{t_1q^{-h_3+l_1+\nu}}{t_3},\frac{t_2q^{-h_3+l_2+\nu}}{t_3},q^{-h_3+l_3+\nu},\frac{t_1q^{h_1+\frac12}}{x},q^{\nu};\frac{xq^{\frac12-h_2}}{t_2}\right),$$

satisfy $\mathcal{H}_3\,y=0$, where $\nu=\frac12\,(h_1+h_2+h_3-l_1-l_2-l_3+1)$, σ_1, $\sigma_2\in\{q^{\frac12-h_1}/t_1,q^{\frac12-h_2}/t_2,q^{\frac12-h_3}/t_3,q^{1-\nu}/x\}$ and

$${}_8W_7(a;b,c,d,e,f;z)=\sum_{n=0}^{\infty}\frac{1-aq^{2n}}{1-a}\frac{(a,b,c,d,e,f)_n}{(q,qa/b,qa/c,qa/d,qa/e,qa/f)_n}z^n.$$

Background

$$\boxed{\text{Painlevé}} \longrightarrow \boxed{\text{Ruijsenaars-van Diejen}} \longrightarrow \boxed{\text{hypergeometric}}$$

specialization of Lax linear equations

specialization like Heun $\to$ Gauss

$$\begin{tikzcd}
\text{elliptic } E_8^{(1)} \arrow[r] \arrow[d] & Ay = Ey \arrow[d] & \\
q\text{-}E_8^{(1)} \arrow[r] \arrow[d] & A^{\langle 1 \rangle} y = Ey \arrow[d] & \\
q\text{-}E_7^{(1)} \arrow[r] \arrow[d] & A^{\langle 2 \rangle} y = Ey \arrow[r] \arrow[d] & \mathcal{H}_3\, y = 0 \arrow[d] \\
q\text{-}E_6^{(1)} \arrow[r] \arrow[d] & A^{\langle 3 \rangle} y = Ey \arrow[r] \arrow[d] & \mathcal{H}_2\, y = 0 \arrow[d] \\
q\text{-}D_5^{(1)} \arrow[r] & A^{\langle 4 \rangle} y = Ey \arrow[r] & \text{Heine}
\end{tikzcd}$$

q-Heun and its variants

A brief review for the Heine's equation

The Heine's q-hypergeometric equation is defined as follows:

$$[(1-T_x)(1-cq^{-1}T_x)-x(1-aT_x)(1-bT_x)]y=0.$$

This equation is a q-analog of the Gauss hypergeometric equation. More precisely, by taking the classical limit $q \to 1$ with $a = q^{\alpha}$, $b = q^{\beta}$, $c = q^{\gamma}$, we get

$$\left[x(1-x)\frac{d^2}{dx^2}+(\gamma-(\alpha+\beta+1)x)\frac{d}{dx}-\alpha\beta\right]y=0.$$

The variants of the q-hypergeometric equation $\mathcal{H}_2$ and $\mathcal{H}_3$ are some extensions for the Heine's equation.

The Gauss equation has Euler type integral solutions and series solutions in terms of the Gauss hypergeometric function ${}_2F_1$:

$$\int_C t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}dt,$$

$${}_2F_1\left(\begin{array}{c}\alpha,\beta\\ \gamma\end{array};x\right)=1+\frac{\alpha\cdot\beta}{\gamma\cdot 1}x+\frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{\gamma(\gamma+1)\cdot 1\cdot 2}x^2+\cdots.$$

Similarly the Heine's equation has solutions in terms of q-analogs of the above functions:

$$\int_C t^{\alpha-1}\frac{(qt)_\infty}{(ct/a)_\infty}\frac{(bxt)_\infty}{(xt)_\infty}d_qt \quad (q^\alpha=a), \qquad \left(\frac{(q^\alpha x)_\infty}{(x)_\infty}\xrightarrow{q\to 1}(1-x)^{-\alpha}\right)$$

$${}_2\varphi_1\left(\begin{array}{c}a,b\\ c\end{array};x\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(q)_n}x^n$$

$$=1+\frac{(1-a)(1-b)}{(1-c)(1-q)}x+\frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-c)(1-cq)(1-q)(1-q^2)}x^2+\cdots.$$

The equation $\mathcal{H}_3\, y = 0$ (recall)

The variant of the q-hypergeometric equation of degree three is defined as follows:

$$
\begin{aligned}
&\mathcal{H}_3\, y = 0,\\
&\mathcal{H}_3 = \prod_{i=1}^{3}(x - q^{h_i+1/2}t_i)\cdot T_x^{-1} + q^{2\alpha+1}\prod_{i=1}^{3}(x - q^{l_i-1/2}t_i)\cdot T_x\\
&\quad - q^{\alpha}\Bigg[(q+1)x^3 - q^{1/2}\sum_{i=1}^{3}(q^{h_i}+q^{l_i})t_i x^2\\
&\qquad + q^{(h_1+h_2+h_3+l_1+l_2+l_3+1)/2}t_1t_2t_3\sum_{i=1}^{3}\frac{q^{-h_i}+q^{-l_i}}{t_i}x\\
&\qquad - q^{(h_1+h_2+h_3+l_1+l_2+l_3)/2}(q+1)t_1t_2t_3\Bigg].
\end{aligned}
$$

Characterization

Suppose an operator $H = \sum_{i=0}^{3} \sum_{j=-1}^{1} a_{i,j} x^i T_x^j$ satisfies the following conditions:

- We rewrite $H = \sum_{i=0}^{3} x^i L_i(T_x)$, then
  - $L_0(y) \propto (y-a)(y-aq)$, $L_1(y) \propto (y-a)$.
  - $L_3(y) \propto (y-b)(y-bq^{-1})$, $L_2(y) \propto (y-b)$.
- We rewrite $H = \sum_{j=-1}^{1} P_j(x) T_x^j$, then
  - $P_{-1}(x) \propto (x-c_1)(x-c_2)(x-c_3)$.
  - $P_1(x) \propto (x-d_1)(x-d_2)(x-d_3)$.

Then the equation $Hy = 0$ is equivalent to the equation $\mathcal{H}_3\, y = 0$.

Classical limit $q \to 1$

By taking the limit $q \to 1$, the equation $\mathcal{H}_3\, y = 0$ becomes a Fuchsian differential equation which has the following Riemann scheme:

$$\left\{ \begin{array}{ccccc} x = 0 & t_1 & t_2 & t_3 & \infty \\ \rho_0 & 0 & 0 & 0 & \rho_\infty \\ \rho_0 + 1 & * & * & * & \rho_\infty + 1 \end{array} \right\}.$$

The points $x = 0$ and ∞ are essentially non-singular (i.e. these points can be transformed to regular points by some gauge factor).
By some gauge transformation, this differential equation is transformed to

$$\left\{ \begin{array}{ccc} x = t_1 & t_2 & t_3 \\ * & * & * \\ * & * & * \end{array} \right\}.$$

This equation is called the Riemann-Papperitz's differential equation.

Solutions for the Riemann-Papperitz's equation

The Riemann-Papperitz's differential equation can be transformed to the Gauss' hypergeometric equation by the Möbius transformation $x \mapsto \dfrac{x-t_1}{x-t_3}\dfrac{t_2-t_3}{t_2-t_1}$.
It is well known that the Gauss' equation has Euler-type integral solutions and series solutions in terms of the hypergeometric function ${}_2F_1$:

$$\int_C t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}dt, \quad {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};x\right).$$

So we get solutions for the Riemann-Papperitz's equation:

$$\text{(gauge factor)} \times \int_C (t-x)^{\nu_0}(t-t_1)^{\nu_1}(t-t_2)^{\nu_2}(t-t_3)^{\nu_3}dt,$$

$$\text{(gauge factor)} \times {}_2F_1\left(\begin{matrix}\mu_1,\mu_2\\ \mu_3\end{matrix};\frac{x-t_1}{x-t_3}\frac{t_2-t_3}{t_2-t_1}\right),$$

where $\nu_0+\nu_1+\nu_2+\nu_3=-2$.

Observation

- The equation $\mathcal{H}_3\, y = 0$ can be regarded as a q-analog of the Riemann-Papperitz's differential system.
- The Riemann-Papperitz's system has integral solutions and series solutions as follows:

$$\text{(gauge factor)} \times \int_C (t-x)^{\nu_0}(t-t_1)^{\nu_1}(t-t_2)^{\nu_2}(t-t_3)^{\nu_3}\,dt,$$

$$\text{(gauge factor)} \times {}_2F_1\left(\begin{matrix}\mu_1,\mu_2\\ \mu_3\end{matrix};\frac{x-t_1}{x-t_3}\frac{t_2-t_3}{t_2-t_1}\right),$$

where $\nu_0+\nu_1+\nu_2+\nu_3=-2$.

$\rightsquigarrow$ Naively it is expected that the equation $\mathcal{H}_3\, y = 0$ has solutions in terms of q-analogs of these functions.

We consider q-analogs of such solutions for the Riemann-Papperitz's system.

- It is difficult to directly consider a q-analog of the series

$$ {}_2F_1\left(\begin{matrix} \mu_1, \mu_2 \\ \mu_3 \end{matrix} ; \frac{x-t_1}{x-t_3}\frac{t_2-t_3}{t_2-t_1} \right). $$

  In the theory of q-difference equations, the points $x = 0$ and ∞ are special points because these are fixed points of the q-shift operator $T_x : x \mapsto qx \mapsto q^2 x \mapsto \cdots$. So we cannot apply some Möbius transformation to q-difference equations.

- It is easy to consider a q-analog of the integral

$$ \int_C (t-x)^{\nu_0}(t-t_1)^{\nu_1}(t-t_2)^{\nu_2}(t-t_3)^{\nu_3}\,dt, $$

  by the q-binomial theorem $\left(\dfrac{(q^\alpha x)_\infty}{(x)_\infty} \xrightarrow{q\to 1} (1-x)^{-\alpha} \right)$.

In short, we should consider the following Jackson integral:

$$\int_C \frac{(q^{-\nu_0}xt)_\infty}{(xt)_\infty}\frac{(q^{-\nu_1}t_1t)_\infty}{(t_1t)_\infty}\frac{(q^{-\nu_2}t_2t)_\infty}{(t_2t)_\infty}\frac{(q^{-\nu_3}t_3t)_\infty}{(t_3t)_\infty}d_qt,$$

where $\nu_0+\nu_1+\nu_2+\nu_3=-2$. This integral is equivalent to

$$\int_C \frac{(Axt)_\infty}{(Bxt)_\infty}\prod_{i=1}^{3}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt,$$

where $Aa_1a_2a_3=q^2Bb_1b_2b_3$.

We will derive a q-difference equation that the integral satisfies.
To derive it, a q-difference equation that <u>the integrand satisfies</u> will be considered.

$$\psi=\psi(x,t)=\frac{(Axt)_\infty}{(Bxt)_\infty}\prod_{i=1}^{3}\frac{(a_it)_\infty}{(b_it)_\infty}.$$

The integrand $\psi = \frac{(Axt)_\infty}{(Bxt)_\infty} \prod_{i=1}^{3} \frac{(a_i t)_\infty}{(b_i t)_\infty}$ satisfies the following system:

$$T_x \psi = \frac{1 - Bxt}{1 - Axt} \psi, \qquad T_x^{-1} T_t \psi = \prod_{i=1}^{3} \frac{1 - b_i t}{1 - a_i t} \psi.$$

So we get the equations:

$$tx(B - AT_x)\psi = (1 - T_x)\psi,$$
$$\sum_{k=0}^{3} (-1)^k t^k \left(e_k(a) T_x^{-1} T_t - e_k(b) \right) \psi = 0.$$

Here e_k is the k-th elementary symmetric function.
By using the first equation, we can delete t^k from the second equation.

We get the following:

$$
\begin{aligned}
&[x^3(B - Aq^2T_x)(B - AqT_x)(B - AT_x)(T_x^{-1}T_t - 1) \\
&\quad - x^2(B - AqT_x)(B - AT_x)(e_1(a)T_x^{-1}T_t - e_1(b))(1 - T_x) \\
&\quad + x(B - AT_x)(e_2(a)T_x^{-1}T_t - e_2(b))(1 - q^{-1}T_x)(1 - T_x) \\
&\quad - (e_3(a)T_x^{-1}T_t - e_3(b))(1 - q^{-2}T_x)(1 - q^{-1}T_x)(1 - T_x)]\psi = 0.
\end{aligned}
$$

By integrating this equation with t, we can delete T_t because

$$
\int_0^{\sigma\infty} T_t f(t) d_q t = q^{-1} \int_0^{\sigma\infty} f(t) d_q t.
$$

So we obtain a linear q-difference equation of rank 4 that the integral $\varphi = \varphi(x, \sigma) = \int_0^{\sigma\infty} \psi(x, t) d_q t$ satisfies.

We find

$$
\begin{aligned}
&[x^3(B - Aq^2T_x)(B - AqT_x)(B - AT_x)(T_x^{-1}q^{-1} - 1) \\
&- x^2(B - AqT_x)(B - AT_x)(e_1(a)T_x^{-1}q^{-1} - e_1(b))(1 - T_x) \\
&+ x(B - AT_x)(e_2(a)T_x^{-1}q^{-1} - e_2(b))(1 - q^{-1}T_x)(1 - T_x) \\
&- (e_3(a)T_x^{-1}q^{-1} - e_3(b))(1 - q^{-2}T_x)(1 - q^{-1}T_x)(1 - T_x)]\varphi(x, \sigma) = 0.
\end{aligned}
$$

This equation is reducible:

$$
\begin{aligned}
&(B - Aq^{-1}T_x)(1 - q^{-2}T_x)H_3\varphi(x, \sigma) = 0, \\
&H_3 = x^3(B - AqT_x)(B - AT_x)T_x^{-1} - x^2(B - AT_x)(e_1(a)T_x^{-1} - qe_1(b)) \\
&\qquad + x(e_2(a)T_x^{-1} - qe_2(b))(1 - T_x) - \frac{a_1a_2a_3}{B}(1 - q^{-1}T_x)(1 - T_x)T_x^{-1}.
\end{aligned}
$$

The equation $H_3\, y = 0$ is equivalent to the variant of the q-hypergeometric equation of degree three $\mathcal{H}_3\, y = 0$.
More precisely, we put $A = q^\nu, B = 1, a_i = t_i q^{h_i + \frac{1}{2}}, b_i = t_i q^{l_i - \frac{1}{2} + \nu}$ and apply some gauge transformation $y \to x^{\nu - \alpha} y$, then $H_3\, y = 0$ becomes $\mathcal{H}_3\, y = 0$.

Now we have $(B - Aq^{-1}T_x)(1 - q^{-2}T_x)H_3\varphi(x, \sigma) = 0$.
So we want to delete the terms $(B - Aq^{-1}T_x)(1 - q^{-2}T_x)$.

- $(B - Aq^{-1}T_x)y = 0 \rightsquigarrow y = Cx^\lambda$, $(q^\lambda = Bq/A)$.
- $(1 - q^{-2}T_x)y = 0 \rightsquigarrow y = Cx^2$.

$\rightsquigarrow$ We get $H_3\varphi(x, \sigma) = C_1 x^\lambda + C_2 x^2$.

By some calculations, we find the following:

$$H_3\varphi(x, \sigma) = (A - B)qx^2, \quad (\sigma \in \{q/a_1, q/a_2, q/a_3, q/(Ax)\}).$$

Integral solutions

Finally we get 6 integral solutions for the equation $H_3\, y = 0$:

$$\int_{\sigma_1}^{\sigma_2} \frac{(Axt)_\infty}{(Bxt)_\infty} \prod_{i=1}^{3} \frac{(a_i t)_\infty}{(b_i t)_\infty} d_q t,$$

where $Aa_1a_2a_3 = q^2 Bb_1b_2b_3$ and σ_1, $\sigma_2 \in \{q/a_1, q/a_2, q/a_3, q/(Ax)\}$.
These are q-analogs of integral solutions

$$\int_C (t-x)^{\nu_0}(t-t_1)^{\nu_1}(t-t_2)^{\nu_2}(t-t_3)^{\nu_3}\, dt,$$

for the Riemann-Papperitz's equation.

From integral solutions to series solutions

Next, we give series solutions for the equation $H_3\, y = 0$.
By the Bailey's formula

$$\int_a^b \frac{(qt/a, qt/b, ct, dt)_\infty}{(et, ft, gt, ht)_\infty} d_q t$$
$$= b(1-q)\frac{(q, bq/a, a/b, cd/eh, cd/fh, cd/gh, bc, bd)_\infty}{(ae, af, ag, be, bf, bg, bh, bcd/h)_\infty}$$
$$\times\, {}_8W_7(bcd/hq; be, bf, bg, c/h, d/h; ah) \quad (cd = abefgh),$$

we get solutions for the equation $H_3\, y = 0$ in terms of the very-well-poised q-hypergeometric function ${}_8W_7$. Here,

$${}_8W_7(a; b, c, d, e, f; z) = \sum_{n=0}^{\infty} \frac{1-aq^{2n}}{1-a} \frac{(a, b, c, d, e, f)_n}{(q, qa/b, qa/c, qa/d, qa/e, qa/f)_n} z^n.$$

Applying the Bailey's formula, we have

$$\int_{q/a_1}^{q/a_3} \frac{(Axt, a_1t, a_2t, a_3t)_\infty}{(Bxt, b_1t, b_2t, b_3t)_\infty} d_qt$$
$$= (\text{const.}) \frac{(Axq/a_3)_\infty}{(Bxq/a_3)_\infty} {}_8W_7\left(\frac{a_2A}{a_3B}; \frac{qb_1}{a_3}, \frac{qb_2}{a_3}, \frac{qb_3}{a_3}, \frac{a_2}{Bx}, \frac{A}{B}; \frac{Bxq}{a_1}\right).$$

Therefore the function

$$\frac{(Axq/a_3)_\infty}{(Bxq/a_3)_\infty} {}_8W_7\left(\frac{a_2A}{a_3B}; \frac{qb_1}{a_3}, \frac{qb_2}{a_3}, \frac{qb_3}{a_3}, \frac{a_2}{Bx}, \frac{A}{B}; \frac{Bxq}{a_1}\right),$$

satisfies the equation $H_3\, y = 0$.

Solutions for $\mathcal{H}_3\, y = 0$

We get integral solutions and series solutions for $H_3\, y = 0$.
So we have solutions for the original equation $\mathcal{H}_3\, y = 0$:

$$x^{\nu-\alpha} \int_{\sigma_1}^{\sigma_2} \frac{(q^{\nu} x t, q^{h_1+\frac{1}{2}} t_1 t, q^{h_2+\frac{1}{2}} t_2 t, q^{h_3+\frac{1}{2}} t_3 t)_\infty}{(xt, q^{\nu+l_1-\frac{1}{2}} t_1 t, q^{\nu+l_2-\frac{1}{2}} t_2 t, q^{\nu+l_3-\frac{1}{2}} t_3 t)_\infty}\, d_q t,$$

$$x^{\nu-\alpha} \frac{(q^{\nu-h_3+\frac{1}{2}} x/t_3)_\infty}{(q^{\frac{1}{2}-h_3} x/t_3)_\infty}$$

$$\times\, {}_8W_7 \left(\frac{t_1 q^{h_1-h_3+\nu}}{t_3}; \frac{t_1 q^{-h_3+l_1+\nu}}{t_3}, \frac{t_2 q^{-h_3+l_2+\nu}}{t_3}, q^{-h_3+l_3+\nu}, \frac{t_1 q^{h_1+\frac{1}{2}}}{x}, q^{\nu}; \frac{x q^{\frac{1}{2}-h_2}}{t_2} \right).$$

This series ${}_8W_7$ is a q-analog of the series solution ${}_2F_1\left(\dfrac{x-t_1}{x-t_3}\dfrac{t_2-t_3}{t_2-t_1}\right)$ (see next page).

Taking the limit $q \to 1$, we have

$$
{}_8W_7\left(\frac{t_1 q^{h_1-h_3+\nu}}{t_3}; \frac{t_1 q^{-h_3+l_1+\nu}}{t_3}, \frac{t_2 q^{-h_3+l_2+\nu}}{t_3}, q^{-h_3+l_3+\nu}, \frac{t_1 q^{h_1+\frac{1}{2}}}{x}, q^{\nu}; \frac{x q^{\frac{1}{2}-h_2}}{t_2}\right)
$$

$$
= \sum_{n=0}^{\infty} \frac{1-(t_2 q^{\bullet}/t_3)q^{2n}}{1-t_2 q^{\bullet}/t_3} \frac{(t_1 q^{\bullet}/t_3, t_1 q^{\bullet}/t_3, t_2 q^{\bullet}/t_3, q^{\bullet}, t_1 q^{\bullet}/x, q^{\bullet})_n}{(q, q^{\bullet}, t_1 q^{\bullet}/t_2, t_1 q^{\bullet}/t_3, x q^{\bullet}/t_3, t_1 q^{\bullet}/t_3)_n} \left(\frac{x q^{\bullet}}{t_2}\right)^n
$$

$$
\xrightarrow{q\to 1} \sum_{n=0}^{\infty} \frac{(-h_3+l_3+\nu; n)(\nu; n)}{(1; n)(h_1-l_1+1; n)} \left(\frac{(1-t_1/t_3)(1-t_1/t_3)(1-t_2/t_3)(1-t_1/x)}{(1-t_1/t_2)(1-t_1/t_3)(1-x/t_3)(1-t_1/t_3)} \frac{x}{t_2}\right)^n
$$

$$
= {}_2F_1\left(\begin{matrix} -h_3+l_3+\nu, \nu \\ h_1-l_1+1 \end{matrix}; \frac{x-t_1}{x-t_3}\frac{t_2-t_3}{t_2-t_1}\right).
$$

Note that $(q^{\alpha})_n/(1-q)^n \xrightarrow{q\to 1} \alpha(\alpha+1)\cdots(\alpha+n-1) = (\alpha; n)$, $(X)_n \xrightarrow{q\to 1} (1-X)^n$.

Summary

In this talk, we give integral solutions and series solutions for the variant of the q-hypergeometric equation of degree three $\mathcal{H}_3\, y = 0$.
The key point is to regard this equation as a q-analog of the Riemann-Papperitz system.

There are many applications:

- q-analogs of Kummer's 24 solutions.
- A new equation that the Askey-Wilson function satisfies.
- A connection problem, and a new linear relation for the Askey-Wilson functions.

There are many future works.

- Some variants of various q-hypergeometric functions/equations.
- Special solutions for the eigenvalue problem of (first degeneration of) the Ruijsenaars-van Diejen operator.

Kummer's 24 solutions are the list of the series solutions for the Gauss' hypergeometric equation. For examples,

$$
{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};x\right),
$$

$$
x^{1-\gamma}{}_2F_1\left(\begin{matrix}\alpha-\gamma+1,\beta-\gamma+1\\ 2-\gamma\end{matrix};x\right),
$$

$$
{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta-\gamma+1\end{matrix};1-x\right).
$$

$$
\begin{tikzcd}
E\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\right) \arrow[d, "\times x^{1-\gamma}"] \arrow[dd, bend left, "x\mapsto 1-x"] \\
E\left(\begin{matrix}\alpha-\gamma+1,\beta-\gamma+1\\ 2-\gamma\end{matrix}\right) \\
E\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta-\gamma+1\end{matrix}\right)
\end{tikzcd}
$$

In the theory of q-difference equations, we can not apply some Möbius transformations. So it is difficult to find q-analogs of the above solutions directly.

By degenerating the equation $\mathcal{H}_3\, y = 0$, we get the variant of the q-hypergeometric equation of degree two $\mathcal{H}_2\, y = 0$ and the Heine's equation:

$$\mathcal{H}_3\, y = 0 \xrightarrow{t_i \to \infty} \mathcal{H}_2\, y = 0 \xrightarrow{t_j \to 0} \text{Heine.}$$

So by taking the same limit for solutions, we obtain solutions for $\mathcal{H}_2\, y = 0$ and the Heine's equation systematically. For integrals, we have

$$\int \prod_{i=0}^{3} \frac{(q^\bullet t_i t)_\infty}{(q^\bullet t_i t)_\infty} d_q t \to \int t^\bullet \prod_{i=0}^{2} \frac{(q^\bullet t_i t)_\infty}{(q^\bullet t_i t)_\infty} d_q t \to \int t^\bullet \prod_{i=0}^{1} \frac{(q^\bullet t_i t)_\infty}{(q^\bullet t_i t)_\infty} d_q t.$$

For series, we have many solutions. We summarize them in next page.

$$\mathcal{H}_3 \longrightarrow \mathcal{H}_2 \longrightarrow \text{Heine}$$

$${}_8W_7$$

$\mathcal{H}_2$	Heine
${}_3\varphi_2\left(\begin{matrix}\bullet,\bullet,\bullet t_1/x\\ \bullet,\bullet t_1/t_2\end{matrix};\bullet\frac{x}{t_2}\right)$	${}_2\varphi_1\left(\begin{matrix}\bullet,\bullet\\ \bullet\end{matrix};x\right)$
${}_3\varphi_2\left(\begin{matrix}\bullet,\bullet,\bullet x/t_1\\ \bullet,\bullet x/t_2\end{matrix};\bullet\frac{t_1}{t_2}\right)$	${}_2\varphi_1\left(\begin{matrix}\bullet,\bullet\\ \bullet\end{matrix};\frac{\bullet}{x}\right)$
${}_3\varphi_2\left(\begin{matrix}\bullet,\bullet,\bullet t_1/t_2\\ \bullet,\bullet x/t_2\end{matrix};q\right)$	${}_3\varphi_2\left(\begin{matrix}\bullet,\bullet,\bullet x\\ \bullet,0\end{matrix};q\right)$
${}_3\varphi_2\left(\begin{matrix}\bullet,\bullet,\bullet t_1/t_2\\ \bullet,\bullet t_1/x\end{matrix};\bullet\frac{t_2}{x}\right)$	${}_3\varphi_2\left(\begin{matrix}\bullet,\bullet,\bullet/x\\ \bullet,0\end{matrix};q\right)$
${}_3\varphi_2\left(\begin{matrix}\bullet,\bullet,\bullet t_1/x\\ \bullet,\bullet t_2/x\end{matrix};q\right)$	${}_2\varphi_2\left(\begin{matrix}\bullet,\bullet\\ \bullet,\bullet x\end{matrix};\bullet x\right)$
${}_3\varphi_2\left(\begin{matrix}\bullet,\bullet,\bullet x/t_1\\ \bullet,\bullet t_2/t_1\end{matrix};q\right)$	${}_2\varphi_2\left(\begin{matrix}\bullet,\bullet\\ \bullet,\bullet/x\end{matrix};\frac{\bullet}{x}\right)$
	${}_3\varphi_2\left(\begin{matrix}\bullet,\bullet,0\\ \bullet,\bullet x\end{matrix};q\right)$
	${}_3\varphi_2\left(\begin{matrix}\bullet,\bullet,0\\ \bullet,\bullet/x\end{matrix};q\right)$

$${}_r\varphi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};x\right) = \sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r)_n}{(q,b_1,\ldots,b_s)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{s+1-r}x^n.$$

On the Askey-Wilson function

The very-well-poised balanced q-hypergeometric function

$$ {}_8W_7\left(a; b, c, d, e, f; \frac{a^2q^2}{bcdef}\right), $$

is well known as the Askey-Wilson function.
Some q-difference equations that the Askey-Wilson function satisfies were obtained (cf. Askey-Wilson, Ismail-Rahman).

The equation $\mathcal{H}_3\, y = 0$ is different from these equations, so we get another equation for the Askey-Wilson function.

On a connection problem

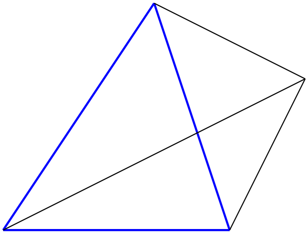
We give 6 integral solutions for the equation $\mathcal{H}_3\, y = 0$:

$$x^{\nu-\alpha}\int_{\sigma_1}^{\sigma_2} \frac{(q^{\nu}xt, q^{h_1+\frac{1}{2}}t_1 t, q^{h_2+\frac{1}{2}}t_2 t, q^{h_3+\frac{1}{2}}t_3 t)_\infty}{(xt, q^{\nu+l_1-\frac{1}{2}}t_1 t, q^{\nu+l_2-\frac{1}{2}}t_2 t, q^{\nu+l_3-\frac{1}{2}}t_3 t)_\infty}\, d_q t,$$

where σ_1, $\sigma_2 \in \{q^{\frac{1}{2}-h_1}/t_1, q^{\frac{1}{2}-h_2}/t_2, q^{\frac{1}{2}-h_3}/t_3, q^{1-\nu}/x\}$.
So there are 4 linear relations for such solutions.
We consider a connection problem:

"Find 4 linear relations for solutions of the equation $\mathcal{H}_3\, y = 0$."

By the definition $\int_{\sigma_1}^{\sigma_2} = \int_0^{\sigma_2} - \int_0^{\sigma_1}$ of the Jackson integral, we have $\int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2}^{\sigma_3} + \int_{\sigma_3}^{\sigma_1} = 0$. Thus we get 3 relations for the above solutions.



Another one is derived by considering Mimachi's relations for the Jackson integral of the Jordan-Pochhammer type:

$$\int_0^{q/(Ax)} t^{\alpha-1} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} \frac{(Axt)_\infty}{(Bxt)_\infty} d_q t$$
$$= \sum_{i=1}^{M} C_i \int_0^{q/(a_i)} t^{\alpha-1} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} \frac{(Axt)_\infty}{(Bxt)_\infty} d_q t$$
$$+ C_{M+1} \int_0^{b_1} t^{\rho} \prod_{i=1}^{M} \frac{(qt/b_i)_\infty}{(qt/a_i)_\infty} \frac{(qt/(Bx))_\infty}{(qt/(Ax))_\infty} d_q t \quad (q^\rho = q^{-\alpha} \frac{a_1 \cdots a_M A}{b_1 \cdots b_M B}).$$

Here $C_1, \ldots, C_{M+1}$ are given by some ratio of the theta function $\theta(x) = (x, q/x)_\infty$. We put the parameters as $M = 3$, $\alpha = 1$, $a_1 a_2 a_3 A = q^2 b_1 b_2 b_3 B$ (and $\rho = 1$). Then the coefficient C_{M+1} is vanished (because $C_{M+1} = \dfrac{\cdots \times \theta(q^\rho)}{\cdots}$).

We have

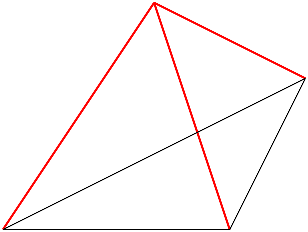
$$\int_0^{q/(Ax)} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} \frac{(Axt)_\infty}{(Bxt)_\infty} d_q t = \sum_{i=1}^{3} C_i \int_0^{q/(a_i)} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} \frac{(Axt)_\infty}{(Bxt)_\infty} d_q t.$$

So we get

$$C \int_0^{q/(Ax)} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} \frac{(Axt)_\infty}{(Bxt)_\infty} d_q t = \sum_{i=1}^{3} C_i \int_{q/(Ax)}^{q/(a_i)} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} \frac{(Axt)_\infty}{(Bxt)_\infty} d_q t,$$

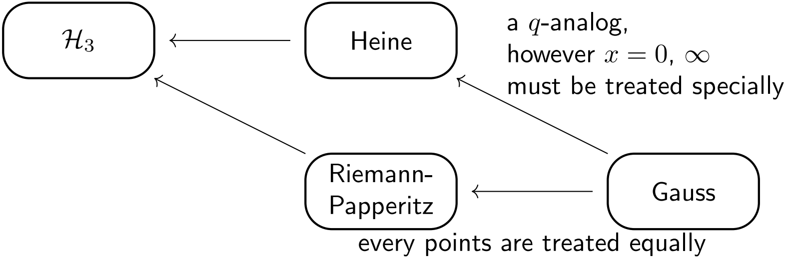
where $C = 1 + C_1 + C_2 + C_3$. The integral of the L.H.S. is not a solution for the equation $H_3\, y = 0$, although the R.H.S. is a solution, so we finally find $C = 0$ and

$$\sum_{i=1}^{3} C_i \int_{q/(Ax)}^{q/(a_i)} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} \frac{(Axt)_\infty}{(Bxt)_\infty} d_q t = 0.$$



Future works: a variant of the q-Appell-Lauricella system

A q-analog of the Riemann-Papperitz system has not been considered because the points $x = 0$ and ∞ are special in the theory of q-difference equations. However we could consider such a q-analog.



We can treat every point equally even in q-difference equations!

It is interesting to consider some q-hypergeometric functions with this slogan. Now I am trying for the q-Appell-Lauricella case.

The q-Appell-Lauricella system

The q-Appell-Lauricella system is defined as follows:

$$[(1-T_i)(1-cq^{-1}T)-x_i(1-b_iT_i)(1-aT)]y=0 \quad (1\le i\le M),$$
$$[x_i(1-b_iT_i)(1-T_j)-x_j(1-b_jT_j)(1-T_i)]y=0 \quad (1\le i<j\le M).$$

The rank of this system is $M+1$, and it has integral solutions and series solutions:

$$\int_C t^{\alpha-1}\frac{(qt)_\infty}{(ct/a)_\infty}\prod_{i=1}^{M}\frac{(b_ix_it)_\infty}{(x_it)_\infty}d_qt,$$

$$\varphi_D\left(\begin{matrix} a;\{b_i\} \\ c \end{matrix};\{x_i\}\right)=\sum_{m_1,\dots,m_M\ge 0}\frac{(a)_{m_1+\cdots+m_M}}{(c)_{m_1+\cdots+m_M}}\prod_{i=1}^{M}\left(\frac{(b_i)_{m_i}}{(q)_{m_i}}x_i^{m_i}\right).$$

We consider the following Jackson integral:

$$\int_{q/a_i}^{q/a_j} \prod_{i=1}^{M+3} \frac{(a_i t)_\infty}{(b_i t)_\infty} d_q t \quad (a_1 \cdots a_{M+3} = q^2 b_1 \cdots b_{M+3}).$$

We find some properties about this integral:

- The integral satisfies a q-difference system of rank $M+1$.
- The integral can be transformed to some multiple q-hypergeometric series. This transformation can be derived by taking some limit for the Kajihara's transformation formula.
- A connection problem associated with the above system is solved.

We expect that many solutions for the q-Appell-Lauricella system can be obtained by degenerating the multiple series. And there are more applications, I think.

A q-analog of the Riemann-Papperitz system has not been considered because the points $x = 0$ and ∞ are special in the theory of q-difference equations. However we could consider such a q-analog.

$$
\begin{tikzcd}
\boxed{\mathcal{H}_3} & \boxed{\text{Heine}} \arrow[l] & \\
 & \boxed{\text{Riemann-Papperitz}} \arrow[ul] & \boxed{\text{Gauss}} \arrow[l] \arrow[ul]
\end{tikzcd}
$$

a q-analog,
however $x = 0$, ∞
must be treated specially

every points are treated equally

We can treat every point equally even in q-difference equations!

Thank you very much for your attention!