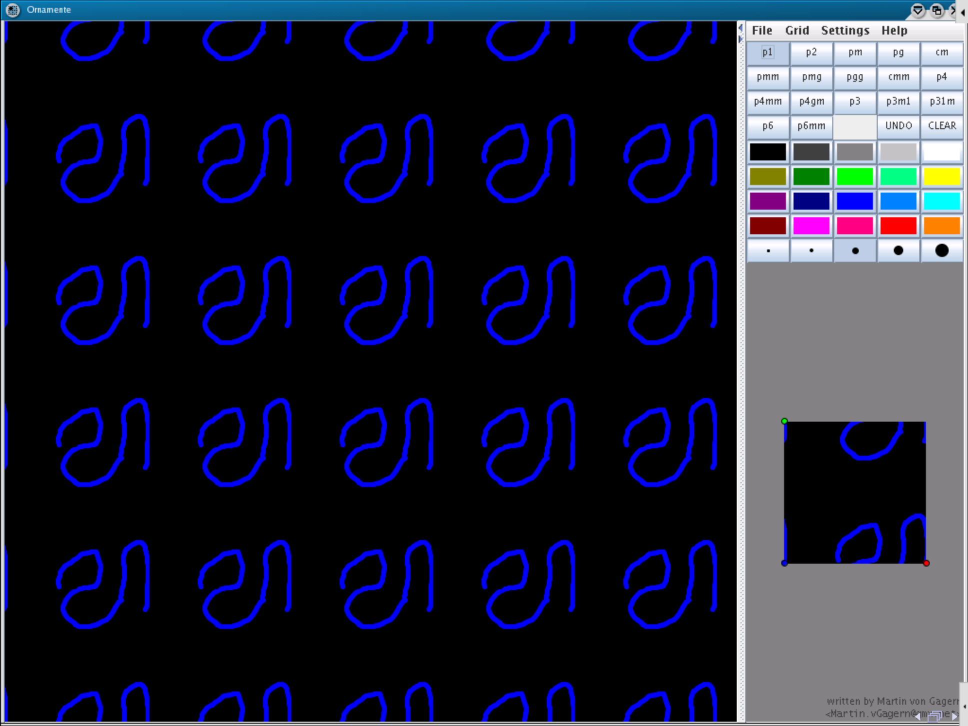
# Hyperbolic Ornaments

## Drawing in Non-Euclidean Crystallographic Groups

Martin von Gagern
*joint work with* Jürgen Richter-Gebert

Technische Universität München

Second International Congress on Mathematical Software,
September 1 2006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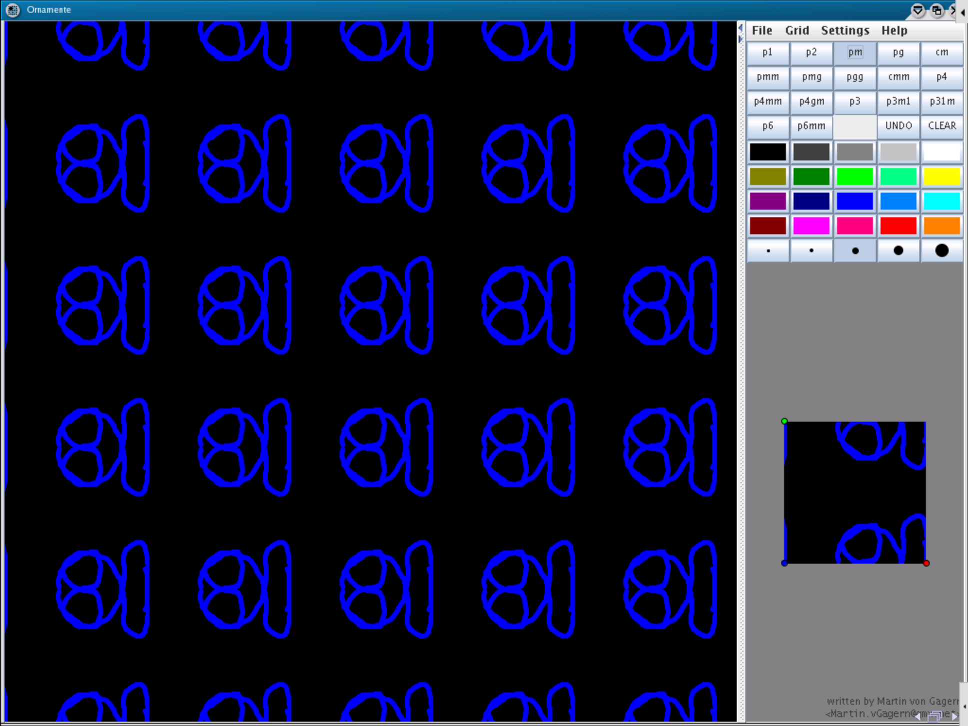
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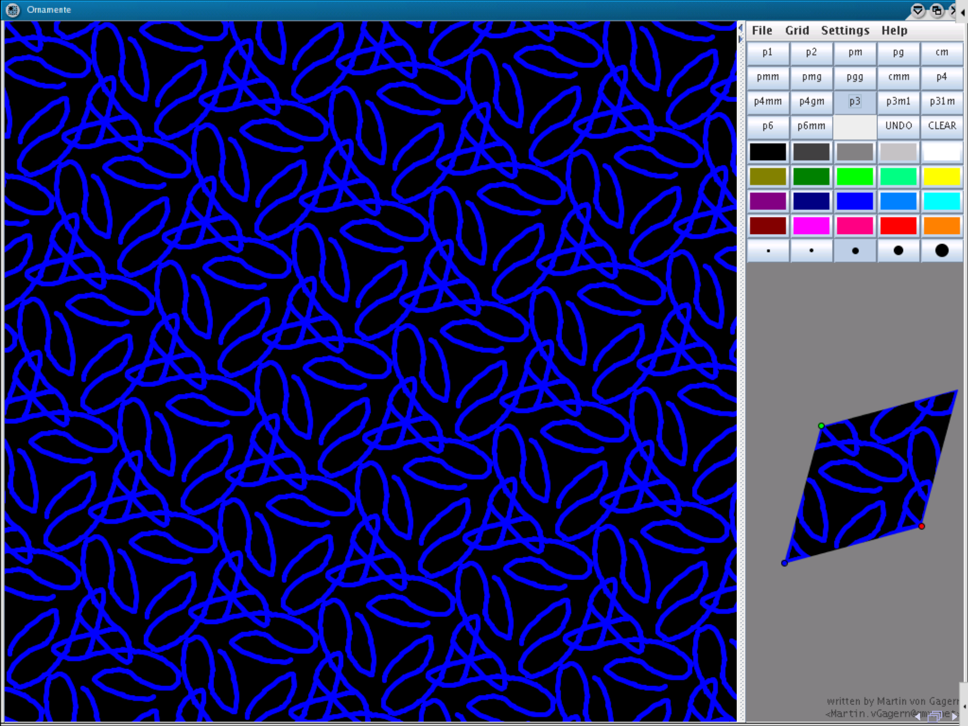
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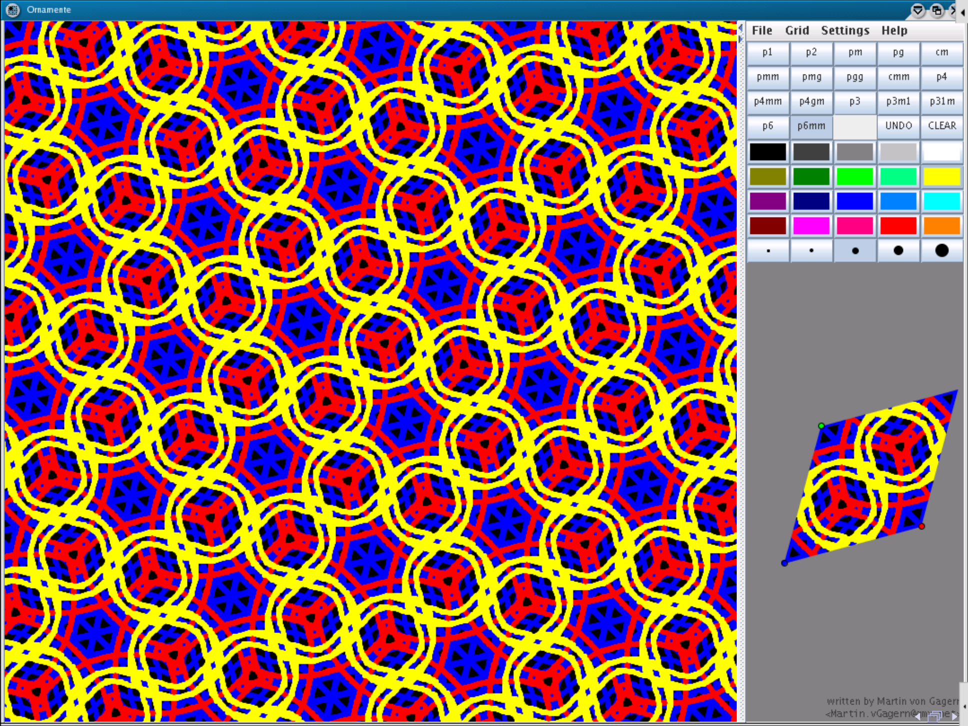
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# Educational Value

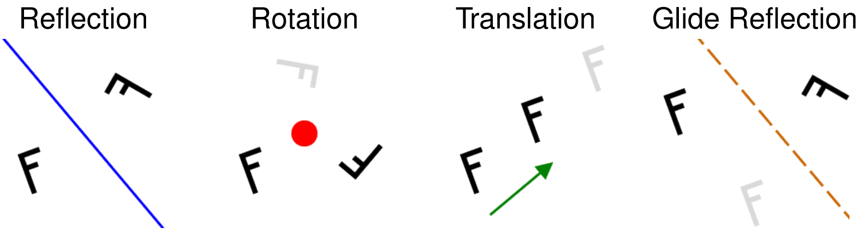
# Escher


# Hyperbolic Escher

# Hyperbolic Escher


# Outline

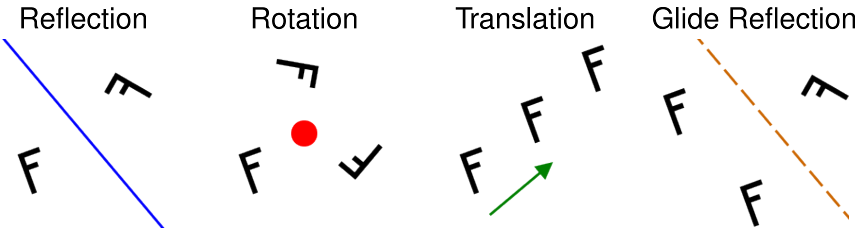
# Rigid Motions




## Definition (Rigid Motion)

Rigid Motions ( = Isometries) are the length-preserving mappings of the plane onto itself.

# Rigid Motions

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# Groups of Rigid Motions

- Group $E(2)$: all euclidean planar isometries
- Discrete Subgroups

**Definition (Discreteness)**

A group $G$ is discrete if around every point $P$ of the plane there is a neighborhood devoid of any images of $P$ under the group operations.

The discrete groups of rigid motions in the euclidean plane:

- 17 Wallpaper Groups
- 7 Frieze Groups
- 2 kinds of Rosette Groups

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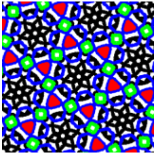


# Anatomy of the Hyperbolic Plane

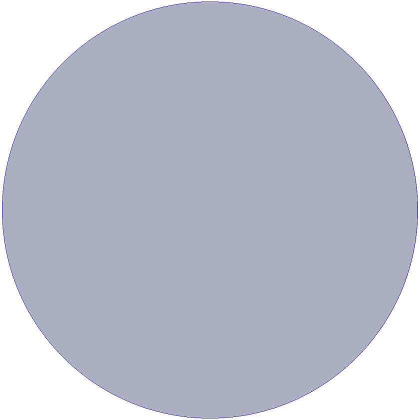
**Definition (Hyperbolic Axiom of Parallels)**

Given a point $P$ outside a line $\ell$
there exist at least two lines through $P$ that do not intersect $\ell$.

- Many facts of euclidean geometry don't rely on the Axiom of Parallels and are true in hyperbolic geometry as well.
- The sum of angles in a triangle is less than $\pi$.
- Lengths are absolute, scaling is not an automorphism.
- Geometry of constant negative curvature.

# Poincaré Disc Model

- hyperbolic points:
  inside of the unit circle
- hyperbolic lines:
  lines and circles
  perpendicular to the unit circle
- hyperbolic angle:
  identical to euclidean angle
- hyperbolic distance:
  changes with
  distance from center



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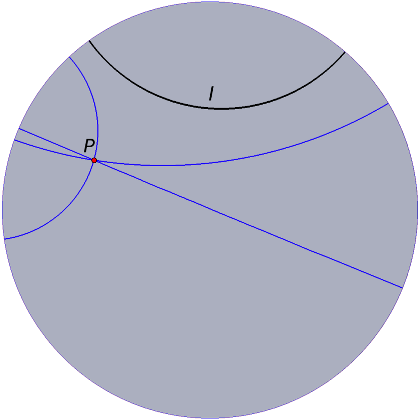

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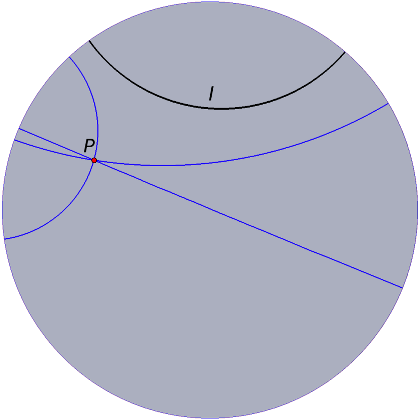

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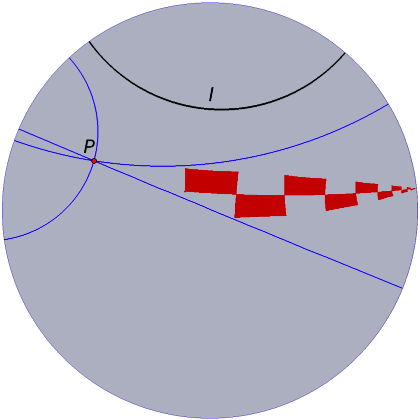
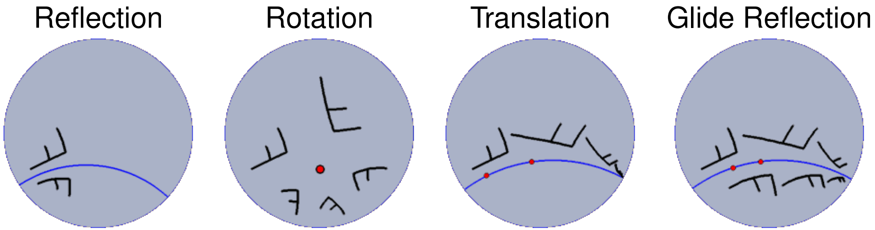


# Hyperbolic Rigid Motions

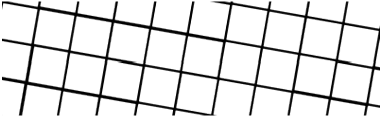



N.B.: translations now have only a single fixed line.

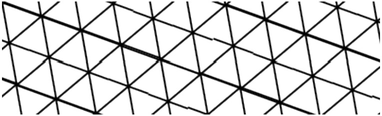
# Outline

# Tilings by regular Polygons

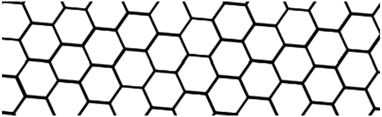
- Square



- Triangular



- Hexagonal



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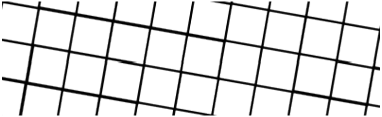

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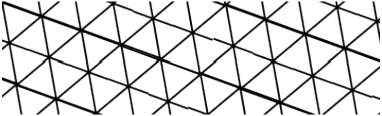

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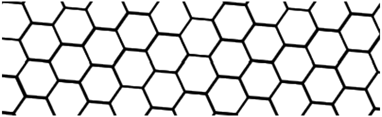
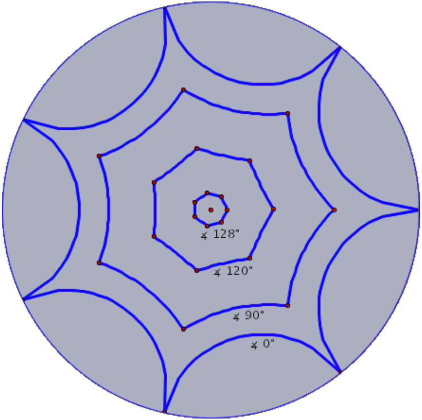
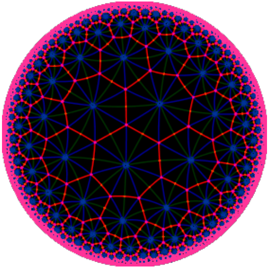




# From regular Polygons to Triangles



regular heptagons

angles $\frac{2\pi}{3}$



$\triangle(2,3,7)$

angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$



regular triangles

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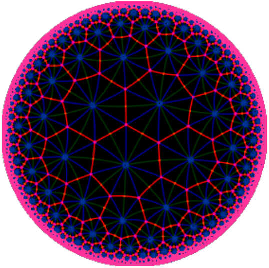

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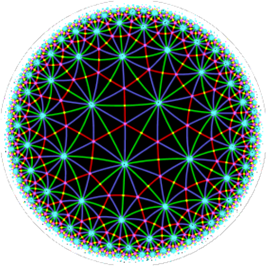

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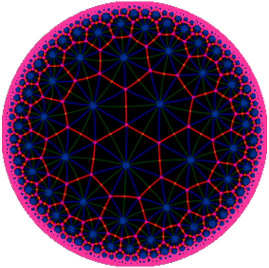

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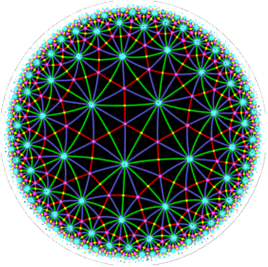

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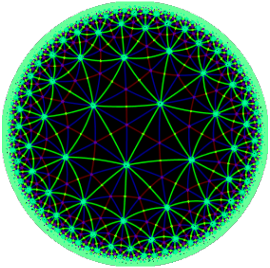

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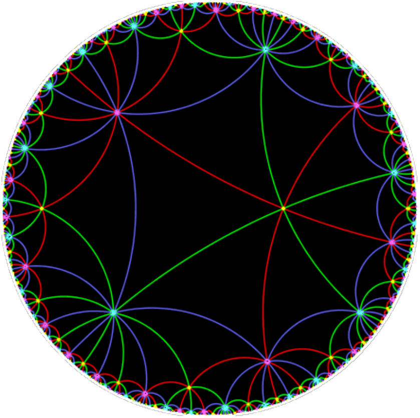
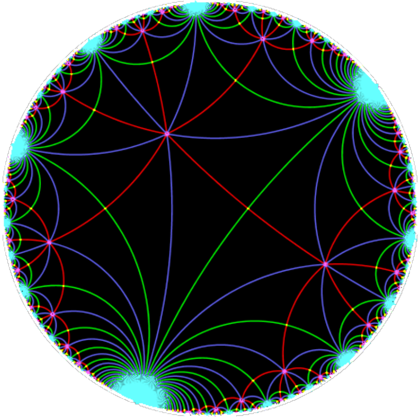
# General Tesselations



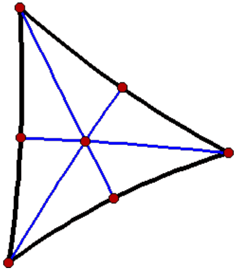
$\triangle(4, 6, 7)$



$\triangle(2, 5, \infty)$

# Why All Angles are Different

- $\triangle(n, \ n, \ n) \subset \triangle(2, \ 3, 2n)$
- $\triangle(n, 2n, 2n) \subset \triangle(2, \ 4, 2n)$
- $\triangle(n, \ m, \ m) \subset \triangle(2, m, 2n)$



$$\triangle(k, m, n) : \frac{\pi}{k} + \frac{\pi}{m} + \frac{\pi}{n} < \pi$$

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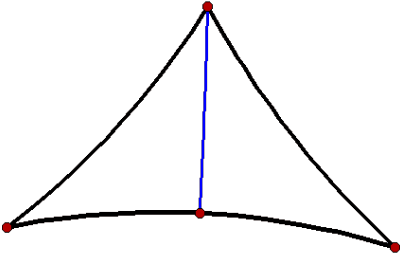

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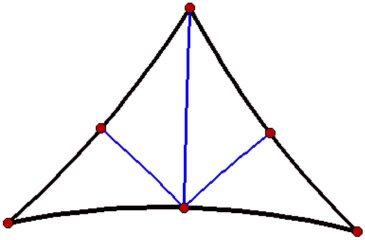

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# Algebraic Calculations

General triangle reflection group $\triangle(k, m, n)$

- Coxeter group (finitely represented group for GAP)
  $\langle a, b, c \mid a^2 = 1, b^2 = 1, c^2 = 1, (ab)^k = 1, (ac)^m = 1, (bc)^n = 1 \rangle$
- Subgroups with finite index are
  non-euclidean crystallographic (N.E.C.) groups
- Orientation preserving subgroups are Fuchsian

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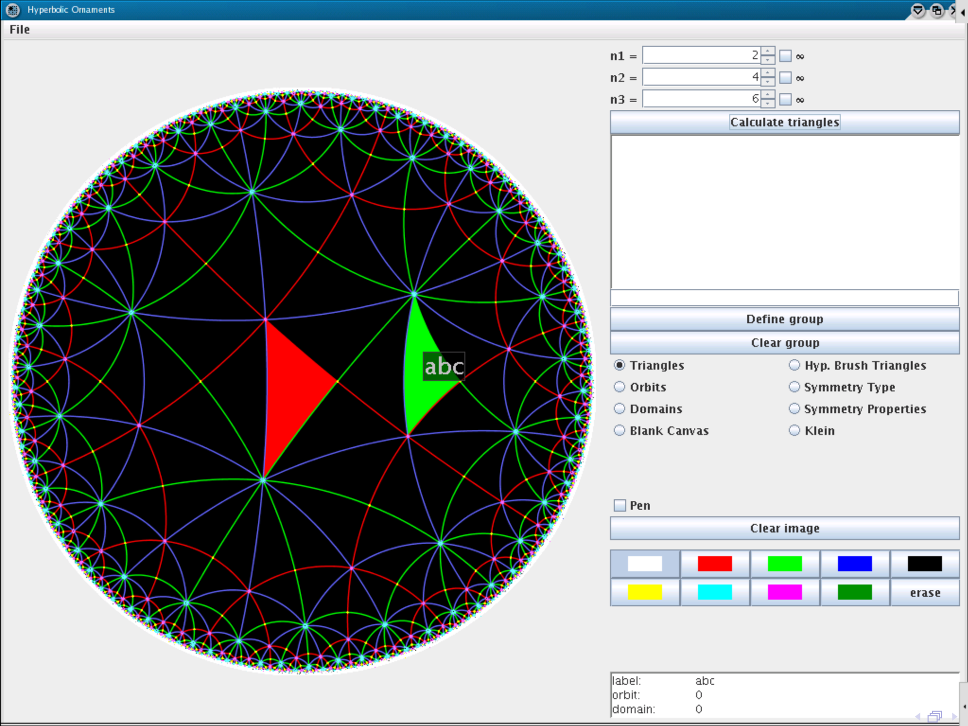
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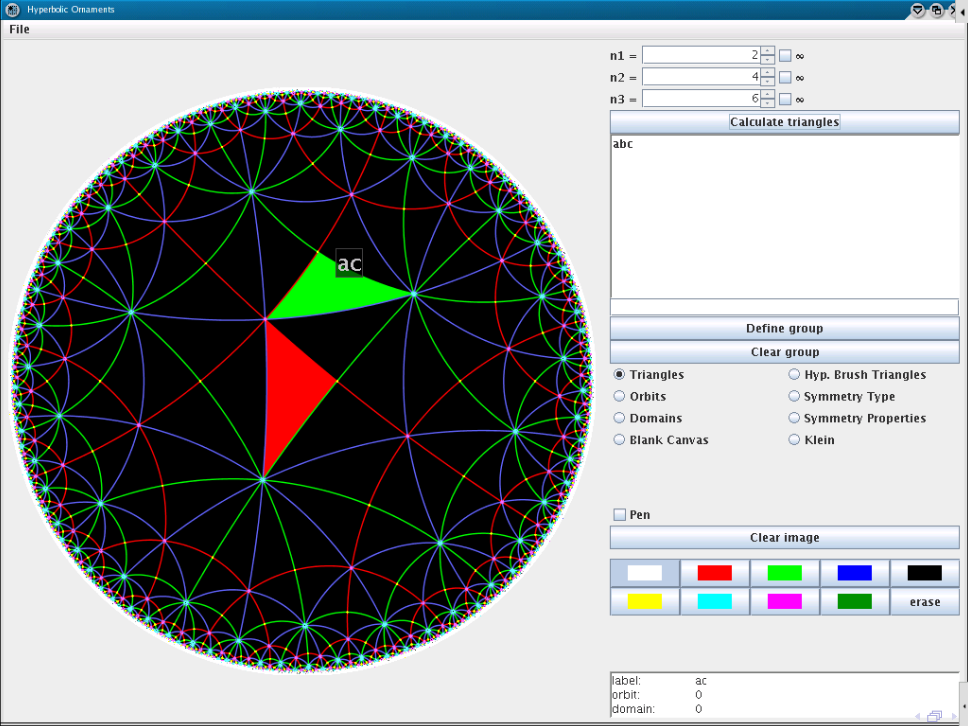
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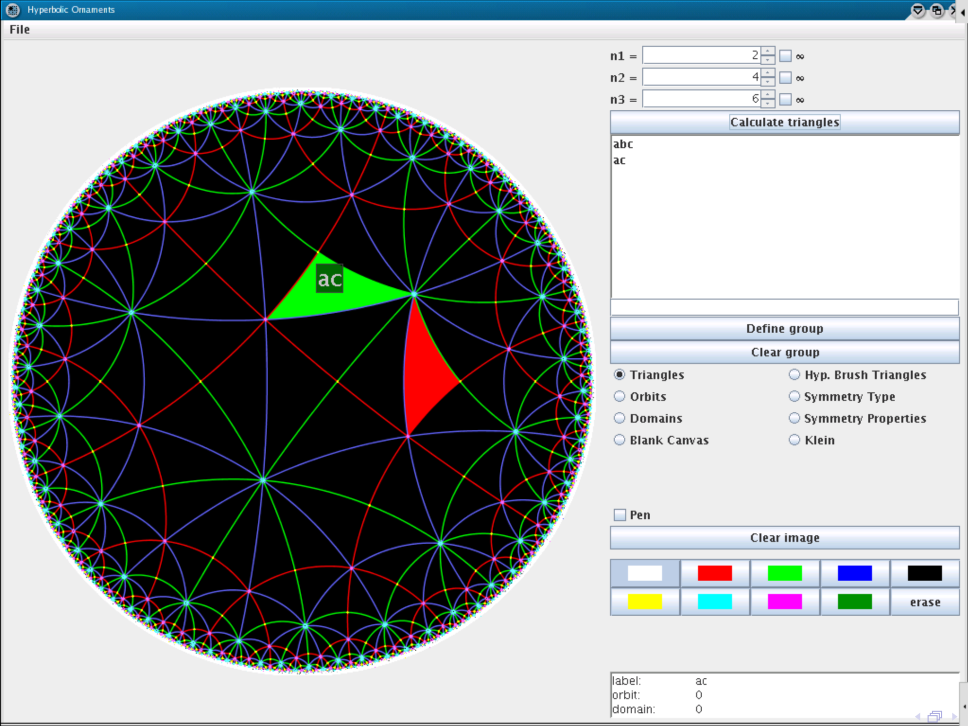
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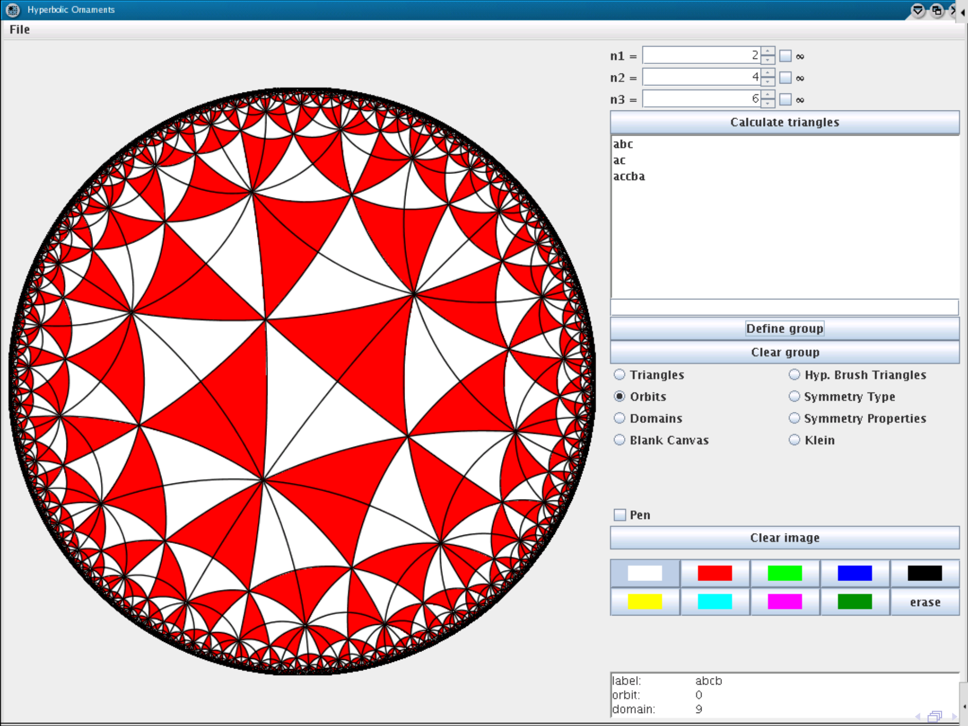
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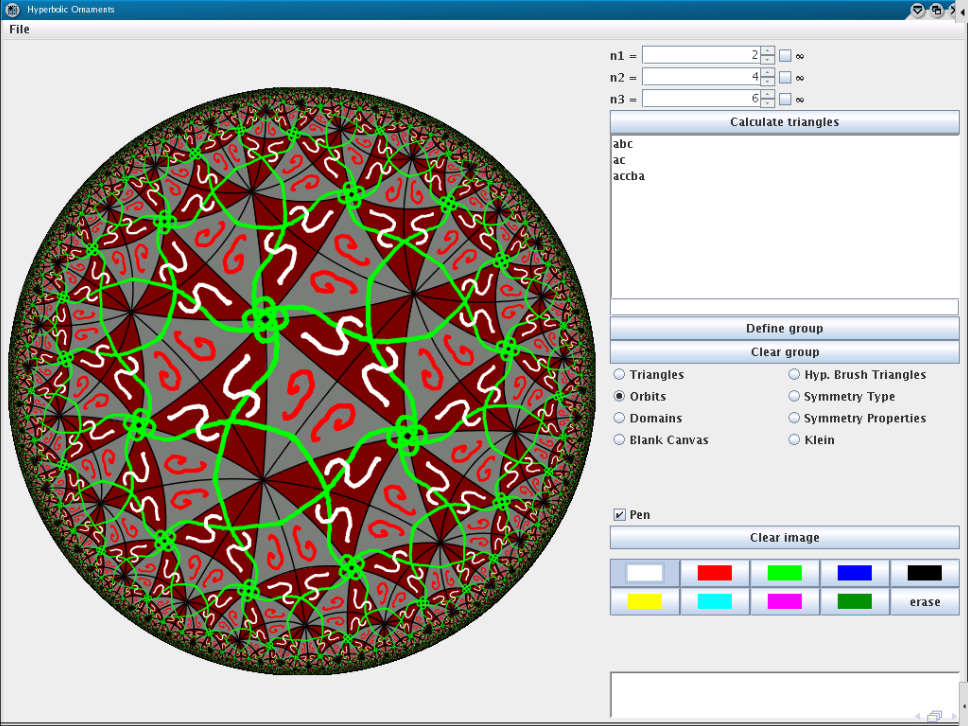
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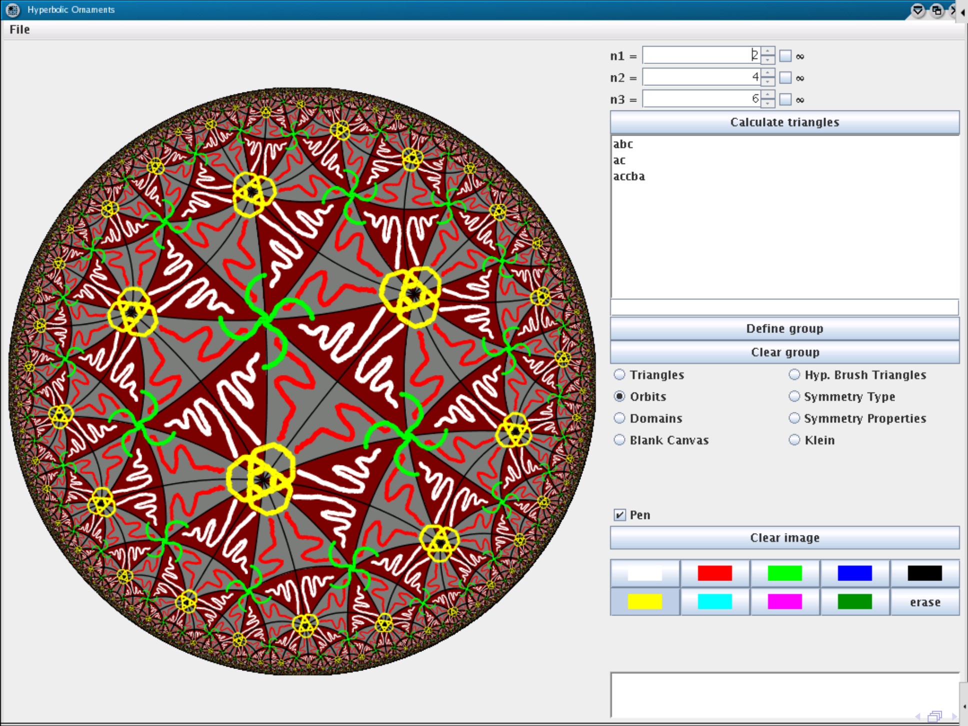
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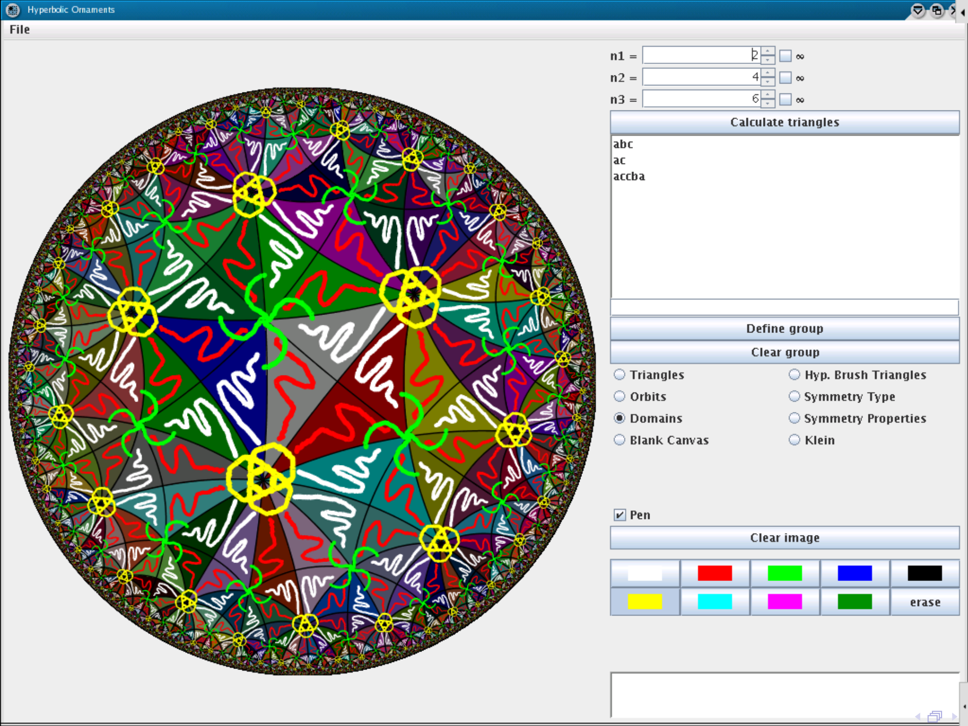
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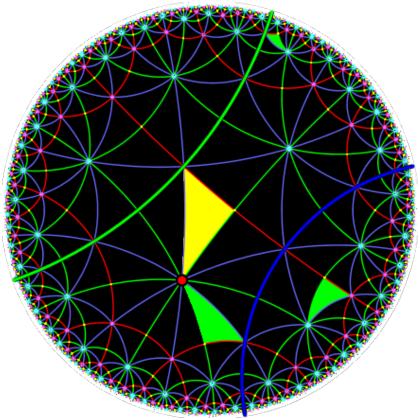
# Group Generation

1. Generator entered by user
2. Add inverse operations
3. Find “all” combinations
   - Group representation
   - Orbit of centerpiece
   - Each element starts a new domain
4. For all triangles that are not yet part of any orbit
   - add triangle to central domain
   - combine triangle with all group elements to calculate its orbit, adding to domains



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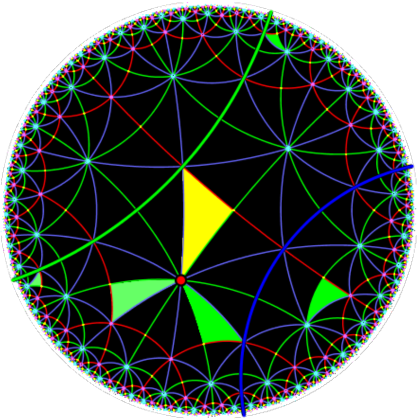

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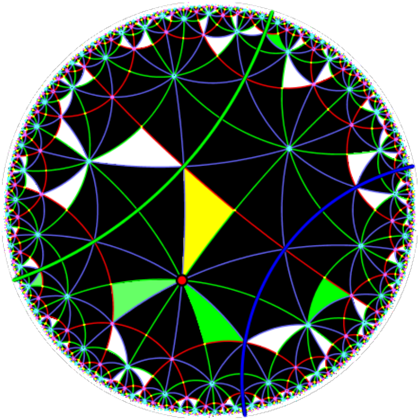

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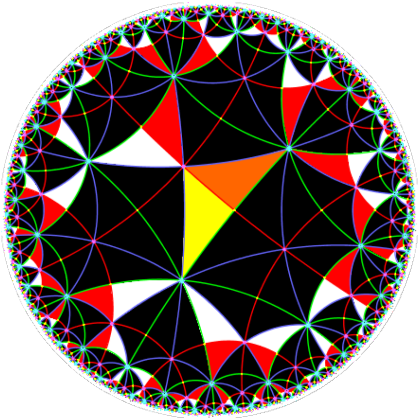

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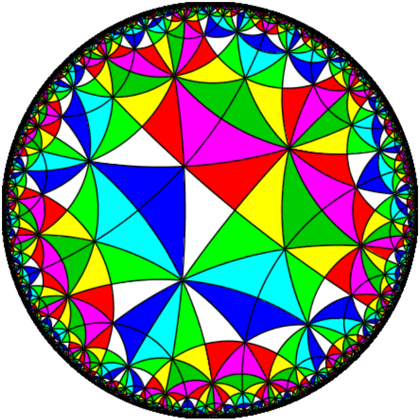
1. Generator entered by user
2. Add inverse operations
3. Find “all” combinations
    - Group representation
    - Orbit of centerpiece
    - Each element starts a new domain
4. For all triangles that are not yet part of any orbit
    - add triangle to central domain
    - combine triangle with all group elements to calculate its orbit, adding to domains



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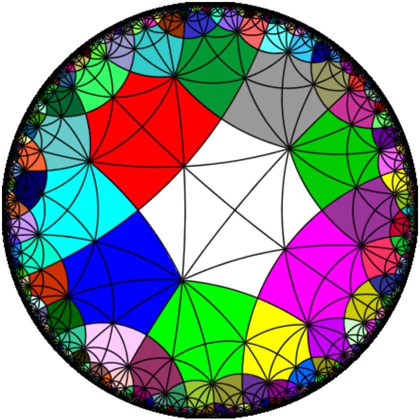
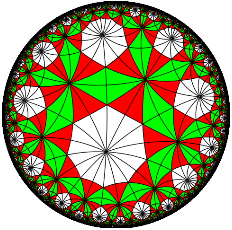
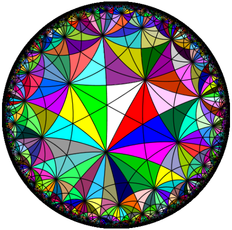
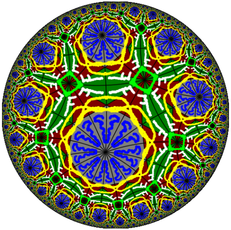
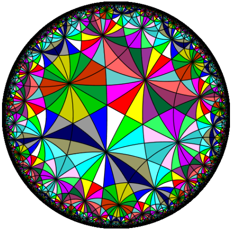
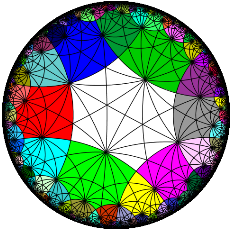


# Group Visualization


# Fast and Perfect Drawing

**Fast** draw smooth lines in real time
**Perfect** image looks as correct as display hardware allows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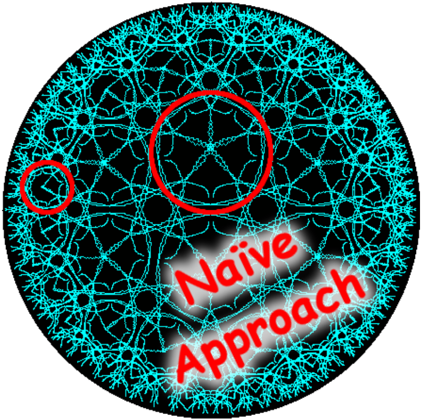


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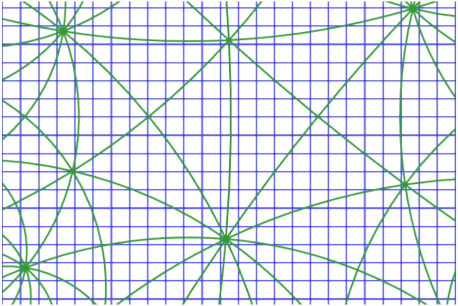
# Reverse Pixel Lookup

1. Scan convert triangles
   *Triangle preprocessing*
2. Map into central domain
   *Group preprocessing*
3. Update only changes
   *Realtime drawing*
4. Supersampling
   *Antialiasing*



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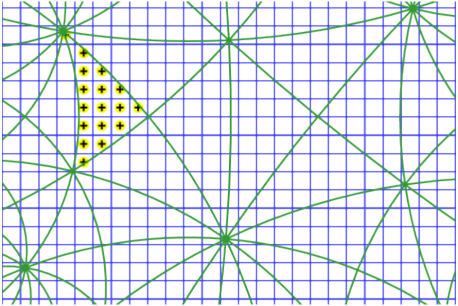

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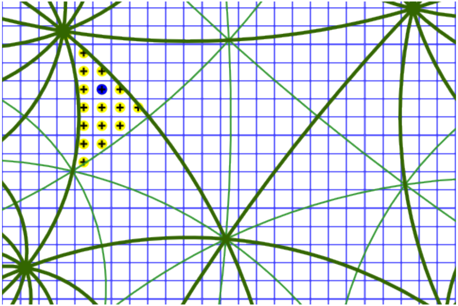

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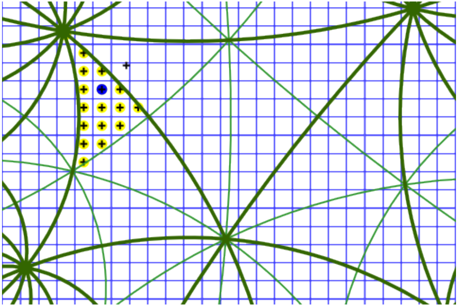

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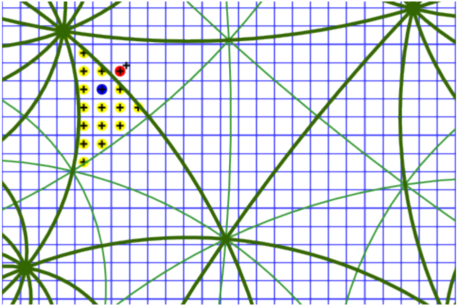

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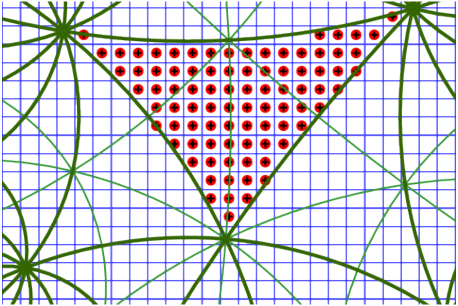

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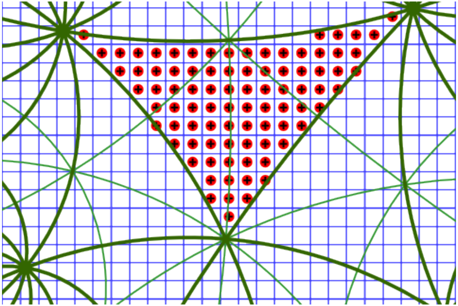

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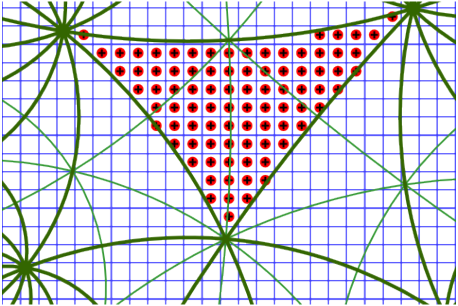
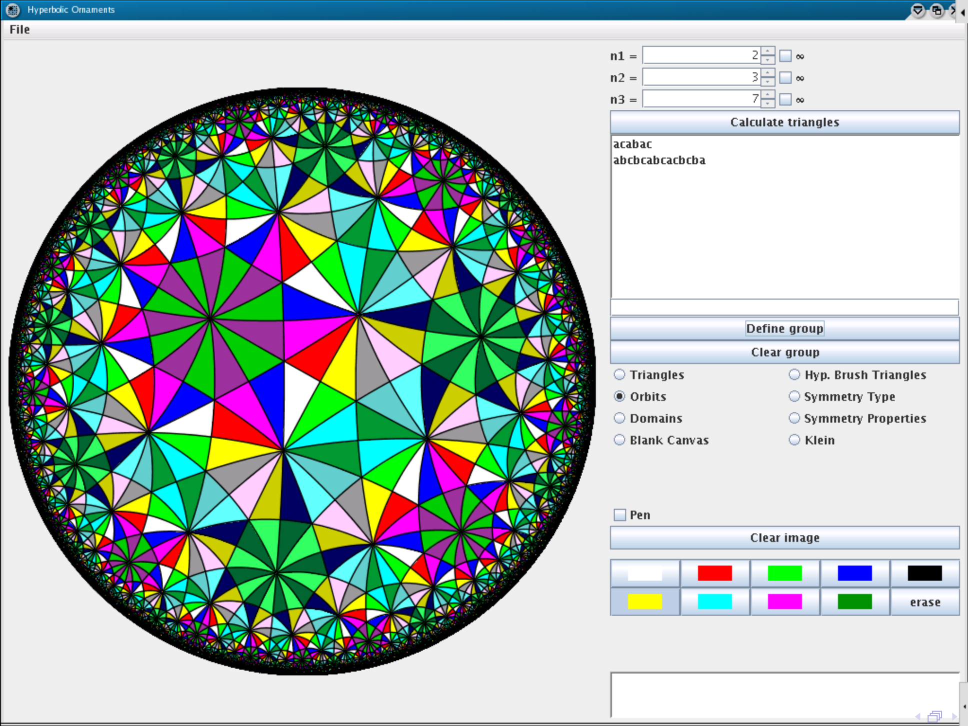
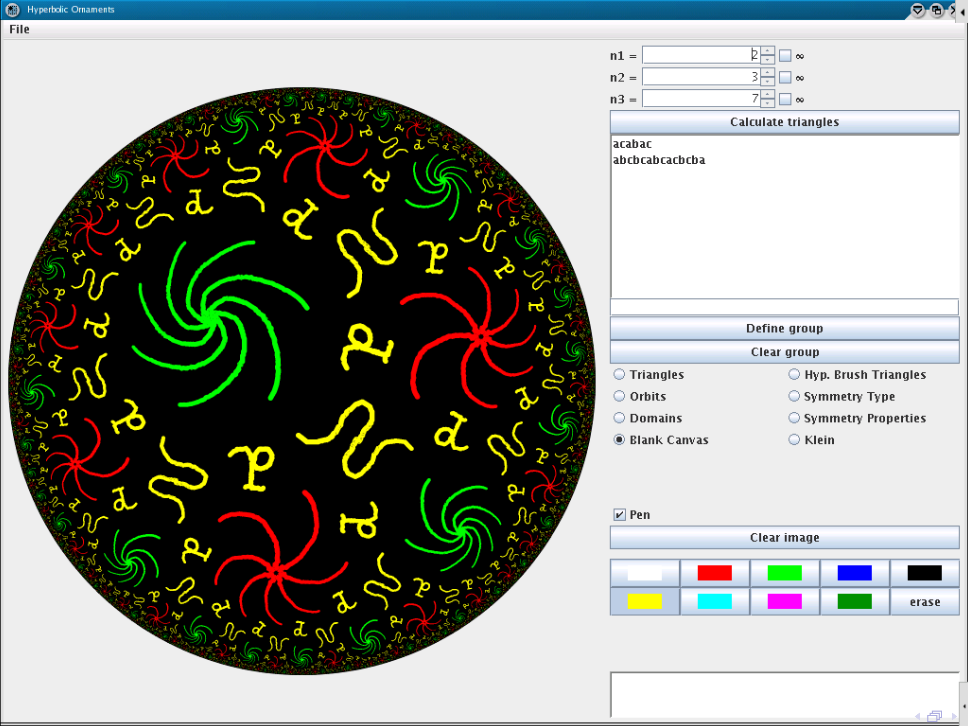



Hyperbolic Ornaments
File
n1 = 2 ∞
n2 = 3 ∞
n3 = 7 ∞
Calculate triangles
acabac
abcbcabcacbcba
Define group
Clear group
Triangles
Orbits
Domains
Blank Canvas
Hyp. Brush Triangles
Symmetry Type
Symmetry Properties
Klein
Pen
Clear image
erase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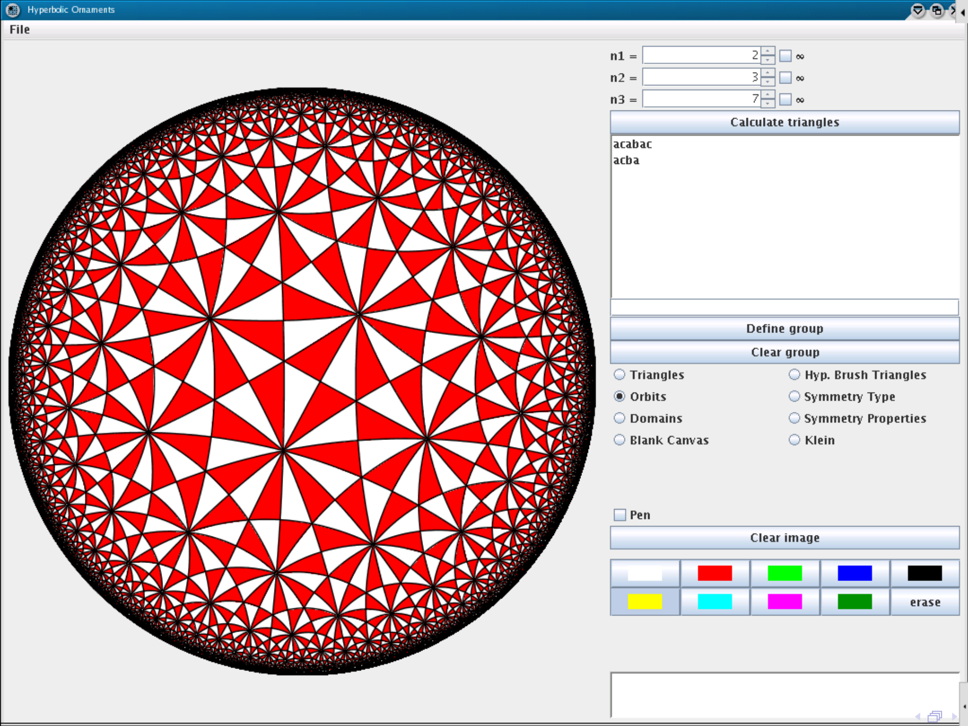
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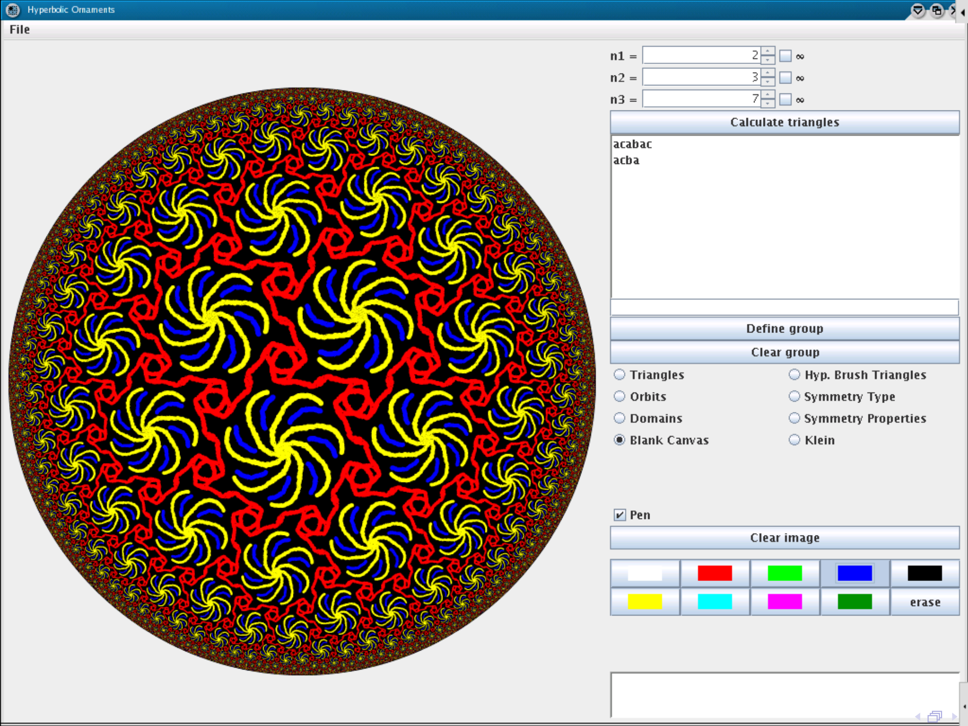
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